High-precision uncertainty propagation involving multimodal probability distributions

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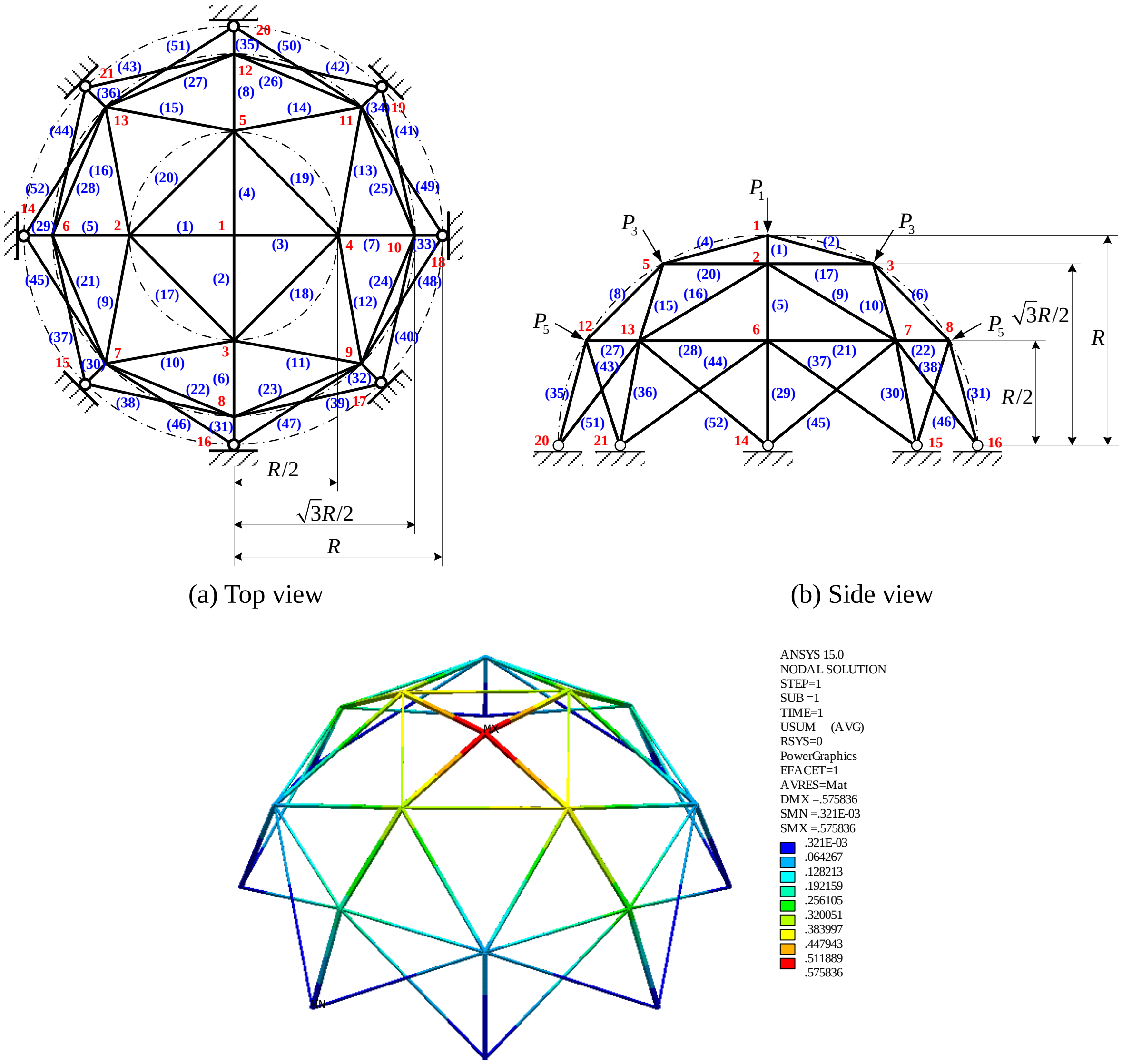
Abstract

In practical engineering applications, random variables may follow multimodal distributions with multiple modes in the probability density functions, such as the structural fatigue stress of a steel bridge carrying both highway and railway traffic and the vibratory load of a blade subject to stochastic dynamic excitations, etc. Traditional probabilistic uncertainty propagation methods are mainly used to treat random variables with only unimodal distributions, which, therefore, tend to result in large computational errors when multimodal distributions are involved. In this paper, a high-precision probabilistic uncertainty propagation method is proposed for problems involving multimodal distributions. Firstly, the multimodal probability density functions of input random variables are constructed based on the Gaussian mixture model. Secondly, the high-order moments of the response function are calculated using the univariate dimension reduction method, based on which the input uncertainty is effectively propagated. Thirdly, the probability density function of the response is estimated using the maximum entropy method. Finally, a convergence mechanism is formulated to help ensure the uncertainty propagation accuracy. Two mathematical problems and two truss structures are investigated to demonstrate the effectiveness of the proposed method.

Keywords: Probabilistic uncertainty propagation; Multimodal distribution; High-order moment;

Example: A 52-bar space truss



(a) Top view (b) Side view

(c) FEM model

Fig. 1. The 52-bar space truss and its FEM model

As shown in Fig. 1, a hemispherical space truss (like a dome) is considered, which contains 52 bars. The cross-sectional areas of bars 1-8 and 29-36 are $A_1 = 2\text{in}^2$. The cross-sectional areas of bars 9-16 are $A_2 = 1.2\text{in}^2$ and that of the other bars are $A_3 = 0.6\text{in}^2$. The radius of the hemispherical space truss is $R = 240\text{in}$. Six external loads are applied on the space truss, which are p_1 in the inner normal direction of point 1, p_2 in the inner normal direction of points 2 and 4, p_3 in the inner normal direction of points 3 and 5, p_4 in the inner normal direction of points 6 and 10, p_5 in the inner normal direction of points 8 and 12, p_6 in the inner normal direction of points 7, 9, 11,13. The response function is defined as follows:

$$\delta_1 = g(E, P_1, P_2, P_3, P_4, P_5, P_6) \tag{1}$$

The detailed information of the random variables $P_1, P_2, P_3, P_4, P_5, P_6$ and E are presented in Table 1.

Table 1 The distribution parameters of random variables $P_1, P_2, P_3, P_4, P_5, P_6$ and E

Random variables	Distribution types	Distribution parameters		
		Coefficients	Mean values	Standard deviations
P_1(kip)	Multimodal	$\boldsymbol{\alpha}$ =(0.2,0.2,0.3,0.3)	$\boldsymbol{\mu}$ =(45,55,75,85)	$\boldsymbol{\sigma}$ =(4,4.5,4,4.5)
P_2(kip)	Multimodal	$\boldsymbol{\alpha}$ = (0.2,0.2,0.3,0.3)	$\boldsymbol{\mu}$ = (40,50,70,80)	$\boldsymbol{\sigma}$ =(4,4.5,4,4.5)
P_3(kip)	Multimodal	$\boldsymbol{\alpha}$ = (0.2,0.2,0.3,0.3)	$\boldsymbol{\mu}$ = (35,45,65,75)	$\boldsymbol{\sigma}$ =(4,4.5,4,4.5)
P_4(kip)	Multimodal	$\boldsymbol{\alpha}$ = (0.18,0.18,0.32,0.32)	$\boldsymbol{\mu}$ =(30,35,55,60)	$\boldsymbol{\sigma}$ =(3,6,3,6)
P_5(kip)	Multimodal	$\boldsymbol{\alpha}$ = (0.18,0.18,0.32,0.32)	$\boldsymbol{\mu}$ = (25,30,50,55)	$\boldsymbol{\sigma}$ =(3,6,3,6)
P_6(kip)	Multimodal	$\boldsymbol{\alpha}$ = (0.25,0.25,0.25,0.25)	$\boldsymbol{\mu}$ = (20,25,45,50)	$\boldsymbol{\sigma}$ =(3,6,3,6)
E(ksi)	Normal	--	μ =2.5e+04	σ =1.0e+03

The PDF results of δ_1 obtained by the proposed method and MCS method are plotted in Fig. 2. It can be observed that the PDF results obtained by the proposed method coincides well with that obtained by MCS, which indicates the high uncertainty propagation accuracy of the proposed method. Especially, the bimodal characteristic of the PDF obtained by MCS is well captured by that obtained by the proposed method. Besides, the CDF results for a series of response functions δ_{10}=$g(P_1, P_2, P_3, P_4) - \overline{\delta}_{10}$ and their relative errors are presented in Table 2. It can be observed that small relative errors are achieved by the proposed method at all cases. For example, the largest relative error of the proposed method is only 3.5088 percents when $\overline{\delta}_{10} = -1.5$ in.

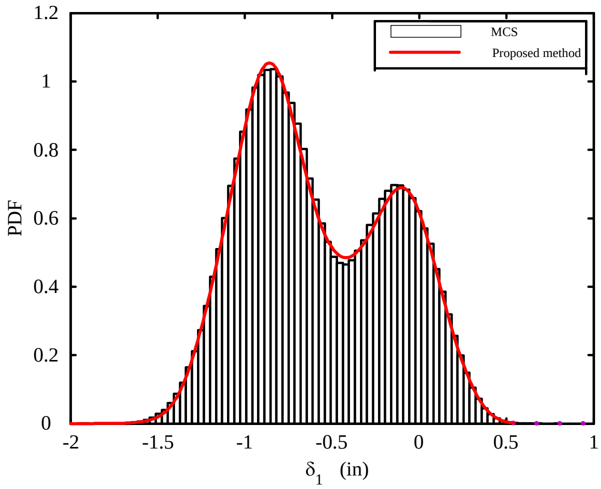


Fig. 2. Comparison of the PDF results between the proposed method and MCS method

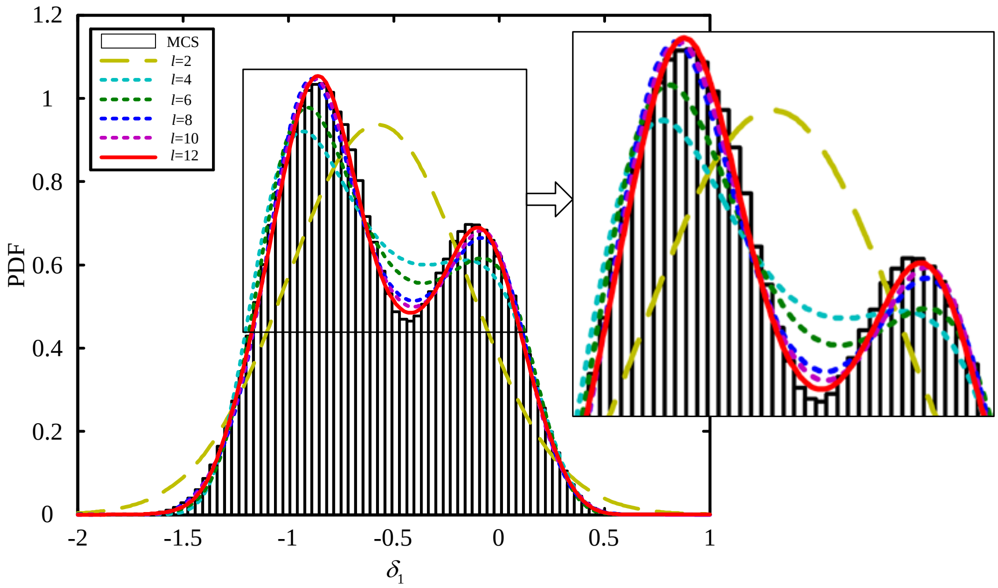
Table 2 Comparison of the CDF results between the proposed method and MCS method

δ_1 (in)	MCS		The proposed method	
	CDF	ε_0(%)	CDF	ε_1(%)
-1.5	1.7100e-03	--	1.6500e-03	3.5088
-1	0.1714	--	0.1694	1.1350
-0.5	0.5947	--	0.5933	0.2337
0	0.8857	--	0.8856	2.9700e-03
0.5	0.9997	--	0.9998	3.5500e-03

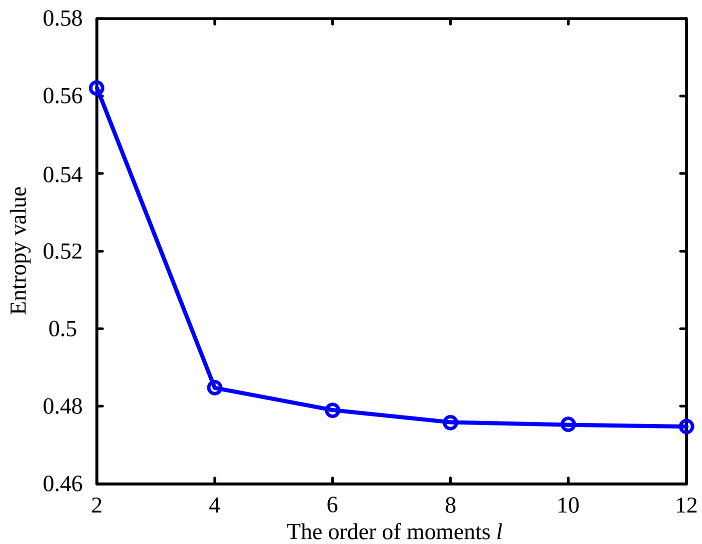
The order of moments that are required for uncertainty propagation is determined as l=12 by the convergence mechanism. The evolution process of the estimated PDF and its Shannon entropy under different order of moments are shown in Fig. 3. It can be found that when l increases from 2 to 12, the response PDF calculated using the proposed method gradually approaches to the reference PDF obtained by MCS. When l=12, the estimated response PDF is of the highest precision. Furthermore, the Shannon entropy of the response PDF gradually converges to a steady value when l increases from 2 to 12. Table 3 presents the first 12 raw moments of the response function calculated by UDRM and their relative errors compared with the results of MCS. It can be observed that the raw moments of the response function are calculated with satisfied accuracy using the UDRM. The largest relative error of the raw moments is just 9.2600 percents, which occurs at the calculation of m_{12} .

Table 4 presents the number of function evaluations of the proposed method and MCS method.

The MCS method is conducted with 1×10^6 function evaluations, while the proposed method operates with only 12×7+1=85 function evaluations. Therefore, the proposed method is of satisfied computational efficiency.



(a)



(b)

Fig. 3. Evolution of the estimated PDF and its Shannon entropy with the variation of *l*

Table 3 The calculated raw moments of the response function

Raw moments	The proposed method	MCS	Relative error（%）
m_1	-0.5758	-0.5764	9.7620e-02
m_2	0.5120	0.5133	0.2672
m_3	0.4327	0.4327	0.5930
m_4	0.4930	0.4981	1.0214
m_5	-0.5243	-0.5327	1.5811
m_6	0.5800	0.5934	2.2272
m_7	-0.6631	-0.6842	3.0931
m_8	0.7807	0.8138	4.0523
m_9	-0.943	-0.9952	5.1494
m_{10}	1.1687	1.2484	6.3843
m_{11}	-1.4789	-1.6033	7.7761
m_{12}	1.9094	2.1043	9.2600

Table 4 Comparison of computational efficiency between different methods

	MCS	The proposed method
Number of function evaluations	1×10^{6}	85