Nonlinear Interaction of Internal Waves Due to Two Point Vortices

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Abstract

In this paper, two-dimensional, two-layer steady stratified flow about a equal-strength counter- or co-rotating vortex pair in the lower layer is concerned. Potential flow theory and boundary integral equation method are applied to establish boundary integral equations about the interfacial wave. These equations are solved numerically based on quasi-Newton method. The effects on interfacial wave profiles of distance of the vortex pair are analyzed. It is found that the wave height oscillates with d and the extreme values are almost the sum of that for two vortices consists of the pair, especially for d large enough. When d is set that the wave height gets its maximum points the wave profiles is about the superposition of that for the two vortices, whereas when d get its minimum points the difference between the wave profiles caused by the pair and the sum of profiles for these two vortices is large.

Keywords: **Stratified flow, Point vortex pair, Boundary integral equation, Nonlinear interfacial waves**

Introduction

When there exists a disturbance source in the steady density-stratified fluids, an internal wave will be generated. Two-layer model of the vertical structure with different densities are often employed as a simplified model of internal waves of steady density-stratified fluids. Many researchers have studied the internal waves generated by various disturbance sources in two-layer fluids such as the moving point source in the upper[1] and lower layers[2], the moving dipole [3], the point vortex in the lower layer[4] and upper layer[5] and the hydrofoil in the lower layer[6].

Forbes applied this boundary integral equation method with arclength parameters to describe the surface while studying non-linear surface wave caused by a submerged point vortex [7] and a submerged hydrofoil[8] in two-dimensional ideal irrotational fluid of infinite depth. The obtained equations were solved numerically based on Newton method. Then this theory is used to study the interfacial waves in two-layer fluids by different disturbance by following researcher[9], [4].

The vortex pair is the basic element of fluid mechanics. Study of it to a large extent comes from the problem of trailing wakes. Many studies are concerned with dynamics and instability of vortex pairs. The literature [10] reviewed the characteristics and the behaviors of vortex pairs. Besides, some researchers focused on the interactions and the flow structures between vortex pairs and other objects like wall[11] and free surface[12],[13].

In this paper, the two-dimensional two-layer steady flow for a submerged vortex pair is considered. Both layers are inviscid and incompressible ideal fluids with consistent flow direction. The upper layer is of finite depth and bounded by a rigid lid, while the lower fluid is infinitely deep in which there exists a vortex pair set on a horizontal fixed position. The structure of this paper is as follows: at first integral-differential equations are established using the potential flow theory and boundary integral equation method. Secondly the problem is solved numerically based on the quasi-Newton method, which has been verified and gives a well performance in DoF. Then we compare the effects of different parameters on the wave profile, including Froude number, vortex strength and distance between two vortices.

Model of the problem

Consider steady two-layer fluids of different densities. Both layers are ideal fluids and irrotational. Their upstream uniform speeds have consistent flow direction. Creating a Cartesian coordinate system such that the x axis is placed at the undisturbed horizontal interface and point in the same direction of upstream uniform speed, as well as the y axis points up vertically. The depth of upper fluid is T and the upper surface satisfies the rigid-lid assumption. The lower fluid is infinitely deep with a point vortex pair placed where its center is at $(0, -H)$. The distance between the two point vortices is $2D$ with circulation $K_1 < 0$ at $(-d, -1$ and $K_2 > 0$ at $(d, -1)$ respectively. In following context we use subscripts 1 and 2 to represent the physical variables associated with the upper fluid and the lower, respectively. Densities and upstream uniform speeds of two layers of fluid are ρ_1, ρ_2, and γ_1, γ_2.

For the convenience of discussion, use γ_2 as the speed scale, H the length scale to get the dimensionless model, then introduce following dimensionless parameters:

$$F = \frac{\gamma_2}{\sqrt{gH}}, \ \epsilon_1 = \frac{K_1}{\gamma_2 H}, \ \epsilon_2 = \frac{K_2}{\gamma_2 H} \rho = \frac{\rho_1}{\rho_2}, \ \gamma = \frac{\gamma_1}{\gamma_2}, \ \lambda = \frac{T}{H} \ d = \frac{D}{H}$$

where F is the Froude number, ϵ_1, ϵ_2 are the dimensionless vortex strengths of the two point vortices, ρ is the ratio of density, γ is the ratio of far upstream uniform speed, λ is the nondimensional depth of the upper layer, and $2d$ is the nondimensional distance between two vortices. The elevation of fluid interface is described by a function $y = \eta(x)$.

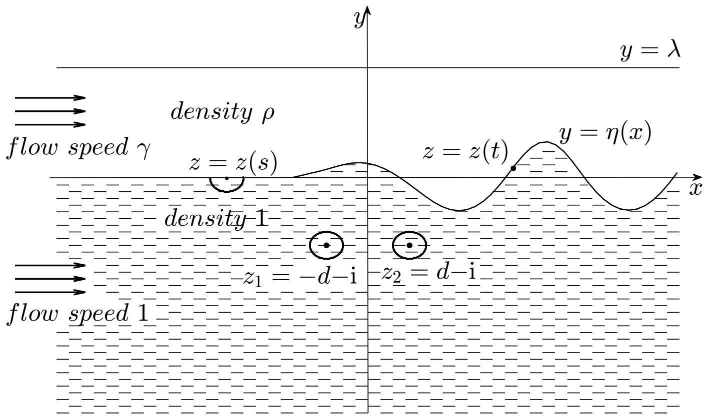


Figure 1. The nondimensional problem of a two-layer flow about a submerged vortex pair located horizontally.

Because two layer fluids are both ideal fluids and flows are irrotational, from potential flow theory two potential functions ϕ_1, ϕ_2, the stream functions ψ_1, ψ_2 satisfy the Cauchy-Riemann equation: $(\partial/\partial x)\phi_j = (\partial/\partial y)\psi_j$, $(\partial/\partial y)\phi_j = -(\partial/\partial x)\psi_j$ $j = 1, 2$. Thus two analytic functions $f_j(z) = \phi_j(x, y) + \mathrm{i}\psi_j(x, y)$, $z = x + \mathrm{i}y$, complex velocity potential functions for upper and lower fluid separately are introduced. $z_1 = -d - \mathrm{i}$, $z_2 = d - \mathrm{i}$, the position of the two point vortices, are two singularities of f_2 where i is the imaginary unit, $\mathrm{i}^2 = -1$. f_2 satisfies

$$f_2 \to z + \frac{i\epsilon_1}{2\pi}\ln(z - z_1) + \frac{i\epsilon_2}{2\pi}\ln(z - z_2), \quad z \to z_1, z_2 \tag{1}$$

at z_1 and z_2. The upstream conditions are

$$f_1 \to \gamma z,\ f_2 \to z, \quad \mathrm{Re}[z] \to -\infty \tag{2}$$

here Re[z] means the real part of z.

The kinematic boundary condition for upper surface

$$\nabla\phi_1 \cdot \mathbf{n} = 0. \tag{3}$$

At the interface $y = \eta(x)$ it is

$$\nabla\phi_j \cdot \mathbf{n} = 0,\ j = 1, 2, \tag{4}$$

where $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$.

Introduce the arclength parameter s to parameterize the fluid interface $y = \eta(x)$, so the fluid interface is represented as $(x, y) = (x(s), y(s))$. The arclength condition is

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1. \tag{5}$$

The fluid interface satisfies the Bernoulli equation

$$\rho\left(\frac{d\phi_1}{ds}\right)^2 - \left(\frac{d\phi_2}{ds}\right)^2 + \frac{2(\rho - 1)y}{F^2} = \rho\gamma^2 - 1, \tag{6}$$

according to setting the pressure and velocity equally on the interface. For more details can be found in [4] and referring in it.

Choose a fixed point $z = z(s) = x(s) + \mathrm{i}y(s)$ randomly on the fluid interface, where s corresponding arclength parameters. The Laplace equation for the upper and lower layer fluid could be replaced by $\oint f(z)dz = 0$ and $\sum \mathrm{Res} f, z_k$. Both two integral equations come form Cauchy integral and residue theorem with respect to integral contour-path on z_1 and z_2. Writing two functions $\phi_j' s$ as forms of integral equations, two integral-differential equations are established for the upper and lower layers by the boundary integral equation method The detailed derivation process is similar to [4].

Introduce the analytic function $G_1(z) = \frac{df_1}{dz} - \gamma$, then apply the Cauchy integral formula and take its imaginary part to establish the governing equation. Applying the equality

$$AB = A \cdot \bar{B} + \mathrm{i}\bar{A} \times B = (ac - bd) + \mathrm{i}\begin{vmatrix} a & -b \\ c & d \end{vmatrix}$$

where $A = a + \mathrm{i}b, B = c + \mathrm{i}d \in \mathbb{C}$, the governing equation can be written as follow

$$\begin{aligned}\pi(\gamma - x'(s)\phi_1'(s)) = \mathrm{Im}\left\{ ⨍_{-\infty}^{+\infty} \overline{G_1(z(t))} \times d(\ln(z(t) - z(s))) \right\} \\ + \mathrm{Im}\left\{ \int_{-\infty}^{+\infty} \overline{G_1(\tilde{z}(t))} \times d(\ln(\tilde{z}(t) - z(s))) \right\}\end{aligned} \tag{7}$$

where $\tilde{z}(t) = x(t) + i(2\lambda - y(t))$ is the mirror point of $z(s)$ about $y = \lambda$. The first integral on the right side of the equation is the singular integral in the sense of the Cauchy principal value.

For the lower layer, introduce the analytic function $G_2(z) = \frac{df_2}{dz} - 1$ similarly, apply the residue theorem, and the equation $\oint_{\Gamma_2} \frac{G_2(\xi)d\xi}{\xi - z(s)} = 2\pi\mathrm{i}\sum_{k=1}^{2} \mathrm{Res}\left\{\frac{G_2(\xi)}{\xi - z(s)}, z_k\right\}$ is obtained, where $z_1 = -d - \mathrm{i}$, $z_2 = d - \mathrm{i}$.

Calculating the residue and integrals in the above formula and taking the imaginary part. The residue contribution obeys the superposition law. The lower layer fluids following governing equation

$$\begin{aligned}\pi(x'(s)\phi_2'(s) - 1) = \mathrm{Im}\left\{ ⨍_{-\infty}^{+\infty} \overline{G_2(z(t))} \times d(\ln(z(t) - z(s))) \right\} \\ + \frac{\epsilon_1 (y(s) - \mathrm{Im}\, z_1)}{|z(s) - z_1|^2} + \frac{\epsilon_2 (y(s) - \mathrm{Im}\, z_2)}{|z(s) - z_1|^2}\end{aligned} \tag{8}$$

Whereas the vortex pair contains two isolated singular points z_1 and z_2, which requires two calculation of residual number and add one more term in the governing equation.

The governing equations (5), (6), (7) and (8) are derived. Based on them and the corresponding boundary condition (2), the unknowns $x(s)$, $y(s)$, $\phi_1(s)$ and $\phi_2(s)$ can be calculated.

Numerical procedure

The numerical calculation method is similar to [4], while the difference is that this paper applies the quasi-Newton iteration method [14, 15] to solve (5), (6), (7) and (8). If y' is determined , then from (5) x' is obtained, as well as $x(s) = \int_{-\infty}^{s} x'(t)dt$ and $y(s) = \int_{-\infty}^{s} y'(t)dt$ are also acquired. (7) can be written as integral equations with respect of ϕ_1', then from (6) ϕ_2' can be solved easily. As y' is unknown, take the approximation of y' as $\widetilde{y'}$ and (8) as the cost function to update the approximation with quasi-Newton method. The following is the detailed process.

The integral area $(-\infty, +\infty)$ is truncated to the finite interval $[s_1, s_N]$, then $N-1$ equally dividing it to get N grid points $s_k = s_1 + (k-1)\Delta s, k = 1, \dots N$. Here $\Delta s = (s_N - s_1)/(N-1)$ means the step size. x_k, y_k, x_k', y_k', $\phi_{1,k}'$, $\phi_{2,k}'$ are the approximation value of the responding unknown quantities. According to the infinity boundary condition (2), the equation (5) and (6) determine $y_1 = y_1' = 0$,, $x_1' = 1$, $x_1 = s_1$, $\phi_{1,1}' = \gamma$, $\phi_{2,1}' = 1$. To eliminate the effects of singularity in the integral, half grid points at at$s_{k-1/2} = (s_{k-1} + s_k)/2$: $x_{k-\frac{1}{2}}$, $y_{k-\frac{1}{2}}$, $x_{k-\frac{1}{2}}'$, $y_{k-\frac{1}{2}}'$, $\phi_{1,k-\frac{1}{2}}'$, $\phi_{2,k-\frac{1}{2}}'$,

$k = 2, \ldots, N$ are also calculated. Here $x_{k-\frac{1}{2}} = (x_{k-1} + x_k)/2$. So as other variables. The initial approximation of $y_2', \cdots, y_N' = 0$.

As $y_2', \cdots, y_N'$ is determined, calculate numerical integration on the finite interval $[s_1, s_N]$ with trapezoidal rule to get discrete equations $A[\phi_{1,1}', \ldots, \phi_{1,N}']^T = c$ while taking the approximation value of quantities at s as its value in (7). Dealing with (8) in a similar way yields the equations of matrix form $E(y_2', \cdots, y_N') = B[\phi_{2,1}', \ldots, \phi_{2,N}']^T - d$, where $\phi_{2,k}'$ calculated from Bernoulli's equation (6). In these equations coefficient matrices A, B, constant terms b, d are all concerned with y_k', x_k', x_k, y_k , which can be calculated by (5) and trapezoidal rules: Finally, we get a system of equations for $y_2', \ldots, y_N'$. Solve it applying quasi-Newton method, and the iteration formula is[14]:

$$\begin{cases} u_{i+1} = u_i - A_i^{-1}E(u_i), \\ A_{i+1} = A_i + (b_i - A_i s_i)s_i^T/(s_i^T s_i) \end{cases} \qquad i = 0, 1, 2, \ldots \tag{9}$$

where $u = (y_2', \ldots, y_N')$, u_i represents ith iteration approximation of u, $E(u) = (E_2[y_2', \ldots, y_N'], \ldots, E_N[y_2', \ldots, y_N'])$, $s_i = u_{i+1} - u_i$, and $b_i = E(u_{i+1}) - E(u_i)$. For i=0, A_0 could be chosen as $\big((E^T(y' + he_i) - E^T(y'))/h\big)$, the $(N-1\times N-1)$ difference matrix of cost function E, $i = 2, \cdots, N$, where $y' = [y_2', \ldots, y_N']$ and e_i is $N-1$ dimensional unit vector. This Calculating progress terminates when Calculate $\|\mathbf{E}\|_2$ $\|\mathbf{E}\|_2$ is less than the given number ε.

Results analysis

In the numerical calculation, the upper depth is set $\lambda = 20$, and far upstream uniform speed ratio $\gamma = 1$, namely two-layer fluids with equal spped. If we set $d = 0, \epsilon_1 < 0, \epsilon_2 = 0$, in fact it is the case for a single vortex $\epsilon < 0$. The calculation error precision is setting to be $\sigma = 10^{-9}$. The calculation domain is [-25, 30] and the grid number $N = 2201$, as well as $\Delta s = 0.025$.

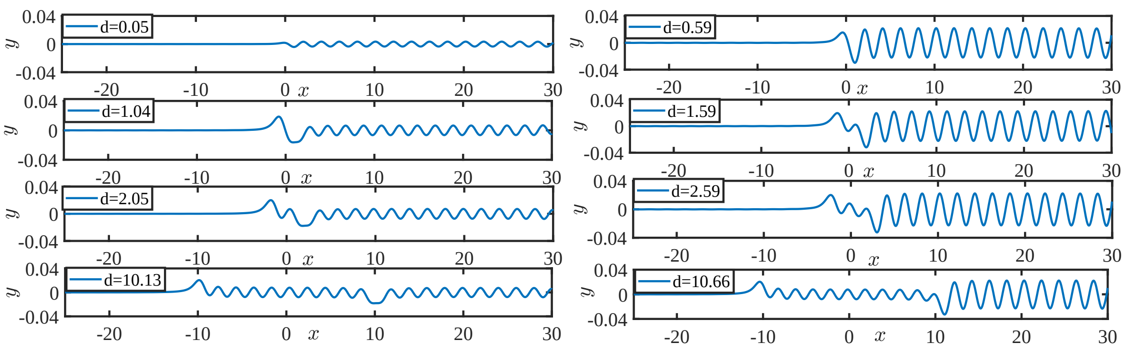


Figure 2. Comparison of interfacial wave profiles when d changes, where parameters $F = 0.13, \rho = 0.9, \epsilon_1 = -0.24, \epsilon_2 = 0.24$

For counter-rotating vortex pair with $\epsilon_1 < 0$ and $\epsilon_2 > 0$, figure 2 represents how wave profiles change with d. As d increasing the amplitude of upstream wave profiles increases and gradually stabilizes, as well as that of downstream steady wave profile oscillates. When d is large enough there's steady wave profile between two vortices, which is close to that for the single vortex $\epsilon < 0$. For $d = 10.13$, the wave height and length of this steady wave profile are $h = 0.01587$, $L = 2.018$, close to that of the steady wave profile $h_- = 0.01580$, $L_- = 2.019$ for a single vortex $\epsilon = -0.24$ at $(0, -1)$. Figure 3 describes the periodical change of wave height h of downstream wave profiles with d clearly. When d changes, the phase difference of two wave profiles caused by two vortices changes periodically. If the phase difference is one/half a period, the amplitude of sum of these two profiles is the maximum of h. The period is close to $L_{-}/2$. That is to say that this period is almost the distance of two vortices(2d). As d is large enough, the extreme values of downstream wave height h are approximately $h_{+} \pm h_{-}$, which are downstream steady wave height for a single vortex $\epsilon = -0.24$ and $\epsilon = 0.24$, respectively. Whereas for $d = 0.51$, $h/(h_{+} + h_{-}) = 0.9775$ and $d = 1.01, h/|h_{+} - h_{-}| = 0.8459$. These values of d are the maximum and minimum of figure 3.

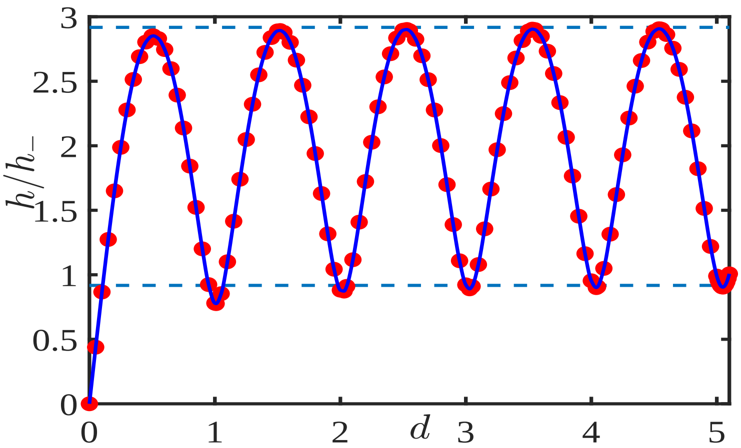


Figure 3. Relationship of h/h_{-} and d. Two horizontal lines of dashes represent $h_{+} \pm h_{-}$. h is the downstream wave height and $h_{-} = 0.01580$, $h_{+} = 0.03038$ are wave height for a single vortex $\epsilon = -0.24$ and $\epsilon = 0.24$, respectively. Other parameters $F = 0.13$, $\rho = 0.9$, $\epsilon_1 = -0.24$, $\epsilon_2 = 0.24$.

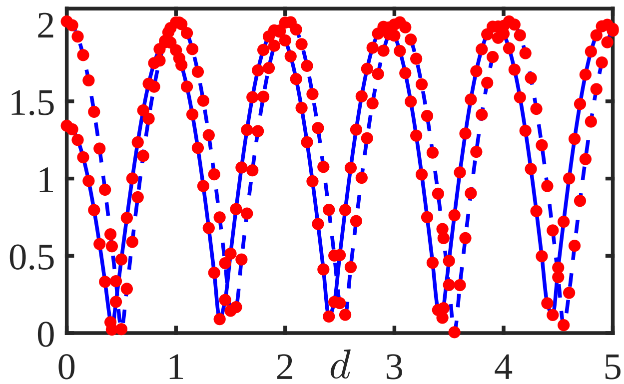


Figure 4. Relationship of h/h_{-}(solid line) and h_{12}/h_{-}(dash line) with d, where h is the downstream wave height and $h_{-} = 0.01580$ is the wave height for a single vortex $\epsilon = -0.23$. Other parameters $F = 0.13$, $\rho = 0.9$, $\epsilon_1 = \epsilon_2 = -0.23$.

For co-rotating pair, figure 4 shows that the wave height h also change with d periodically and the extreme values are almost the sum and difference of wave height for the single vortex except $d = 0.413$ and 0.93. On the other hand, let $(x(s), y(s))$, $(x_1(s), y_1(s))$ and $(x_2(s), y_2(s))$ represent the interfaces for the vortex pair ϵ_1 and ϵ_2, the single vortex at z_1 and z_2, respectively. Figure 4 shows the variance of wave height of $y_1(s)) + y_1(s))$ (written as h_{12}) oscillates like h with d and moves backward except for about $d < 0.5$. To examine this behavior, try to construct an approximate analytic expression considering that for a single vortex $\epsilon < 0$ the wave profile consists of a large crest like a solitary wave and the downstream steady waves. Assume that this crest can be written as $y = Asech^{\alpha}(x - x_0)$ and the downstream wave $y = Asin((2\pi)x/L + \phi)$, calculate curvature of highest point, the amplitude and wavelength and the positions of the maximum points to obtain the expression $y = Asech^{\alpha}(x-x_0)+0.0794sin(2\pi/2.025+0.562)H(x-x_1)$, where $x_0 = 0.275$, $A = 0.0205$, $\alpha = 2.957$ and $x_1 = (\pi - \phi)L/2\pi$. Figure 5 describes the

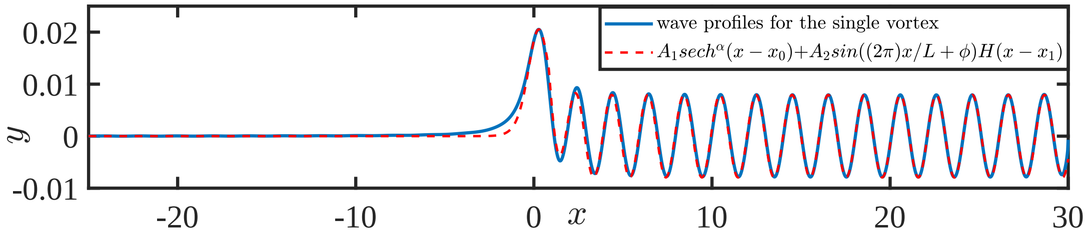
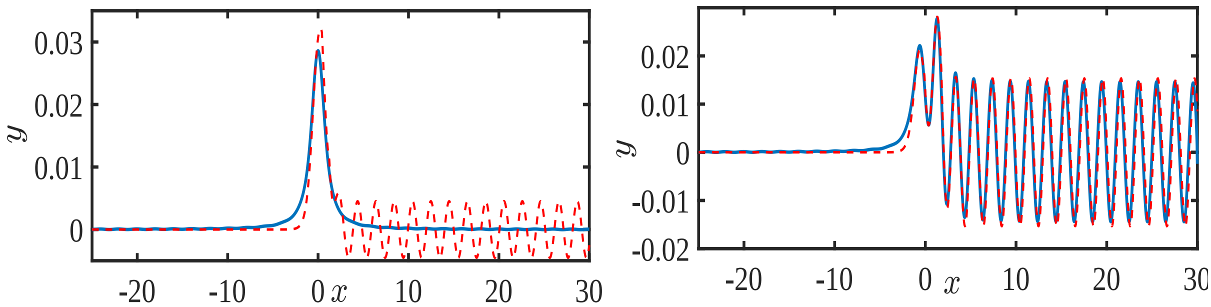


Figure 5. Comparison of interfacial wave profiles calculating by boundary integral method(solid line) and the expression patched (dash line), where parameters $F = 0.13$, $\rho = 0.9$, $\epsilon = -0.23$.

fitting effects. From figure 6, the wave profile is close to the superposition of two wave profiles



(a) d = 0.413

(b) d = 0.93

Figure 6. Comparison of interfacial wave profiles calculating by boundary integral method(solid line) and the expression patched (dash line) for (a) $d = 0.413$, (b) $d = 0.93$.

for the single vortex as d makes h around its maximum (for instance $d = 0.93$) and significantly different as h gets is minimum points(for example $d = 0.413$).

Conclusion

In the steady two-dimensional two-layer flow with ideal irrotating fluids, a vortex pair submerged in lower layer generates interfacial waves . In this paper, two integral equations coupled with Bernoulli equations of nonlinear boundary waves are established by applying potential flow theory and boundary integral equation method, then a numerical method based on quasi-Newton method is carried out. The influences of d, which is the half of distance between two vortices on symmetric and asymmetric pair is discussed.

For symmetric/antisymmetric vortex pair, as d increases the wave height of downstream wave h oscillates and the extreme values are close to the sum/difference of wave heights for two vortices of the pair. When d is taken near its maximum points, the wave profiles is close to the superposition of that for these two vortices, whereas if d is taken other values the difference is large.

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