A Mesh Refinement Algorithm for Mixed Boundary Value Elastic Problems

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Abstract

The finite difference method (FDM) is a renounced numerical method for solving of complex problems of numerous fields. However, the efficacy of this method depends on the resolution of the mesh i.e. the size of the mesh used to obtain the solutions. In general, very small sized mesh, i.e., high mesh resolution is necessary to obtain an acceptable solution for various multi-scale physical problems. This high resolution of mesh consumes a significant amount of computational memory. Thus, huge wastage of computational resources occurs in refinement of sections of the domain where computation of the solution does not require high resolutions. This problem is effectively addressed by mesh refinement (MR) technique, a technique of local refinement of mesh only in sections where needed, thus allowing concentration of effort where it is required. The objective of this paper is to develop a mesh refinement algorithm for fourth order biharmonic equation which is widely used to solve boundary value elastic problems by using finite difference approach. Initial tests using the MR algorithm establish that the model adopted has considerable potential for mixed boundary value elastic problems. The results of initial test also show that consumption of computational resources is significantly less compare to uniform mesh (UM) while maintaining the quality of the solution.

Keywords: Finite Difference Method; Mesh Refinement; Fourth Order Biharmonic Equation; Mixed Boundary Value Problems

Introduction

Finite difference method is extensively used to solve mixed boundary value elastic problems because of the simplicity of this method[1],[2],[3],[4],[5]. Usually mixed boundary value elastic problems are governed by fourth order biharmonic equation (FOBE) of potential function, Ψ. Analytical solution is not possible for FOBE which makes numerical method very popular for mixed boundary value problems. For instance, Ahmed et al. [1] analyzed stress-strain distributions of a both end fixed deep beams with mixed boundary conditions by applying FDM. Later, a generalized mathematical model for the solution of mixed-boundary-value elastic problems is depicted by Hossain et al. [4]. However, in these examples, the physical domains are discretized with high resolution uniform mesh, which consumes a significant amount of computational resources to store data. Moreover, high resolution of mesh involves the solution of a large matrix which ultimately accumulates huge amount of round of error during computation.

Previously, remedies of fine uniform mesh induced problems are sought by zooming the critical region, stress concentrated area [6]. However, the adaptability of this method is limited by the requirement of solutions for several times. Moreover, for this method, the boundary conditions for current solution step depend on the previous step solution. So, if the solution of previous step is not acceptable due to less resolution, then current step solution would not be also acceptable. Recently, memory exhausting problems is significantly reduced with the development of powerful computers, however, an algorithm for the solution of mixed boundary value problems with less memory consumption is still demanding. To address this memory exhausting problem, the mesh refinement technique is extensively used in various field of study[7],[8],[9],[10].

Mesh refinement is a technique of local refinement of a mesh to allow computational resources and efforts where it is required. Sections of the physical domain needing high resolution are generally determined by means various criteria which includes comparing the solution to a threshold or the local rate of change to a solution. A mesh refinement algorithm based on the idea of multiple component grids for the solution of fourth order biharmonic equation using finite difference techniques is presented in this article. The solution of this equation is often smooth and easily approximated over large portions of their domains if there is no steep gradients, cracks or other discontinuity in the solution. However, most often the physical problem contains support boundary or locally isolated internal regions with steep gradients, cracks, or discontinuities, where the solution is difficult to approximate. We place locally finer grids in these regions over a coarse grid covering the domain. The solution on each line sub grid can then be approximated by standard finite difference techniques, as done on the coarse grid.

Mathematical Formulation

In terms of potential function, Ψ under plain stress or plain strain condition, solution of boundary value elastic problems requires to solve the following fourth order biharmonic equations with appropriate boundary conditions [1].

$$\frac{\partial^4\Psi}{\partial x^4}+\frac{\partial^4\Psi}{\partial x^2\partial y^2}+\frac{\partial^4\Psi}{\partial y^4}=0 \tag{1}$$

The relation between potential function and displacement components are as follows

$$u_x=\frac{\partial^2\Psi}{\partial x\partial y} \tag{2}$$

$$u_y=-\frac{1}{1+\mu}\left[2\frac{\partial^2\Psi}{\partial x^2}+(1-\mu)\frac{\partial^2\Psi}{\partial y^2}\right] \tag{3}$$

where u_x and u_y are the displacement components in the *x*- and *y*- directions respectively. The relation between stress components, displacement components are as follows

$$\sigma_x(x,y)=\frac{E}{1-\mu^2}\left(\frac{\partial u_x}{\partial x}+\mu\frac{\partial u_y}{\partial y}\right)=\frac{E}{(1+\mu)^2}\left(\frac{\partial^3\Psi}{\partial x^2\partial y}-\mu\frac{\partial^3\Psi}{\partial y^3}\right) \tag{4}$$

$$\sigma_y(x,y)=\frac{E}{1-\mu^2}\left(\frac{\partial u_y}{\partial y}+\mu\frac{\partial u_x}{\partial x}\right)=-\frac{E}{(1+\mu)^2}\left(\frac{\partial^3\Psi}{\partial y^3}+(2+\mu)\frac{\partial^3\Psi}{\partial x^2\partial y}\right) \tag{5}$$

$$\sigma_{xy}(x,y)=\frac{E}{2(1+\mu)}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)=\frac{E}{(1+\mu)^2}\left(\mu\frac{\partial^3\Psi}{\partial x\partial y^2}-\frac{\partial^3\Psi}{\partial x^3}\right) \tag{6}$$

Since, the target of this paper is to develop a mesh refinement algorithm for fourth order biharmonic equation, we select a very simple mixed boundary value elastic problem as shown in Fig. 1. In the considered problem, a simple elastic member of length '2b' and width 'a' has an embedded crack under the uniform axial loading. For simplicity, we considered there is no crack growth under this uniform loading condition. The material geometry of the problem is taken as a/b=1.0 and size of the crack is taken as one fourth (a/4) of the width of the member.

Referring to Fig. 1, for this problem, both the top and bottom edges are free and the both lateral edges are subjected to uniform tensile loading. Taking the advantages of symmetry, the right half section of the elastic member is solved under MR and finite element method (FEM) with necessary BCs as shown in Fig. 2.

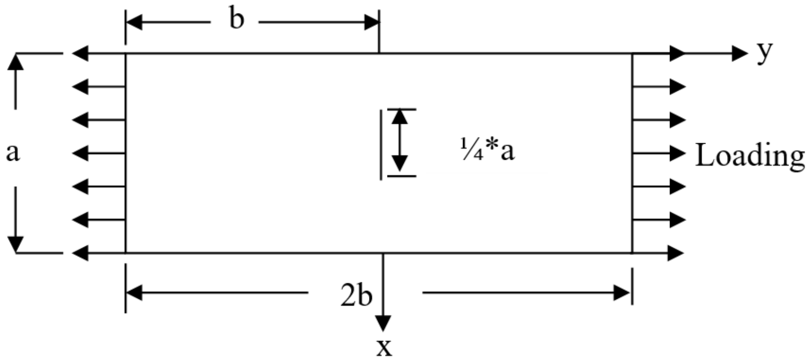

Figure 1: Simple bar with embedded crack under uniform tensile stress

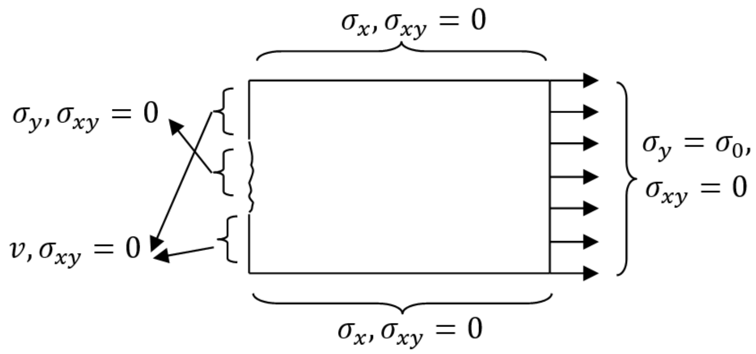

Figure 2: Half section of the problem with necessary boundary conditions

As far as numerical solution is concerned, it is evident from the expression of boundary conditions (please see Fig. 2) that all the boundary conditions of interest can easily be discretized in terms of displacement function, Ψ by the finite-difference method.

Numerical Method

The replacement process of continuous problem by a discrete problem whose solution approximates the solution of continuous problem under numerical method is known discretization. Under mesh refinement technique, first the domain is discretized with a coarse grid. Then, finer grids are added in the region which requires more resolution. An example of discretization of the considered problem under mesh refinement is shown in Fig. 3. From theory of elasticity we know that the crack location is the critical region for this problem. So, under the mesh refinement technique, the finer mesh is taken in that region. Since, no time scale is associated with the problem, instead of adaptive mesh refinement, we introduce statistically refined mesh. Under any numerical method, the governing equation (Eq. 1) must be satisfied each interior nodal point. Since, the domain is discretized into variable sized mesh, uniform grid-based stencil (Stencil-1 of Fig. 4) cannot be used throughout the domain. Thus, to satisfy the governing equation throughout the domain, several stencils are formulated as shown Fig. 4 (Stencils 2-6). Details of the stencil formulation can be found in reference These stencils can fully satisfy the governing equation throughout the domain.

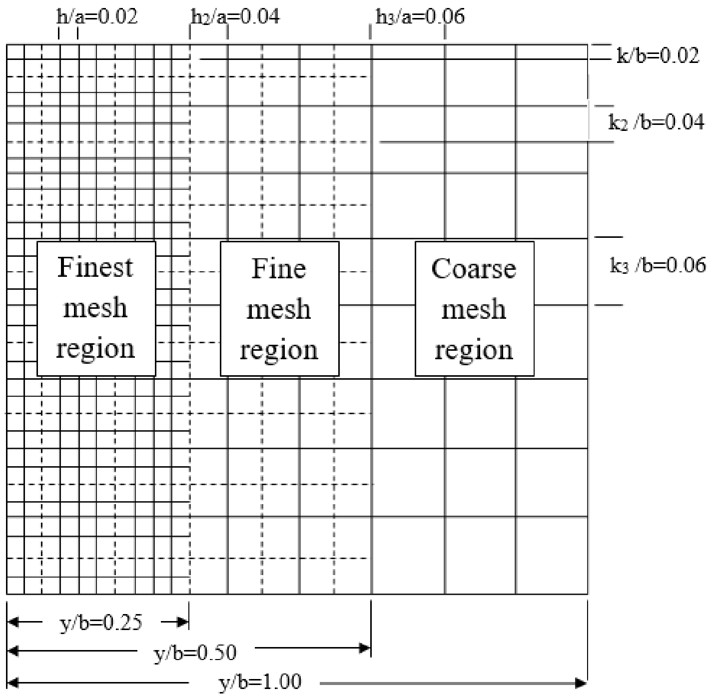

Figure 3: Discretization of domain under mesh refinement technique with three different size of mesh.

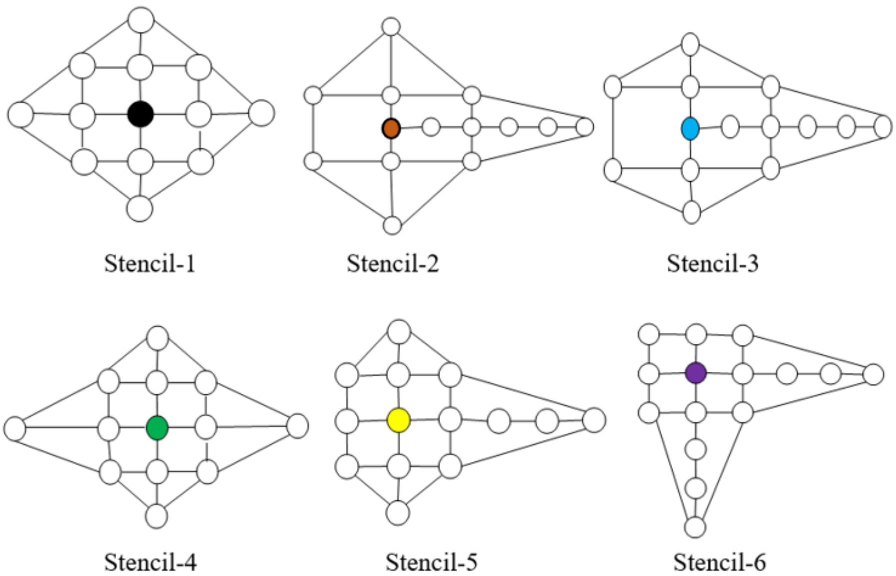

Figure 4: Types of stencils for governing equation

As seen from the problem definition, each physical boundary is defined by two conditions. This double conditions problem is satisfied by bringing an imaginary boundary [4]. The stencils of various boundary parameters over uniform mesh are shown in Fig. 5. However, these uniform mesh-based stencils are not applicable on the transition node, a node that connects two sizes of mesh. Special stencils are required for these transition nodes which is shown in Fig. 6.

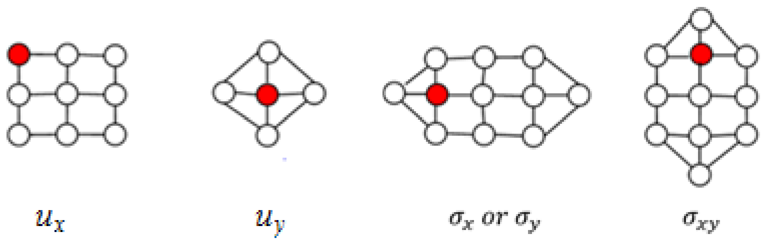

Figure 5: Stencils for displacement and stress components

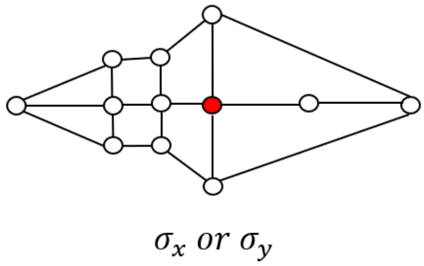

Figure 6: Special stencil for stress component

Results and Discussions

In MR FDM, finest mesh size is taken as 0.01×0.01, fine mesh size is taken as 0.02×0.02 and coarse mesh size is taken as 0.04×0.04. A uniform mesh of size 0.01×0.01 is taken for FEM discretization. Mesh sensitivity analysis is performed for both methods (data not shown). The displacement components at y/b = 0.0 with MR FDM with refine mesh (RM) and FEM are shown in Fig. 7 as a comparative study. Fig. 7a shows displacement component, u_x/a distribution at section y/b = 0.0. Except at the tip of the crack both methods show same amount of displacement in x-direction. At the tip of the crack FEM shows a little bit higher displacement, however, this disagreement is not significant. Fig. 7b shows displacement component u_y/b distribution at section y/b=0.0 and the results of both are in good agreement. In other section of the member, both methods provide exactly same amount of displacement (data not shown). In every case, MR results are as good as FEM results although greater no of nodal points is considered under FEM discretization.

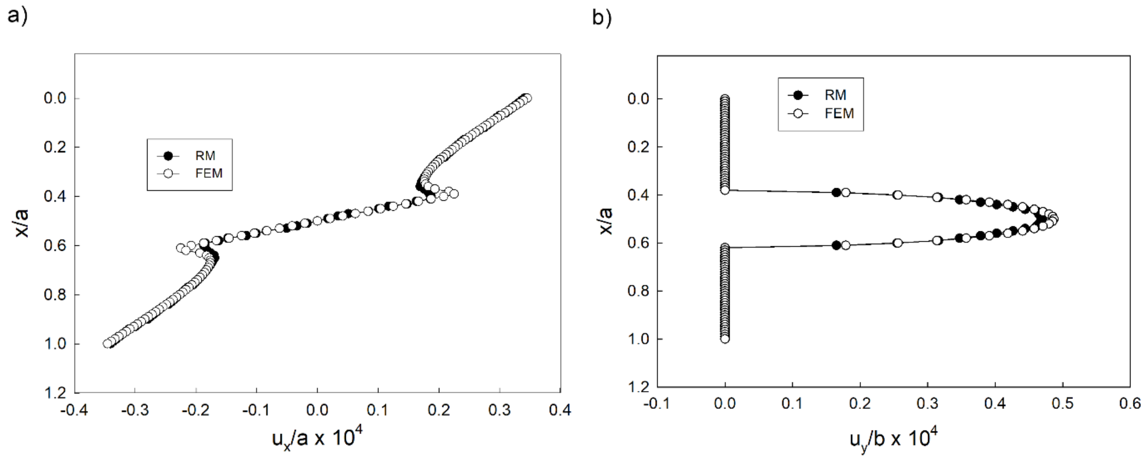

Figure 7: Comparison of MR FDM results with FEM results at section y/b=0.0: a) u_x distribution and b) u_y distribution.

The comparison of stress components at y/b = 0.0 is shown in Fig. 8. For this type of problems, the most desired parameter is the stress component in direction to applied stress and, in this case, it is σ_y which is shown in Fig. 8a. From this figure it is seen that results of both methods are in good agreement. The maximum stress is observed as 3.75 times of applied stress for FEM and around 3.8 times of applied stress for MR FDM. The normal stress, σ_x distribution is shown in Fig. 8b. From figure 8b, it is seen that the pattern of distribution is similar for both methods, but FEM give somewhat larger stress than that of mesh refinement technique. As stated in earlier example, this discrepancy arises due to the application of three BCs at the singularity points.

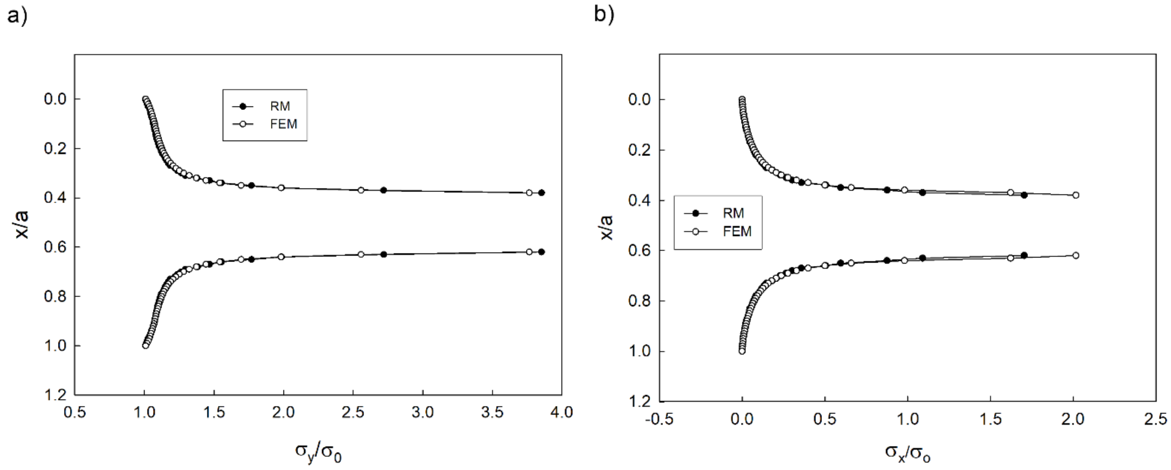

Figure 8: Comparison of MR FDM results with FEM results at y/b=0.0 in terms of stress components: a) σ_y/σ_o distribution and b) σ_x/σ_o distribution.

Conclusions

A mesh refinement algorithm is developed for fourth order biharmonic equation which is widely used to investigate the displacements and stress analysis of mixed boundary value problems. The governing equation is discretized by finite difference method in various way to develop various stencils which are required to satisfy the governing equation throughout the domain. Due to mesh refinement, the boundaries are also discretized into irregular meshes. As a result, the boundary conditions also need to be discretized in different way than regular mesh. Our results show that the developed method can easily be used to obtain the solution of mixed boundary value elastic problems in terms of displacement and stress components. Our results also show that a reduced number of nodes can yields results as good as finite element method.

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