A novel substructural damage detection approach for shear structures based on the combination of ARMAX model residual and Kullback-Leibler divergence

***Liu Mei[1,2], Huaguan Li[1,2], Yunlai Zhou[3], and †Wujian Long[1,2]**

[1]Guangdong Provincial Key Laboratory of Durability for Marine Civil Engineering, Shenzhen University, Shenzhen, 518060, P.R. China

[2]College of Civil and Transportation Engineering, Shenzhen University, Shenzhen, 518060, P.R. China

[3]Department of Civil and Environmental Engineering, The Hong Kong Polytechnic University, Hong Kong SAR, China

*Presenting author: meiliu@szu.edu.cn

†Corresponding author: longwj@szu.edu.cn

Abstract

In this paper, a novel substructural damage detection method combining autoregressive moving average with exogenous inputs (ARMAX) model residual and Kullback-Leibler divergence (KLD) is proposed to identify the damages of shear building structures. Firstly, based on the partition strategy of multi-input multi-output (MIMO) model, the overall structure is divided into series of substructures such that structural damage diagnosis process is able to be implemented on each substructure independently. For the sake of better resisting noise interference and enhancing damage detection robustness, every substructure is modeled by autoregressive-moving average with exogenous inputs (ARMAX) model, and its model residuals contain sensitive structural damage feature characterized by the changes in chi-square distribution function (CSDF) of the model residuals. Furthermore, KLD is utilized to measure the similarity between two probability distributions and used as structural damage indicator to quantify the damage in detail. Numerical simulation is conducted to evaluate the performance of proposed damage identification approach, and it shows the satisfactory results of structural damage localization and quantification.

Keywords: Damage detection; shear structure; substructure; ARMAX model residual; chi-square distribution function (CSDF); Kullback-Leibler divergence (KLD)

1. Introduction

In recent years, due to the aging of aerospace, civil, and mechanical infrastructures especially for the structures serving human society activities for a long time, structural health monitoring (SHM) has become an essential research field in maintaining the integrity of structures. Structural damage detection generally provides the fundamental information for SHM practices and is one of the most challenging components in the construction of SHM system [1].

As the nondestructive evaluation (NDE) techniques, vibration based damage detection methods have become more effective and flexible than the traditional detection approaches in engineering applications in the latest decades. Modal properties are easily obtained from structural responses, and modal frequencies, mode shapes or mode shape curvatures are chosen as damage sensitive features for wide applications of damage detection. In addition, substructuring technique has been also developed for subtly designing and analyzing the complex large-scale structures in an efficient way that the whole structure is decomposed into a series of smaller substructures. By adopting a strategy of ‘divide-and-conquer’, the performance of dynamic system model fitting and the accuracy of structural parameters identification are not reduced especially for actual large-scale engineering structures, such as high-rise building structures and long span bridges. For damage detection, most damage indicators of previous substructuring method are based on modal parameters. A substructuring method combined with the difference between squared original frequency and squared damaged frequency was proposed in [2]-[3] for damage identification of shear structures. However, modal parameters usually represent the property of the whole structure such that the modal parameters based damage indicators indicate the global structural damage, which is not sensitive enough for local damage identification under complicated environmental conditions [4]. In contrast to modal parameters identification based detection methods, the time domain or frequency domain methods extract local damage sensitive features via signal processing and only concentrating on measured data of structural responses, meanwhile, these methods are in a data-driven way without a model of the structure and different from the model based method requiring an accurate finite element model. For linear time-invariant (LTI) systems, autoregressive process can well model the structural systems and distinguish various system dynamics through the ‘black box’ model structure containing system inputs and outputs; besides, the autoregressive model based methods are more intuitionistic than the frequency domain method by generating underlying observations process directly from the model parameters without spectral representation. The autoregressive model parameters, such as the model residuals and model coefficients, have been well utilized as the damage sensitive indexes for local damage detection. In the previous researches, most model residuals-based damage detection methods are based on pattern recognition approaches [5], and their damage indicators usually rely on pattern recognition tools, which often require a large amount of training data to extract damage sensitive features and inevitably produces huge computational complexity.

In order to improve the computational flexibility of existing methods and potentially locate and quantify damages for shear structures, this paper proposes a new substructural damage detection method based on autoregressive moving average with exogenous inputs (ARMAX) model and Kullback-Leibler divergence (KLD). At first, the substructural division strategy in [2]-[3] is employed so that the damage detection process can be carried out on each substructure independently, which is suitable for a parallel and distributed SHM system. The ARMAX model combined with substructure of MIMO system is built to remove strong correlation of the responses and needed not to use pre-whitening filter, which is more convenient in signal processing than the autoregressive (AR) method of single output [6]-[7]. Besides, an ARMAX model is able to enhance the noise immunity of damage detection results by its moving average term of model residual. Furthermore, this study proposes an innovative damage indicator by incorporating ARMAX model residual and Kullback-Leibler divergence (KLD) for sensitive damage quantification in a data-driven way. KLD is an index widely used for measuring the similarity between two probability distributions in statistics; the value of KLD is close to zero when the two probability distributions are similar, otherwise, the value of KLD is close to one [8]. In this study, the distribution of model residual can be well described by the chi-square distribution function (CSDF), the KLD value between CSDFs in damage state and undamaged state can clearly indicate the damage, including the location and extent of substructural damage. Simulation of six degrees of freedom (DOFs) shear building structure subjected to mutually correlative white noises is conducted to verify the performance of proposed damage substructural damage detection approach, and specific conclusions are finally discussed.

2. Theoretical fundamentals

2.1. Dynamic system modeling with ARMAX model

For the linear discrete-time system with multiple-input and multiple-output, the dynamic process can be described by an ARMAX model as following:

$$y(t) + \sum_{k=1}^{n_a} a_k y(t-k) = \sum_{k=1}^{n_b} b_k u(t-n_k-k+1) + \sum_{k=1}^{n_c} c_k e(t-k) \tag{1}$$

where $y(t)$ represents the system output at time t, $u(t)$ denotes the system input; a_k, b_k, and c_k indicate the coefficients of autoregressive term, system input term, and moving-average term, respectively, n_a, n_b and n_c depict their corresponding model orders, respectively, n_k means the time delay steps; $e(t)$ are the residuals of the estimation process at time t; the ARMAX model is efficient for its flexibility to availably handle the disturbance modeling through its moving-average coefficient c_k.

2.2. Substructure division

Generally, the shear building structure can be simulated as a one-dimensional shear model with lumped masses through the below motion equation:

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\ddot{\boldsymbol{x}} + \boldsymbol{K}\ddot{\boldsymbol{x}} = -\boldsymbol{M}\boldsymbol{r}\ddot{\boldsymbol{x}}_{\boldsymbol{g}} \tag{2}$$

where $\boldsymbol{M}_{n\times n}$, $\boldsymbol{K}_{n\times n}$, and $\boldsymbol{C}_{n\times n}$ respectively depicted the mass, stiffness, and damping matrixes, n depicts the number of DOFs, $\boldsymbol{r}$ denotes the $n \times 1$ unit vector ($\boldsymbol{r} = [1 \cdots 1]^T$), $\boldsymbol{x}$ indicates the displacement vector of lateral vibration relative to the ground, $\ddot{x}_g$ means the ground acceleration.

The motion of each DOF is affected by the motion of adjacent DOFs; every mass and its adjacent masses are separated from the overall structure to construct series of substructures, as shown in Fig. 1. According to the principle of force balance at the lateral direction, the motion equation of substructure i ($1 \le i \le n$-1) can be expressed as

$$m_i\ddot{y}_i + (c_i + c_{i+1})\dot{y}_i + (k_i + k_{i+1})y_i = -m_i\ddot{z}_{i-1} + c_{i+1}\dot{y}_{i+1} + k_{i+1}y_{i+1} \tag{3}$$

where m_i is the i^{th} story mass, k_i is the stiffness coefficient of the i^{th} story, c_i is the damping coefficient of the i^{th} story; y_i represents the displacement of the i^{th} story relative to the $(i\text{-}1)^{th}$ story; $\ddot{z}_i$ means the absolute acceleration of the i^{th} story, and especially $\ddot{z}_0$ denotes the ground acceleration $\ddot{x}_g$. Considering that the top mass m_n is the free end and only one mass is adjacent to it, the motion equation of the top substructure is represented by following:

$$m_n\ddot{y}_n + c_n\dot{y}_n + k_n y_n = -m_n\ddot{z}_{n-1} \tag{4}$$

Introducing the difference expression

$$\dot{y}_i(t) = \frac{y_i(t+T) - y_i(t-T)}{2T} \tag{5}$$

$$\ddot{y}_i(t) = \frac{y_i(t+T) - 2y_i(t) + y_i(t-T)}{T^2} \tag{6}$$

where $\dot{y}_i$ and $\ddot{y}_i$ means the velocity and acceleration of the i^{th} story relative to the $(i\text{-}1)^{th}$ story, respectively, t represents the time index, T depicts the sampling interval. By substituting Eq. (5) and Eq. (6) into Eq. (3), the motion equation of substructure i ($1 \le i \le n$-1) can be rewritten as

$$\begin{aligned}\ddot{y}_i(t) + a_1\ddot{y}_i(t-1) + a_2\ddot{y}_i(t-2) = {} & b_{11}\ddot{z}_{i-1}(t-1) + b_{12}\ddot{z}_{i-1}(t-2) \\ & + b_{21}\ddot{y}_{i+1}(t-1) + b_{22}\ddot{y}_{i+1}(t-2) \\ & + c_1 e(t-1) + c_2 e(t-2)\end{aligned} \tag{7}$$

In this, Eq. (7) can be regarded as an ARMAX model with two-input ($\ddot{z}_{i-1}$ and $\ddot{y}_{i+1}$) and single-output ($\ddot{y}_i$) [2]-[3], where $e(t)$ represents the ARMAX model residuals. Likewise, the motion equation of top substructure n can be rewritten by substituting Eq. (5) and Eq. (6) into Eq. (4), and it can be identified as a single-input ($\ddot{z}_{n-1}$) and single-output ($\ddot{y}_n$) ARMAX model, that is

$$\ddot{y}_n(t) + a_1\ddot{y}_n(t-1) + a_2\ddot{y}_n(t-2) = b_1\ddot{z}_{n-1}(t-1) + b_2\ddot{z}_{n-1}(t-2) + c_1 e(t-1) + c_2 e(t-2) \quad (8)$$

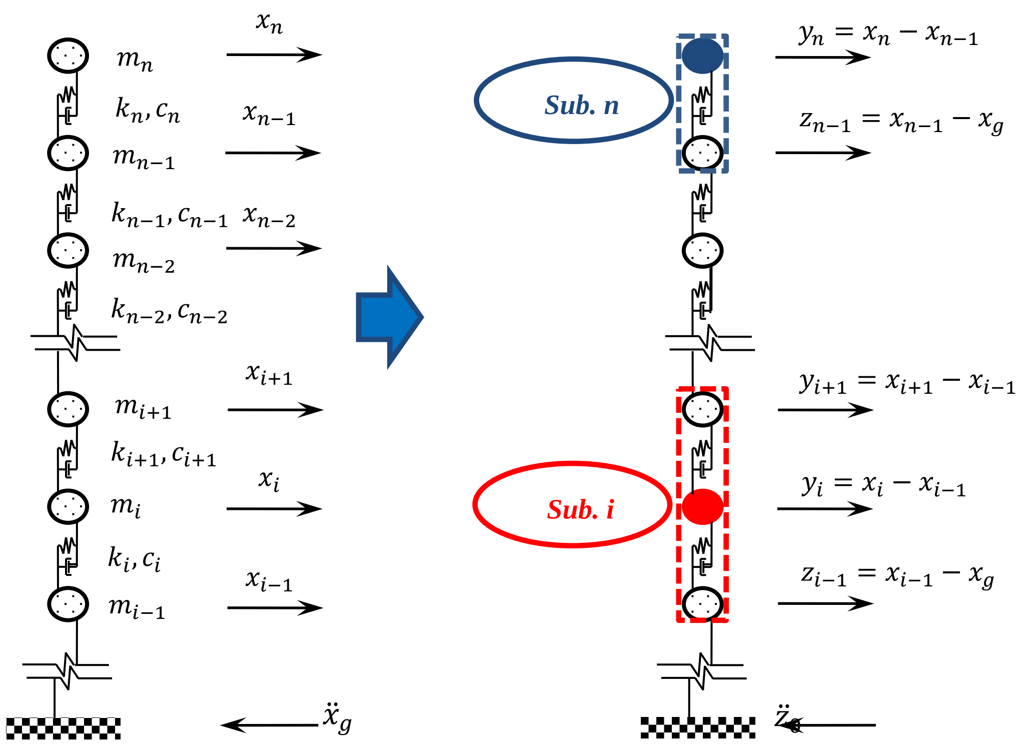

Figure 1. Substructure division method

Herein, in order to characterize each substructure with MIMO model and promptly acquired essential structural features, only three accelerometers are needed to establish the related ARMAX model while two is enough for the top substructure.

2.3. Damage indicator

In this paper, a novel structural damage indicator based on ARMAX model residual and Kullback-Leibler divergence is proposed to identify the damages of shear structures. Initially, the predicted system output $\hat{y}_u(t)$ modeling with ARMAX model in undamaged state can be calculated as

$$\hat{y}_u(t) = -\sum_{k=1}^{n_a} a_k y(t-k) + \sum_{k=1}^{n_b} b_k u(t-n_k-k+1) + \sum_{k=1}^{n_c} c_k\, e(t-k) \quad (9)$$

The model residuals of the undamaged state and damaged state can be generated between the measurement system output y and the predicted system output $\hat{y}$ by comparing with the reference ARMAX model of undamaged state, that is

$$e_u(t) = y_u(t) - \hat{y}_u(t) \tag{10}$$

$$e_d(t) = y_d(t) - \hat{y}_u(t) \tag{11}$$

where $y_u(t)$ and $y_d(t)$ are the measurement output of undamaged and damaged state from the substructure being analyzed, respectively. For damage case caused by the degradation of story stiffness, structural responses generated from damaged system generally vary from responses of undamaged system, and it is hard to fit the structural responses in damaged state well by using the reference ARMAX model in undamaged state. In other words, model residuals from damaged system responses (Eq. (11)) are different from residuals of responses in undamaged state (Eq. (10)), which contains important structural information for damage examination. In addition, the model residual vector are normalized to a dimensionless vector so as to remove the effects of various response amplitudes, as shown as follows

$$\bar{\boldsymbol{e}}_{\boldsymbol{u}} = \frac{\boldsymbol{e}_{\boldsymbol{u}}}{\|\boldsymbol{y}_{\boldsymbol{u}}\|} \tag{12}$$

$$\bar{\boldsymbol{e}}_{\boldsymbol{d}} = \frac{\boldsymbol{e}_{\boldsymbol{d}}}{\|\boldsymbol{y}_{\boldsymbol{d}}\|} \tag{13}$$

where $\boldsymbol{e}_{\boldsymbol{u}}$ and $\boldsymbol{e}_{\boldsymbol{d}}$ represent the ARMAX model residual vector in the undamaged and damaged state, respectively; $\|\boldsymbol{y}_{\boldsymbol{u}}\|$ and $\|\boldsymbol{y}_{\boldsymbol{d}}\|$ mean the norm of output response vector in the undamaged and damaged state, respectively, $\bar{\boldsymbol{e}}_{\boldsymbol{u}}$ and $\bar{\boldsymbol{e}}_{\boldsymbol{d}}$ denote the corresponding normalized dimensionless residual vector, respectively. On the other hand, the discrepancy between the distributions of residual vectors in undamaged and damaged state is able to qualitatively reflect the existence of structural damage, and in this work we utilize the chi-square distribution function (CSDF) to characterize ARMAX residual vectors for structural damage identification:

$$f(x) = \begin{cases} \dfrac{x^{\frac{k}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{k}{2}} \Gamma(\frac{k}{2})} & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{14}$$

where x represent the random variable, $f(.)$ denotes the chi-square distribution function, $\Gamma(.)$ indicates the gamma distribution function, k depicts a positive integer that specifies the number of degrees of freedom and affects the shape of the chi-square distribution function curves of residual vectors. Moreover, KLD is utilized to quantify the difference of distributions of ARMAX model residuals and used as the structural damage indicator in this study. At first, for the discrete random variable $X = \{x_1, x_2, \ldots, x_n\}\ (n \geq 2)$ and

$Y = \{y_1, y_2, \dots, y_n\}$ from an uncertainty system, their corresponding probability distribution of each element are given as

$$P(X) = \{p_1(x), p_2(x), \dots, p_n(x)\} \quad (15)$$

$$Q(Y) = \{q_1(y), q_2(y), \dots, q_n(y)\} \quad (16)$$

where $p_{\mathrm{i}}(x)$ and $q_i(y)$ represent the probability distribution function of the element x_i and y_i, respectively; and $0 \le p_i(x), q_i(y) \le 1$, $\sum_{i=1}^{n} p_i(x)$ (or $q_i(y)) = 1$, $i = 1, \dots, n$. The KLD [8] between the probability distributions of discrete random variables *X* and *Y* is defined as

$$D_{KL}(P(X) \| Q(Y)) = \sum_{i=1}^{n} p_i(x) \ln \frac{p_i(x)}{q_i(y)} \quad (17)$$

For the discrete random variables of ARMAX model residuals applied in linear time-invariant system,

$$\boldsymbol{e_u} = \{e_u(t), e_u(t-1), \dots, e_u(t-n+1)\} \quad (18)$$

$$\boldsymbol{e_d} = \{e_d(t), e_d(t-1), \dots, e_d(t-n+1)\} \quad (19)$$

where t indicates time index, $n \ge 2$ denotes the length of the residual vector, the corresponding probability distributions of each element of undamaged and damaged state are described as

$$P(\boldsymbol{e_u}) = \{p_1(e_u), p_2(e_u), \dots, p_n(e_u)\} \quad (20)$$

$$P(\boldsymbol{e_d}) = \{p_1(e_d), p_2(e_d), \dots, p_n(e_d)\} \quad (21)$$

Eventually, the structural damage indicator using KLD between distributions of ARMAX model residuals is defined as following:

$$D_{KL}(P(\tilde{\boldsymbol{e}}_{\boldsymbol{u}})\|P(\tilde{\boldsymbol{e}}_{\boldsymbol{d}})) = \sum_{i=1}^{n} p_i(\tilde{\boldsymbol{e}}_{\boldsymbol{u}})\ln\frac{p_i(\tilde{\boldsymbol{e}}_{\boldsymbol{u}})}{q_i(\tilde{\boldsymbol{e}}_{\boldsymbol{d}})} \tag{22}$$

$$\tilde{\boldsymbol{e}}_{\boldsymbol{u}} = \mathrm{sort}(\mathrm{abs}(\bar{\boldsymbol{e}}_{\boldsymbol{u}})) \tag{23}$$

$$\tilde{\boldsymbol{e}}_{\boldsymbol{d}} = \mathrm{sort}(\mathrm{abs}(\bar{\boldsymbol{e}}_{\boldsymbol{d}})) \tag{24}$$

where $\tilde{\boldsymbol{e}}_{\boldsymbol{u}}$ and $\tilde{\boldsymbol{e}}_{\boldsymbol{d}}$ represent the rearrangement vectors of absolute value of $\bar{\boldsymbol{e}}_{\boldsymbol{u}}$ and $\bar{\boldsymbol{e}}_{\boldsymbol{d}}$ in ascending order, respectively; $P(.)$ means the corresponding chi-square distribution function.

3. Numerical simulation

3.1. Simulation setup

In order to substantiate the performance of proposed method for damage detection, numerical simulation of damage detection on a six-story shear building structure has been conducted. As is depicted in Fig. 2, it is a six-story shear building model which can be simplified as a 6-DOF structure system, and the structure system is subjected to white noise excitation. The structural parameters are given as follows: the mass of every story is 1×10^2 kg, and the lateral stiffness is 1×10^6 N/m; damping ratio is assumed to be 3% for all modes; the first six natural frequencies of the shear model in undamaged state are given as 3.84 Hz, 11.29 Hz, 18.08 Hz, 23.83 Hz, 28.18 Hz and 30.91 Hz for the 1^{st} mode to the 6^{th} mode, respectively; the data sampling frequency is 200 Hz; taking into account the influence of environmental disturbance, measurement noises of 5% noise level are added into the acceleration data of all stories; there are totally 5×6=30 damage cases which consist of 10%, 20%, 30%, 40% and 50% reduction of lateral stiffness on every story. Fig. 3 shows the time series excitation of white noise.

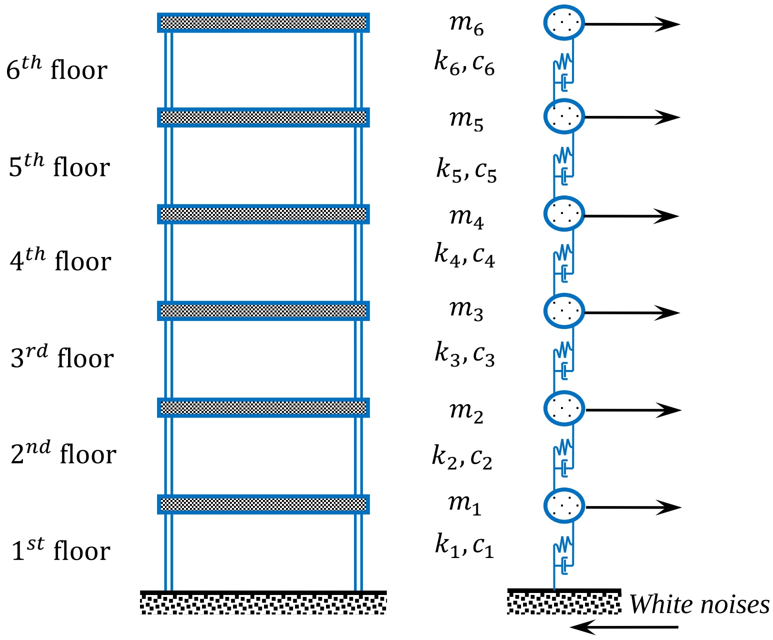

Figure 2. 6-story shear building structure subjected to white noises excitaion

3.2. Procedure and results

Primarily, the overall structure is divided into 6 substructures using the partition method mentioned in Section 2.2 (Eqs. (2) ~ (8)), as shown in Fig. 4. For each substructure from number 1 to 5, it can be modeled in good condition as a 2-input and 1-output ARMAX model, while the substructure 6 is modeled as 1-input and 1-output ARMAX model. For example, the ground acceleration ($\ddot{z}_g$) and the acceleration of the 2^{rd} DOF relative to the ground ($\ddot{y}_2$) are modeled as the input of substructure 1 while the 1^{st} DOF relative to the ground ($\ddot{y}_1$) is modeled as the output. Besides, the absolute acceleration ($\ddot{z}_{i-1}$) ($2 \leq i \leq 5$) of the $(i\text{-}1)^{th}$ DOF and the acceleration of the $(i+1)^{th}$ DOF relative to $(i\text{-}1)^{th}$ DOF ($\ddot{y}_{i+1}$) are modeled as the input of substructure i while the acceleration of the i^{th} DOF relative to the $(i\text{-}1)^{th}$ DOF ($\ddot{y}_i$) is modeled as the output. Especially for the top substructure, the absolute acceleration ($\ddot{z}_5$) of the 5^{th} DOF is modeled as the input and the acceleration of the 6^{th} DOF relative to 5^{th} DOF ($\ddot{y}_6$) is modeled as the output of substructure 6.

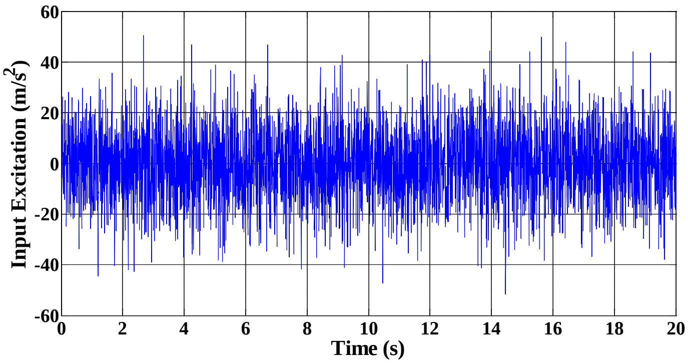

Figure 3. Input excitation of white noises

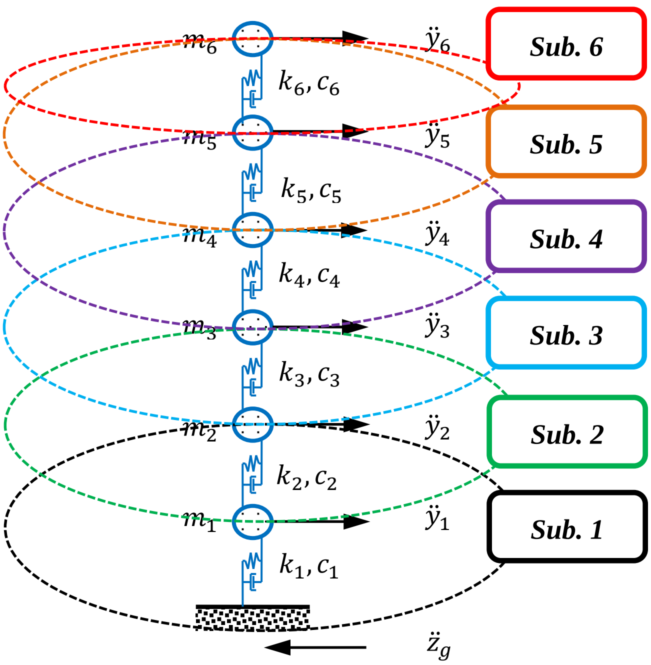

Figure 4. Substructure division for 6-DOF simulated shear building structure

The undamaged limit of each substructure is independently calculated through the mean value of KLD values between 10 data subsets in undamaged state, and the time duration of each subset is 20 secs. The changes in ARMAX model residuals from the undamaged system to the damaged systems can be reflected by its chi-square distribution function. All CSDF curves of 10% damage at the 1^{st} ~ 6^{th} floors are shown in Fig. 5 to indicate the damages existing in the structure. It can be observed from Fig. 5 that the CSDF curves shapes of damaged floors are conspicuous compared with others of undamaged floors. This is in good agreement with the theoretical expectation that ARMAX model residual of substructure in damaged state varies from the one of substructure in undamaged state since the substructural responses in damaged state cannot fit well the ARMAX model in undamaged state, as shown in Eqs. (9) ~ (14). Nevertheless, it is hard to exactly calculate the gap between the CSDF curves of different damage cases by only using the distinction of curves shapes. In view of this, KLD derived from Eqs. (15) ~ (24) is adopted to quantify the difference of CSDF curves between undamaged state and damaged state, and the complete identification of 10%, 20%, 30%, 40% and 50% damage cases is shown in the bar plots of Fig. 6. As a result, it reveals that there exists evident regularity in the damage location while the damage indicator can clearly quantify the damage with the damage degree increasing though it was interfered by the 5% noise. Therefore, it is explicitly reasonable that the proposed residual-based KLD is acceptably to reveal the linear relationship between the values of damage indicator and the structural stiffness reductions even in the case of a high severity of damage.

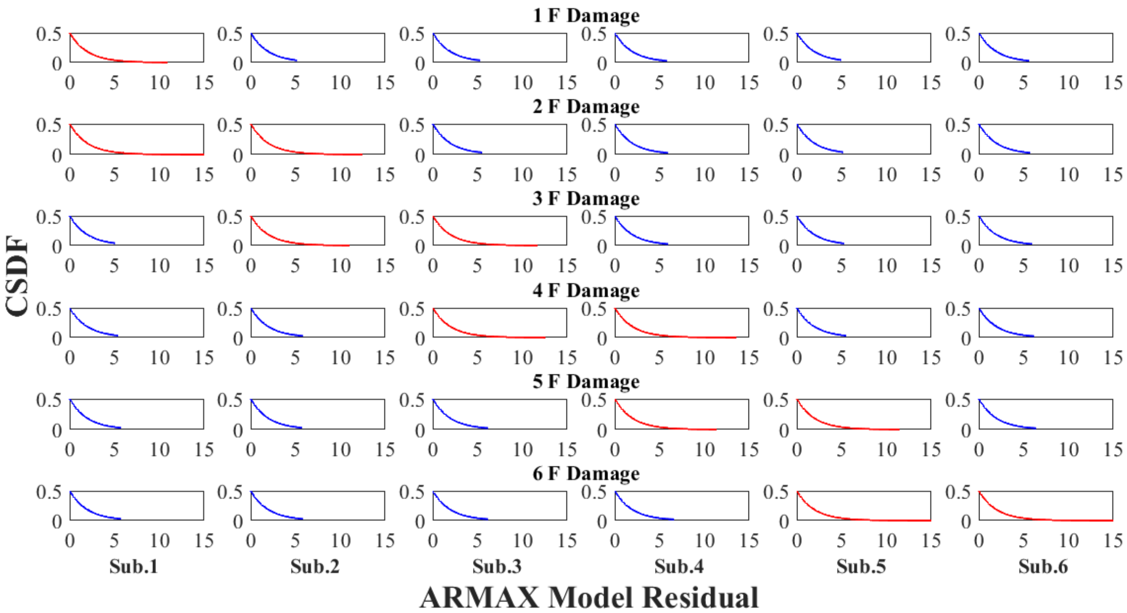

Figure 5. Chi-square distribution function (CSDF) of ARMAX model residual (white noise excitation, 5% noise, ARMAX model, data length = 4000, na = 2, nb = 3, nc = 3, and nk = 0; k = 2 (number of DOFs of CSDF))

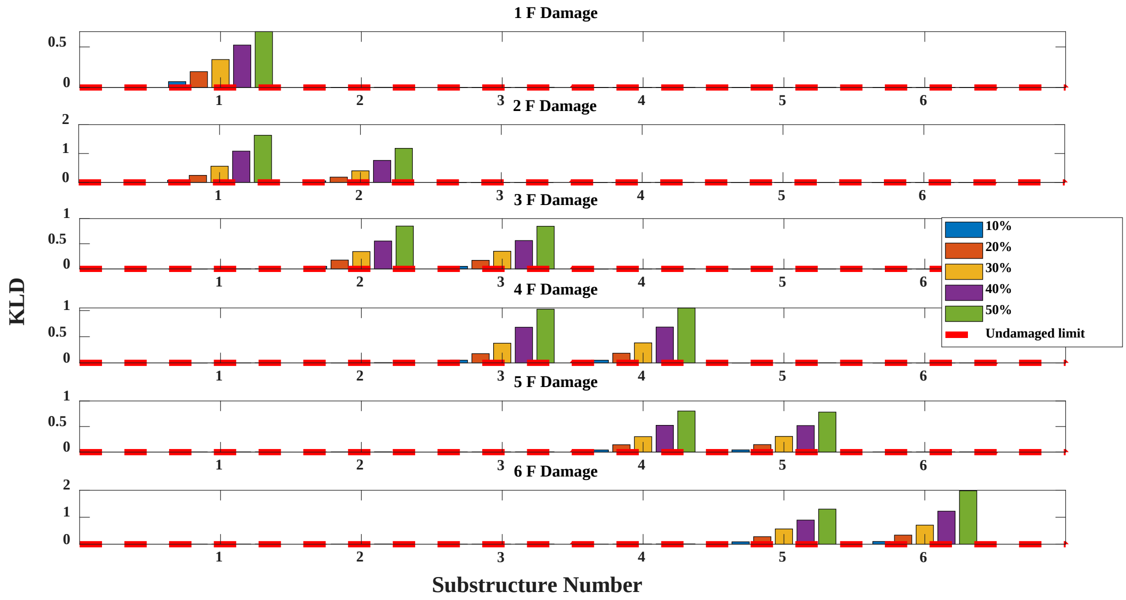

Figure 6. Damage indicator of Kullback-Leibler divergence (KLD) (white noise excitation, 5% noise, ARMAX model, data length = 4000, na = 2, nb = 3, nc = 3, and nk = 0; k = 2 (number of DOFs of CSDF))

4. Conclusions

This paper proposed an innovative substructural damage detection method based on damage indicator of ARMAX model residual-based KLD. Simulation of damage identification on a six-story shear building structure subjected to white noise is conducted to evaluate the performance of proposed damage detection strategy and damage indicator, and the results show that it can locate and quantify the damages of shear structures effectively by the proposed method. Due to the damage detection procedure can be implemented on each substructure independently, which suits for monitoring of key areas of actual engineering structure. On the other hand, the proposed CSDF curves of ARMAX model residual can clearly locate the structural damages in a visualized way with its distinguished tendencies; the proposed damage indicator of residual-based KLD can locate and quantify the damages in a data-driven way, which is suitable for local damage detection and does not rely on previous training data of various damage patterns. These mean that the proposed substructural damage detection approach is easy and efficient for local substructure damage detection of shear structures. In the following research, it is needed to further investigate about the identification of nonlinear damage in complex engineering structures with the proposed substructure damage identification method.

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