# Stability Investigation of Direct Integration Algorithms Using Lyapunov-Based Approaches

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## Abstract

In structural dynamics, direct explicit and implicit integration algorithms are commonly used to solve the temporally discretized differential equations of motion for linear and nonlinear structures. The stability of different integration algorithms for linear elastic structures has been extensively studied for several decades. However, investigations of the stability applied to nonlinear structures are relatively limited and rather challenging. Recently, the authors proposed two systematic approaches using Lyapunov stability theory to investigate the stability property of direct integration algorithms of nonlinear dynamical systems. The first approach is a numerical one that transforms the stability analysis to a problem of convex optimization. The second approach investigates the Lyapunov stability of explicit algorithms considering the strictly positive real lemma. This paper reviews and compares these two Lyapunov-based approaches in terms of their merits and limitations.

**Keywords:** Convex optimization, Direct integration algorithm, Lyapunov stability, Nonlinear, Strictly positive real lemma, Structural dynamics.

## Introduction

In structural dynamics, direct integration algorithms are commonly used to solve the differential equations of motion after they are temporally discretized to estimate dynamic responses of structures, e.g., seismic responses of bridges [1]. Integration algorithms are categorized into either implicit or explicit. An integration algorithm is explicit when the responses of the next time step depend on the responses of previous and current time steps only. Otherwise, it is implicit. Numerous implicit and explicit direct integration methods have been developed, including the Newmark family of algorithms [2], the TRBDF2 algorithm [3], and the Operator-Splitting (OS) algorithms [4]. Liang et al. [5,6] investigated the suitability of the OS algorithms for efficient nonlinear seismic response of multi-degree of freedom (MDOF) reinforced concrete highway bridge systems and promising results in terms of accuracy and numerical stability were obtained. The stability of different integration algorithms for linear elastic structures has been studied extensively for several decades, e.g., [7]. Studies related to the stability properties of these integration algorithms applied to nonlinear dynamic analysis are relatively limited and, unlike linear ones, are rather complicated and challenging. This is attributed to specific properties of the nonlinear systems. For example, initial conditions affect the stability of nonlinear systems and the principle of superposition does not hold.

Lyapunov stability theory [8,9], developed by the Russian mathematician Aleksandr Lyapunov in [10], is the most complete framework of stability analysis for dynamical systems. It is based on constructing a function of the system state coordinates that serves as a generalized norm of the solution of the dynamical system. The most important property of Lyapunov stability theory is the fact that conclusions about the stability behavior of the dynamical system can be obtained without

actually computing the system solution trajectories. As such, Lyapunov stability theory has become one of the most fundamental and standard tools of dynamical systems and control theory.
Generally speaking, constructing the above-mentioned energy function for the nonlinear system is not readily available. To address this difficulty, the authors proposed two approaches. In the first, a numerical approach is proposed to transform the problem of seeking a Lyapunov function to a convex optimization problem [11,12], which can solve the problem in a simple and clear manner. Convex optimization minimizes convex functions over convex sets, in which a wide range of problems can be formulated in this way. In this optimization, any local minimum must be a global one, which is an important property leading to reliable and efficient solutions using, e.g., interior-point methods, which are suitable for computer-aided design or analysis tools [13]. The second approach proposed by the authors is based on formulating a generic explicit integration algorithm into a nonlinear system governed by a nonlinear function of the basic forces. This enables investigating the Lyapunov stability of explicit algorithms by means of the strictly positive real lemma [11,14]. The study for nonlinear single degree of freedom (SDOF) systems in [14] was extended to MDOF ones in [15]. This approach transforms the stability analysis of the formulated nonlinear system to investigating the strictly positive realness of its corresponding transfer function matrix. This is further equivalent to a problem of convex optimization that can be solved numerically.

This paper reviews and compares these previously discussed two Lyapunov-based approaches in terms of their merits and limitations. The first numerical approach is shown to be generally applicable to implicit and explicit direct integration algorithms for various nonlinear force-deformation relationships. Moreover, this approach can potentially be extended to nonlinear MDOF systems but may involve extensive computations. The second approach is applicable to explicit algorithms without adopting any approximation and is computationally efficient even for MDOF systems.

## Integration Algorithm

The discretized equations of motion of a MDOF system under an external dynamic force excitation is expressed as follows:

$$\mathbf{m}\ddot{\mathbf{u}}_{i+1} + \mathbf{c}\dot{\mathbf{u}}_{i+1} + \mathbf{f}(\mathbf{u}_{i+1}) = \mathbf{p}_{i+1} \tag{1}$$

where $\mathbf{m}$ and $\mathbf{c}$ are the mass and damping matrices, and $\ddot{\mathbf{u}}_{i+1}$, $\dot{\mathbf{u}}_{i+1}$, $\mathbf{f}_{i+1}$, and $\mathbf{p}_{i+1}$ are respectively the acceleration, velocity, restoring force, and external force vectors at time step $i+1$. The restoring force $\mathbf{f}(\mathbf{u})$ is generally defined as a function of the displacement vector $\mathbf{u}$. It is to be noted that bold-faced symbols indicate arrays, either vectors or matrices.

A single-step direct integration algorithms (explicit or implicit) are collectively defined in this paper using the following difference equations:

$$\mathbf{u}_{i+1} = \mathbf{u}_i + (\Delta t)\dot{\mathbf{u}}_i + \eta_1(\Delta t)^2\ddot{\mathbf{u}}_i + \eta_2(\Delta t)^2\ddot{\mathbf{u}}_{i+1} \tag{2}$$

$$\dot{\mathbf{u}}_{i+1} = \dot{\mathbf{u}}_i + \eta_3(\Delta t)\ddot{\mathbf{u}}_i + \eta_4(\Delta t)\ddot{\mathbf{u}}_{i+1} \tag{3}$$

In general, Eqs. (1)–(3) require an iterative solution, which forms the basis of the implicit algorithms. On the other hand, these algorithms become explicit when $\eta_2 = 0$. For example, $[\eta_1,\eta_2,\eta_3,\eta_4] = [1/4,1/4,1/2,1/2]$ leads to implicit Newmark with constant average acceleration, $[\eta_1,\eta_2,\eta_3,\eta_4] = [1/2,0,1/2,1/2]$ transforms the integration to the explicit Newmark algorithm [2].

## Lyapunov-Based Numerical Approach

For each direct integration algorithm of SDOF systems, the relationship between the kinematic quantities at time steps $i+1$ and $i$ can be established as follows:

$$\boldsymbol{x}_{i+1} = \mathbf{A}_i \boldsymbol{x}_i + \mathbf{L}_i \tag{4}$$

where $\boldsymbol{x}_i = \left[(\Delta t)^2 \ddot{u}_i \;\; (\Delta t)\dot{u}_i \;\; u_i\right]^T$. It is noted that $\mathbf{A}_i$ and $\mathbf{L}_i$ are the approximation operator and the loading vector at the time step $i$, respectively. The loading vector, $\mathbf{L}$, is generally bounded and independent of the vector of kinematic quantities, $\boldsymbol{x}$, and does not affect the Lyapunov stability of the direct integration algorithms. Therefore, $\mathbf{L}$ can be set to zero in the sequel of this paper.

For linear structures, the approximation operator, $\mathbf{A}$, remains constant. The stability criterion of linear systems is obvious and well-known, namely the spectral radius of the approximation operator $\rho(\mathbf{A})$ must be less than or equal to 1.0. In contrast, for nonlinear structures, methods that are applicable to linear ones generally do not work. For example, the spectral radius and frequency domain methods basically convey nothing about the stability properties of algorithms. Instead, we turned to Lyapunov stability theory, based on which a numerical approach was proposed. This approach transforms the stability analysis to a problem of convex optimization, which is applicable to direct integration algorithms used to solve nonlinear problems.

As discussed above, we are investigating the system in Eq. (4) with the loading vector $\mathbf{L} = \mathbf{0}$, i.e.,

$$\boldsymbol{x}_{i+1} = \mathbf{A}_i \boldsymbol{x}_i \tag{5}$$

where $\mathbf{A}_i$ is a function of $\delta_{i+1}$ which is the tangent stiffness at time step $i+1$ normalized by the initial stiffness. Detailed derivations of $\mathbf{A}_i$ for different algorithms are given in [11,12].

One standard Lyapunov function $v_{i+1}$ at the time step $i+1$ is defined in [16] as follows:

$$v_{i+1} = \boldsymbol{x}_{i+1}^T \mathbf{M}_{i+1} \boldsymbol{x}_{i+1} \tag{6}$$

where the positive definite matrix $\mathbf{M}_{i+1} = \mathbf{M}_{i+1}^T$ is a function of $\delta_{i+1}$. A sufficient condition for the system and thus the direct integration algorithm to be stable is as follows:

$$\begin{aligned} \Delta v_{i+1} &= v_{i+1} - r_t v_i \\ &= \boldsymbol{x}_{i+1}^T \mathbf{M}_{i+1} \boldsymbol{x}_{i+1} - r_t \boldsymbol{x}_i^T \mathbf{M}_i \boldsymbol{x}_i \\ &= \boldsymbol{x}_i^T \left( \mathbf{A}_i^T \mathbf{M}_{i+1} \mathbf{A}_i - r_t \mathbf{M}_i \right) \boldsymbol{x}_i \\ &= \boldsymbol{x}_i^T \mathbf{P}_{i+1} \boldsymbol{x}_i \le 0 \end{aligned} \tag{7}$$

where $0 < r_t \le 1$ controls the rate of convergence, i.e., the smaller the $r_t$, the faster the convergence. Eq. (7) lead to the negative semi-definiteness of $\mathbf{P}_{i+1}$, i.e., $\mathbf{P}_{i+1} \prec= \mathbf{0}$. For a direct integration algorithm, $\mathbf{M}_{i+1}$ can be expressed as:

$$\mathbf{M}_{i+1} = \sum_{j=1}^{B} \alpha_j \mathbf{\Phi}_j(\delta_{i+1}) \tag{8}$$

where $\alpha_j$ and $\mathbf{\Phi}_j(\delta_{i+1})$ are the *j*–th constant coefficient and base function, respectively, and $B$ is the total number of base functions. One example set of base functions is given in [11] where the set of base functions of $\mathbf{\Phi}_1$ to $\mathbf{\Phi}_6$ represent constant $\mathbf{M}_{i+1}$, $\mathbf{\Phi}_7$ to $\mathbf{\Phi}_{12}$ constitute the base functions that treat $\mathbf{M}_{i+1}$ as a linear function of $\delta_{i+1}$, and nonlinear relationship between $\mathbf{M}_{i+1}$ and $\delta_{i+1}$ are considered by base functions $\mathbf{\Phi}_{13}$ to $\mathbf{\Phi}_{18}$.

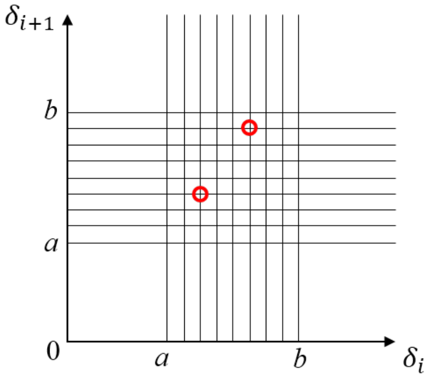


**Figure 1. Schematic illustration of discretization process**

With the range of $\delta_i$ and $\delta_{i+1}$ given, e.g., $\delta_i, \delta_{i+1} \in [a,b]$, points can be discretized within this range (Figure 1), e.g., sampling $p+1$ points in $[a,b]$ with interval $\Delta\delta = (b-a)/p$. This yields $(p+1)^2$ possible pairs of $(\delta_i, \delta_{i+1})$. Accordingly, the stability analysis becomes a problem of convex optimization that seeks the determination of the coefficients $\alpha_j$ by minimizing their norm for the selected base functions $\mathbf{\Phi}_j(\delta_{i+1})$ where $j:1 \to B$, subjected to the following conditions on the $(p+1)^2$ possible pairs of $(\delta_i, \delta_{i+1})$:

$$\begin{aligned} &\delta_i, \delta_{i+1} \in [a,b], \quad \Delta\delta = (b-a)/p \\ &\mathbf{A}_i^T \mathbf{M}_{i+1} \mathbf{A}_i - r_t \mathbf{M}_i = \mathbf{A}_i^T(\delta_{i+1}) \left( \sum_{j=1}^{B} \alpha_j \mathbf{\Phi}_j(\delta_{i+1}) \right) \mathbf{A}_i(\delta_{i+1}) - r_t \sum_{j=1}^{B} \alpha_j \mathbf{\Phi}_j(\delta_i) \prec= \mathbf{0} \\ &\mathbf{M}_i = \sum_{j=1}^{B} \alpha_j \mathbf{\Phi}_j(\delta_i) \succ \mathbf{0}, \quad \mathbf{M}_{i+1} = \sum_{j=1}^{B} \alpha_j \mathbf{\Phi}_j(\delta_{i+1}) \succ \mathbf{0} \end{aligned} \tag{9}$$

Moreover, with prior knowledge about the variation of $\delta_{i+1}$, the range of $|\delta_{i+1} - \delta_i|$ can be specified, e.g., $|\delta_{i+1} - \delta_i| < \varepsilon$, where $\varepsilon$ is an optional parameter that is not necessarily small. For example, suppose we are interested in investigating the stability of a certain algorithm in the range of $\delta_i, \delta_{i+1} \in [1,2]$, and $\delta_i = 1.5$ at the *i*-th time step. If prior knowledge is known such that $\varepsilon = 0.3$, i.e., $\delta_{i+1} \in (1.2, 1.8)$, fewer possible pairs of $(\delta_i, \delta_{i+1})$ that require less computational effort can be considered. The problem of convex optimization can be solved numerically by CVX, a software package for specifying and solving convex programs [17].

Two examples of the softening and the stiffening cases for the implicit Newmark algorithm with constant average acceleration are presented to illustrate this approach. The following conditions are considered in these examples:

$$\zeta = 0.05 \quad \mu = 0.05/(2\pi) \quad n = 20 \quad \varepsilon = 0.05 \quad r_t = 1.0 \tag{10}$$

where $\zeta = c/(2m\omega_n)$, $\omega_n^2 = k_I/m$, $\mu = \Delta t/T_n$, $T_n = 2\pi/\omega_n = 2\pi\sqrt{m/k_I}$. The set of base functions $\mathbf{\Phi}_1$ to $\mathbf{\Phi}_{12}$ in [12] is used.

*Softening Example*

Suppose we are interested in investigating the stability of the implicit Newmark algorithm in the range of $\delta_i, \delta_{i+1} \in [0.9, 1.0]$, therefore $\Delta\delta = (b-a)/p = 0.005$. The coefficients $\alpha_j$, $j:1 \to 12$, are:

$$\begin{aligned}
&\alpha_1 = 1.90 \times 10^{-8}, \quad \alpha_2 = 2.46 \times 10^{-9}, \quad \alpha_3 = 1.70 \times 10^{-10}, \quad \alpha_4 = -2.25 \times 10^{-9}, \\
&\alpha_5 = -2.70 \times 10^{-10}, \quad \alpha_6 = -4.60 \times 10^{-10}, \quad \alpha_7 = 1.76 \times 10^{-8}, \quad \alpha_8 = 1.05 \times 10^{-9}, \\
&\alpha_9 = 6.00 \times 10^{-11}, \quad \alpha_{10} = -3.35 \times 10^{-9}, \quad \alpha_{11} = 4.30 \times 10^{-10}, \quad \alpha_{12} = -2.00 \times 10^{-10}
\end{aligned} \tag{11}$$

*Stiffening Example*

Analogous to the procedure of the previous softening example, suppose the range of interest for the stiffening case is $\delta_i, \delta_{i+1} \in [1.0, 1.1]$, the obtained coefficients $\alpha_j$, $j:1 \to 12$, are:

$$\begin{aligned}
&\alpha_1 = 9.81 \times 10^{-7}, \quad \alpha_2 = 2.28 \times 10^{-10}, \quad \alpha_3 = 1.93 \times 10^{-8}, \quad \alpha_4 = -2.25 \times 10^{-8}, \\
&\alpha_5 = -2.30 \times 10^{-9}, \quad \alpha_6 = -1.53 \times 10^{-9}, \quad \alpha_7 = 1.01 \times 10^{-6}, \quad \alpha_8 = 3.39 \times 10^{-11}, \\
&\alpha_9 = 7.03 \times 10^{-9}, \quad \alpha_{10} = -8.94 \times 10^{-8}, \quad \alpha_{11} = 7.20 \times 10^{-9}, \quad \alpha_{12} = -5.01 \times 10^{-10}
\end{aligned} \tag{12}$$

The set of $\alpha_j$ in Eqs. (11) and (12) from many determined sets has the minimum 2-norm $\alpha$, i.e. $\min \sqrt{\sum_{j=1}^{12} \alpha_j^2}$, explaining the listed small values of $\alpha_j$. The existence of such set of $\alpha_j$ implies the existence of $\mathbf{M}_{i+1}$ in Eq. (8) that satisfies the inequality in Eq. (7), which signifies that the implicit Newmark algorithm is stable for the conditions in Eq. (10) in the range of $\delta_i, \delta_{i+1} \in [0.9, 1.0]$ based on Eq. (11) or in the range of $\delta_i, \delta_{i+1} \in [1.0, 1.1]$ based on Eq. (12). Several other examples are provided in [9,10].

The approach discussed above can be applied to investigate the stability of different direct integration algorithms considering various nonlinear effects, e.g., stiffening ($\delta_{i+1} > 1$) and softening ($\delta_{i+1} < 1$) force-deformation relationships. Thus, this approach is generally applicable to direct integration algorithms as long as they can be expressed as given by Eq. (5). Moreover, this approach can potentially be extended to MDOF systems. For $m$ DOF systems, the $3m \times 3m$ approximation operator is a function of $\delta_{i+1}^j$, where $j:1 \to m$ denotes the *j*-th DOF, and thus $(m+1)(9m^2 + 3m)/2$ selected base functions and corresponding coefficients are needed if $\mathbf{M}_{i+1}$ is expressed as an affine function of $\delta_{i+1}^j$, $j:1 \to m$. Thus, this approach involves extensive computations for MDOF systems.

### Lyapunov-Based Approach Considering Strictly Positive Real Lemma

This approach was proposed to deal with stability issues of explicit direct integration algorithms, i.e., $\eta_2 = 0$ in Eq. (2). As mentioned previously in the introduction, it transforms the stability analysis of the formulated MDOF nonlinear system to investigating the strictly positive realness of its corresponding transfer function matrix.

For a MDOF system with $n$ DOFs, the *j*–th term of the restoring force vector, $f^j, j \in [1, n]$, can be expressed as a linear combination of $N$ basic resisting forces of the system, $q^l, l \in [1, N]$, i.e.,

$$f^j = \sum_{l=1}^{N} \alpha_l^j q^l = \boldsymbol{\alpha}^j \mathbf{q} \tag{13}$$

where $\mathbf{q}^T = \left[q^1, q^2, \ldots, q^N\right]$ and $\boldsymbol{\alpha}^j = \left[\alpha_1^j, \alpha_2^j, \ldots, \alpha_N^j\right]$. Therefore,

$$\mathbf{f} = \left[f^1, f^2, \ldots, f^n\right]^T = \boldsymbol{\alpha}\mathbf{q} \tag{14}$$

where $\boldsymbol{\alpha} = \left[\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2, \ldots, \boldsymbol{\alpha}^n\right]^T$ is an $n \times N$ matrix. In general, $N$ is the summation of the number of the basic resisting forces from each element that contribute to the $n$ DOFs of the system. For the special case of a shear building, $N = n$ because of its assumed shear mode behavior. The *l*–th basic resisting force, $q^l$, is here defined as a function of $\bar{u}^l$, which is in itself a linear combination of the displacement of each DOF, $u^j$, $j \in [1, n]$, i.e.,

$$\bar{u}^l = \sum_{j=1}^{n} \beta_j^l u^j = \boldsymbol{\beta}^l \mathbf{u} \tag{15}$$

where $\mathbf{u} = [u^1, u^2, \ldots, u^n]$ and $\boldsymbol{\beta}^l = [\beta_1^l, \beta_2^l, \ldots, \beta_n^l]$. Therefore,

$$\bar{\mathbf{u}} = \left[\bar{u}^1, \bar{u}^2, \ldots, \bar{u}^N\right]^T = \boldsymbol{\beta}\mathbf{u} \tag{16}$$

where $\boldsymbol{\beta} = \left[\boldsymbol{\beta}^1, \boldsymbol{\beta}^2, \ldots, \boldsymbol{\beta}^N\right]^T$ is an $N \times n$ matrix. Detailed explanation of $N$ defining the number of columns and rows of the matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively, are available in [15]. Moreover, the *l*–th basic resisting force, $q^l$, is a sector-bounded nonlinearity and is restricted to the following range:

$$\bar{k}_{Min}^l \left(\bar{u}^l\right)^2 \le q^l \bar{u}^l \le \bar{k}_{Max}^l \left(\bar{u}^l\right)^2 \tag{17}$$

where $\bar{k}_{Min}^l \bar{u}^l$ and $\bar{k}_{Max}^l \bar{u}^l$ are the minimum and maximum bounds of $q^l$, respectively. Define

$$\bar{\mathbf{k}}_{Min} = \mathrm{diag}\left[\bar{k}_{Min}^1, \bar{k}_{Min}^2, \ldots, \bar{k}_{Min}^N\right] \tag{18a}$$

$$\bar{\mathbf{k}}_{Max} = \mathrm{diag}\left[\bar{k}_{Max}^1, \bar{k}_{Max}^2, \ldots, \bar{k}_{Max}^N\right] \tag{18b}$$

$$\bar{\mathbf{k}} = \bar{\mathbf{k}}_{Max} - \bar{\mathbf{k}}_{Min} = \mathrm{diag}[\bar{k}^1, \bar{k}^2, \ldots, \bar{k}^N] \tag{18c}$$

After some manipulation [15], both stiffening and softening systems can be expressed in Eq. (19) with coefficients $\mathbf{A}_e, \mathbf{B}_e$ and $\mathbf{q}_e$ summarized in Table 1.

$$\boldsymbol{x}_{i+1} = \mathbf{A}_e \boldsymbol{x}_i - \mathbf{B}_e \mathbf{q}_e \tag{19}$$

**Table 1. Coefficients of MDOF stiffening and softening systems**

| Matrix | Stiffening Systems | Softening Systems |
|---|---|---|
| $\mathbf{A}_e$ | $\mathbf{A}_{e1} = \mathbf{A} - \mathbf{B}_1 \boldsymbol{\alpha} \bar{\mathbf{k}}_{Min} \mathbf{C}$ | $\mathbf{A}_{e2} = \mathbf{A} + \mathbf{B}_2 \boldsymbol{\alpha} \bar{\mathbf{k}}_{Max} \mathbf{C}$ |
| $\mathbf{B}_e$ | $\mathbf{B}_1 \boldsymbol{\alpha}$ | $\mathbf{B}_2 \boldsymbol{\alpha}$ |
| $\mathbf{q}_e$ | $\mathbf{q}_{e1} = \mathbf{q}_{i+1} - \bar{\mathbf{k}}_{Min} \mathbf{C} \boldsymbol{x}_i$ | $\mathbf{q}_{e2} = \bar{\mathbf{k}}_{Max} \mathbf{C} \boldsymbol{x}_i - \mathbf{q}_{i+1}$ |

where the terms in Table 1 are expressed as follows:

$$\mathbf{B}_1 = -\mathbf{B}_2 = \left[(\Delta t)^2 \mathbf{m}_{eff}^{-1} \quad \eta_3(\Delta t)^2 \mathbf{m}_{eff}^{-1} \quad \mathbf{0}\right]^T, \quad \mathbf{m}_{eff} = \mathbf{m} + \eta_3(\Delta t)\mathbf{c} \tag{20a}$$

$$\mathbf{C} = \boldsymbol{\beta}\tilde{\mathbf{C}}, \quad \tilde{\mathbf{C}} = [\eta_1\mathbf{I} \quad \eta_0\mathbf{I} \quad \mathbf{I}], \quad \mathbf{I} = \text{Identity matrix} \tag{20b}$$

Similar to the first numerical approach, the Lyapunov function $v_{i+1}$ at the time step $i+1$ is chosen as:

$$v_{i+1} = \mathbf{x}_{i+1}^T \mathbf{M} \boldsymbol{x}_{i+1} \tag{21}$$

The constraints that the basic forces are sector-bounded lead to

$$\Delta v_{i+1} = v_{i+1} - v_i \le -(\mathbf{W}\mathbf{q}_e - \mathbf{L}\boldsymbol{x}_i)^T(\mathbf{W}\mathbf{q}_e - \mathbf{L}\boldsymbol{x}_i) \le 0 \tag{22}$$

where there exist matrices $\mathbf{M}$, $\mathbf{L}$ and $\mathbf{W}$ such that

$$\mathbf{M} = \mathbf{A}_e^T \mathbf{M} \mathbf{A}_e + \mathbf{L}^T \mathbf{L} \tag{23a}$$

$$\mathbf{0} = \mathbf{B}_e^T \mathbf{M} \mathbf{A}_e - \boldsymbol{\lambda}\bar{\mathbf{k}}\mathbf{C} + \mathbf{W}^T \mathbf{L} \tag{23b}$$

$$\mathbf{0} = \boldsymbol{\lambda} + \boldsymbol{\lambda}^T - \mathbf{B}_e^T \mathbf{M} \mathbf{B}_e - \mathbf{W}^T \mathbf{W} \tag{23c}$$

where $\boldsymbol{\lambda}$ is a constant diagonal matrix of arbitrary positive coefficients. Derivations from Eq. (21) to Eqs. (23) can be found in [15]. Based on the generalized strictly positive real lemma [18], the stability analysis reduces to seeking $\bar{\mathbf{k}}$ such that the transfer function matrix $\mathbf{G}(z)$ in Eq. (24) is strictly positive real.

$$\mathbf{G}(z) = \boldsymbol{\lambda} + \boldsymbol{\lambda}\bar{\mathbf{k}}\mathbf{C}(\mathbf{I}z - \mathbf{A}_e)^{-1}\mathbf{B}_e \tag{24}$$

For SDOF systems, the matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ become 1, based on [11,14], Eq. (24) reduces to

$$G(z) = 1 + \bar{k}\mathbf{C}(\mathbf{I}z - \mathbf{A}_e)^{-1}\mathbf{B}_e \tag{25}$$

The strictly positive realness of $G(z)$ can be guaranteed by the asymptotical stability of $\mathbf{A}_e$ and

$$\mathrm{Re}[G(z)] > 0 \tag{26}$$

which leads to

$$\mathrm{Re}[H(z)] > -1/\bar{k} \tag{27}$$

where

$$H(z) = \mathbf{C}(\mathbf{I}z - \mathbf{A}_e)^{-1}\mathbf{B}_e \tag{28}$$

The Nyquist plot [16] can be used to plot $H(e^{j\theta})\ \forall\ \theta \in [0, 2\pi]$. From this plot, the minimum value of $\mathrm{Re}[H(z)]$ that is corresponding to the $-1/\bar{k}$ can be obtained.

For MDOF systems, based on [19], the strictly positive realness of $\mathbf{G}(z)$ in Eq. (24) becomes equivalent to Eq. (29) with $\mathbf{P} = \mathbf{P}^T \succ \mathbf{0}$:

$$\begin{bmatrix} \mathbf{A}_e^T \mathbf{P} \mathbf{A}_e - \mathbf{P} & \mathbf{A}_e^T \mathbf{P} \mathbf{B}_e - (\boldsymbol{\lambda}\bar{\mathbf{k}}\mathbf{C})^T \\ \left[\mathbf{A}_e^T \mathbf{P} \mathbf{B}_e - (\boldsymbol{\lambda}\bar{\mathbf{k}}\mathbf{C})^T\right]^T & -(\boldsymbol{\lambda}^T + \boldsymbol{\lambda}) + \mathbf{B}_e^T \mathbf{P} \mathbf{B}_e \end{bmatrix} \prec \mathbf{0} \tag{29}$$

Eq. (29) is a linear matrix inequality (LMI) over variables $\mathbf{P}$ and $\bar{\mathbf{k}}$ [20]. This problem of convex optimization, which seeks $\bar{\mathbf{k}}$ and the corresponding $\mathbf{P}$ by minimizing certain convex cost function, subjected to the constraints of $\mathbf{P} = \mathbf{P}^T \succ \mathbf{0}$ and $\bar{\mathbf{k}} \succ \mathbf{0}$, can be solved numerically by CVX [17].

Multi-story shear buildings with stiffening and softening structural behaviors are used as examples to illustrate this approach. A general multi-story shear building structure is depicted in Figure 2. The detailed derivation of $\mathbf{q}$ and $\bar{\mathbf{u}}$ as well as the corresponding matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ for this shear building is given in [15]. Accordingly, the maximum, $\bar{k}^j_{Max}$, and minimum, $\bar{k}^j_{Min}$, stiffness values of the *j*–th story, where $j:1 \rightarrow n$ and the number of stories is $n$, for stable (in the sense of Lyapunov) stiffening and softening multi-story shear building systems, respectively, are to be determined.

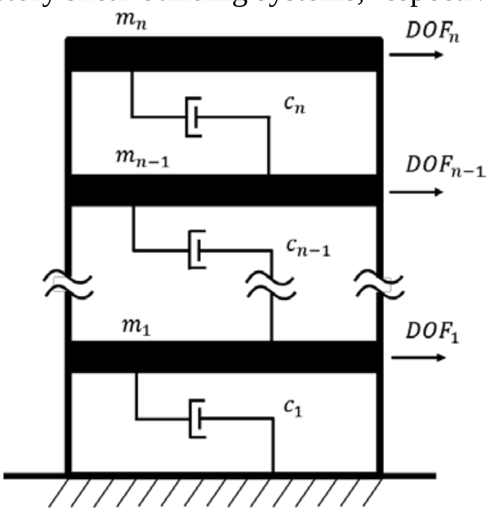


**Figure 2. A general multi-story shear building structure**

The stability analysis is conducted for the following numerical values:

$$m_j = 0.5, \quad \zeta = 0.05, \quad \bar{k}^j_I = 1000.0 \tag{30a}$$

$$\lambda_j = \omega_j^2 \Big/ \left( \sum_{j=1}^{n} \omega_j^2 \right), \quad \omega_j = 2\pi / T_j, \quad \mu = (\Delta t)/T_n = 0.01 \tag{30b}$$

where $T_j$ is the period of the *j*–th mode of vibration of the analyzed structure. The initial bound matrix is $\bar{\mathbf{k}}_I = \mathrm{diag}[\bar{k}^1_I, \bar{k}^2_I, \ldots, \bar{k}^n_I]$, i.e., $\bar{\mathbf{k}}_{Min}$ and $\bar{\mathbf{k}}_{Max}$ for stiffening and softening systems, respectively. A 20-story (Figure 2 with $n = 20$) shear building is used to investigate the Lyapunov stability analysis of the explicit Newmark algorithm, i.e. $[\eta_1, \eta_2, \eta_3, \eta_4] = [1/2, 0, 1/2, 1/2]$. Lyapunov stability analysis following the approach previously discussed in this section is conducted for the analyzed this 20-story shear building with stiffening or softening behavior. The cost function for this building structure is selected as $\min(-\sum_{j=1}^{20} \bar{k}^j)$, which is equivalent to $\max(\sum_{j=1}^{20} \bar{k}^j)$. In this cost function, $\bar{k}^j = \bar{k}^j_{Max} - \bar{k}^j_{Min}$ is the difference of the upper and lower bounds of the basic resisting force $q^j$ associated with the j–th story, where $j:1 \rightarrow n$. Table 2 shows that the difference of the upper and lower bounds, $\bar{k} = \bar{k}_{Max} - \bar{k}_{Min}$, of each resisting force for the explicit Newmark algorithm to be stable (in the sense of Lyapunov) for both stiffening, $\bar{\mathbf{u}}^T \mathbf{q} \in [\bar{\mathbf{u}}^T \bar{\mathbf{k}}_I \bar{\mathbf{u}}, \bar{\mathbf{u}}^T \left( \bar{\mathbf{k}}_I + \bar{\mathbf{k}} \right) \bar{\mathbf{u}}]$, and softening, $\bar{\mathbf{u}}^T \mathbf{q} \in [\bar{\mathbf{u}}^T \left( \bar{\mathbf{k}}_I - \bar{\mathbf{k}} \right) \bar{\mathbf{u}}, \bar{\mathbf{u}}^T \bar{\mathbf{k}}_I \bar{\mathbf{u}}]$, systems.

**Table 2. The $\bar{k}$ of each basic resisting force for the 20-story shear building**

| Story Number | Stiffening systems | Softening systems | Story Number | Stiffening systems | Softening systems |
|---|---|---|---|---|---|
| 1 | 716.1 | 203.7 | 11 | 31.3 | 35.9 |
| 2 | 125.1 | 149.6 | 12 | 25.3 | 30.8 |
| 3 | 98.8 | 150.4 | 13 | 21.8 | 28.5 |
| 4 | 133.0 | 166.5 | 14 | 19.6 | 26.7 |
| 5 | 163.4 | 140.0 | 15 | 17.3 | 24.0 |
| 6 | 119.7 | 97.7 | 16 | 15.0 | 21.2 |
| 7 | 76.9 | 74.8 | 17 | 14.2 | 20.4 |
| 8 | 56.5 | 64.1 | 18 | 16.9 | 23.7 |
| 9 | 46.7 | 55.3 | 19 | 29.6 | 37.3 |
| 10 | 39.0 | 44.8 | 20 | 116.1 | 106.7 |

More Examples are given in [10,14,15] to illustrate this approach for different direct explicit integration algorithms applied to different structures (buildings and bridges) with stiffening and softening force-deformation relationships.

**Summary and Concluding Remarks**

This paper reviewed and compared two recently proposed Lyapunov-based approaches of stability analysis in terms of their merits and limitations. Interested readers should consult references [11,12,14,15] for detailed derivations and examples.

The first approach transforms the stability analysis to a problem of existence, that can be solved via convex optimization, over the discretized domain of interest of the restoring force. As such, this approach is a numerical one with certain approximations. It is shown to be generally applicable to both implicit and explicit direct integration algorithms for various nonlinear force-deformation relationships, including stiffening and softening ones. References [11,12] considered nonlinear SDOF systems. This approach can potentially be extended to nonlinear MDOF systems but extensive computations are involved and can be overcome by some methods, e.g., parallel computing [21].

The second approach is specifically applicable to explicit algorithms for nonlinear SDOF and MDOF systems considering strictly positive real lemma. In this approach, a generic explicit algorithm was formulated for a nonlinear system governed by a nonlinear function of the basic force without adopting any approximations. Starting from this formulation and based on Lyapunov stability theory, the stability analysis of the formulated nonlinear system is transformed to investigating the strictly positive realness of its corresponding transfer function matrix. Furthermore, this is equivalent to a problem of convex optimization that can be solved numerically. The basic force in this study was limited to the sector-bounded nonlinearity, including stiffening, softening and even hysteretic force-deformation relationships as long as they are within the sector bounds. Moreover, this approach is more computationally efficient than the first numerical one, especially for MDOF systems. Comparisons between these two approaches are listed in Table 3. It should be emphasized that Eqs. (7) and (22) are sufficient conditions for dynamical systems to be stable. Therefore, both approaches provide a sufficient condition for the direct integration algorithm to be stable. In other words, neither of these two approaches can indicate the condition of instability of the investigated algorithms. For example, having some basic resisting force vector $\mathbf{q}$ that may fall outside the range in Table 2 does not indicate the instability of the explicit Newmark algorithm for the analyzed 20-story shear building.

**Table 3. Comparisons between the two approaches**

| Property | First approach | Second approach |
|---|---|---|
| Algorithm | Implicit & Explicit | Explicit |
| Nonlinearity | No restriction | Sector-bounded |
| Condition | Sufficient | Sufficient |
| Approximation | Yes | No |
| MDOF | Potentially | Yes |
| Computational effort | Extensive | Efficient |

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