Application of the R-function Theory for the Bending Problem of Shallow Spherical Shells with Concave Boundary

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Abstract

The R-function theory is applied to describe a shallow spherical shells on Winkler foundation with concave boundary, and then a quasi-Green's function is established by using the fundamental solution and the normalized boundary equation. The quasi-Green's function satisfies the homogeneous boundary condition of the problem. The differential equation of the problem is reduced to two simultaneous Fredholm integral equations of the second kind by the Green's formula. The singularity of the kernel of the integral equation is overcome by choosing a suitable form of the normalized boundary equation. A comparison with the ANSYS finite element solution shows a good agreement, and it demonstrates the feasibility and efficiency of the present method.

Keywords: Green's function, R-function, integral equation, bending of shallow spherical shell, concave boundary

Introduction

As a kind of structural forms, the shells and plates are widely used in various fields, such as, in the large-span roof, the underground foundation engineering, the hydraulic engineering, the large container manufacturing, the aviation, the shipbuilding, the missiles, the space technology, the chemical industry, and so on. Only few problems of the shells and plates with a regular geometric boundary and a simple differential equation can be solved with an analytical or a half analytical method. For most these problems with a geometry of arbitrary shape and a complex boundary condition, only numerical methods can be used to solve the problems, such as the boundary element method[1], the finite element method[2] and the finite difference method[3].

In the present paper, the R-function theory and the quasi-Green's function method (QGFM) proposed by Rvachev [4] are utilized. The bending problem of simply supported dodecagon shallow spherical shells on Winkler foundation with concave boundary is studied. The governing differential equation of the problem is decomposed into two simultaneous differential equations of lower order by utilizing an intermediate variable. A quasi-Green's function is established by using the fundamental solution and the boundary equation of the problem. This function satisfies the homogeneous boundary condition of the problem, but it does not satisfy the fundamental differential equation. The key point of establishing the quasi-Green's function consists in describing the boundary of the problem by a normalized equation $\omega = 0$ and the domain of the problem by an inequality $\omega > 0$. There are multiple choices for the normalized boundary equation. Based on a suitably chosen normalized boundary equation, a new normalized boundary equation can be established such that the singularity of the kernel of the integral equation is overcome. For any complicated domain, a normalized boundary equation can always be found according to the R-function theory. Thus, the problem can always be reduced to two simultaneous Fredholm integral equations of the second kind without the singularity. Using the R-function theory, Li and Yuan described successfully the rectangular, trapezoidal, triangular and parallelogrammic domains of plates[5][6] and shallow spherical shells[7][8]. For the first time, the R-function theory is applied to describe the dodecagon domain of the shallow spherical shells with concave boundary. The numerical example demonstrates the feasibility and efficiency of the present method. The R-function theory can be used to describe any more complex domains of the plates and shells.

Fundamental equations

The governing differential equations of the bending problem of simply supported shallow spherical shells on Winkler foundation[9] can be expressed as follows.

$$\nabla^4\varphi(\mathbf{x}) - \frac{Eh}{R}\nabla^2 w(\mathbf{x}) = 0\,, \qquad \mathbf{x}\in\Omega \tag{1}$$

$$D\nabla^4 w(\mathbf{x}) + \frac{1}{R}\nabla^2\varphi(\mathbf{x}) + kw(\mathbf{x}) = P_Z\,, \qquad \mathbf{x}\in\Omega \tag{2}$$

where $\nabla^4 = (\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2)^2$ is the biharmonic operator, φ is the stress function, w is the radial deflection of the shell, R is the radius of curvature of the shell, k is the elastic coefficient of the foundation, $\mathbf{x} = (x_1, x_2)$, Ω is the domain of the trapezoid of shallow spherical shells in Cartesian coordinates, P_Z is the radial load; and $D = Eh^3/(12(1-\nu^2))$ is the flexural rigidity of the shell, in which h is the thickness of the shell, and E and ν are Young's modulus and Poisson's ratio, respectively.
The simply supported boundary conditions can be written as.

$$w\big|_\Gamma = \nabla^2 w\big|_\Gamma = \varphi\big|_\Gamma = \nabla^2\varphi\big|_\Gamma = 0 \tag{3}$$

where $\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is the Laplace operator, and $\Gamma = \partial\Omega$ is the boundary of the domain Ω .Making use of Eqs.(1) and (3), we can easily obtain.

$$\nabla^2\varphi = wEh/R \tag{4}$$

Substituting Eq.(4) into Eq.(2) yields.

$$D\nabla^4 w + wEh/R^2 + kw = P_Z \tag{5}$$

To decompose Eq.(5), let us introduce the following intermediate variable.

$$M = (M_1 + M_2)/(1+\nu) \tag{6}$$

where $M_1 = -D(\partial^2 w/\partial x_1^2 + \nu\partial^2 w/\partial x_2^2)$ and $M_2 = -D(\partial^2 w/\partial x_2^2 + \nu\partial^2 w/\partial x_1^2)$.
Then, substituting Eq.(6) into Eq.(5), we obtain the following two simultaneous differential equations of second rank.

$$\nabla^2 M = -P_Z + wEh/R^2 + kw \text{ and } \nabla^2 w = -M/D\,, \qquad \mathbf{x}\in\Omega \tag{7}$$

The displacement and the bending moment should be equal to zero along the simply supported boundary of shallow spherical shells on Winkler foundation, which can be written as.

$$w = 0 \text{ and } M = 0, \quad \mathbf{x} \in \Gamma \tag{8}$$

Integral equations

Let $\omega = 0$ be the normalized boundary equation of the first-order on the boundary Γ, i.e.[4]

$$\omega(\mathbf{x}) = 0, \ |\nabla \omega| = 1, \quad \mathbf{x} \in \Gamma \text{ and } \omega(\mathbf{x}) > 0, \quad \mathbf{x} \in \Omega \tag{9}$$

The quasi-Green's function can be established as follows.

$$G(\mathbf{x}, \boldsymbol{\xi}) = \frac{1}{2\pi} \ln r - \frac{1}{2\pi} \ln R_1 \tag{10}$$

where $r = \|\boldsymbol{\xi} - \mathbf{x}\| = \sqrt{(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2}$ and $R_1 = \sqrt{r^2 + 4\omega(\boldsymbol{\xi})\omega(\mathbf{x})}$, in which $\mathbf{x} = (x_1, x_2)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$. Obviously, the quasi-Green's function $G(\mathbf{x}, \boldsymbol{\xi})$ satisfies the following condition.

$$G(\mathbf{x}, \boldsymbol{\xi})\big|_{\boldsymbol{\xi} \in \partial\Omega} = 0 \tag{11}$$

To reduce the boundary value problems Eqs.(7) and (8) into the integral equations, the following Green's formula of sets of function $C^2(\Omega)$, i.e., U and $V \in C^2(\Omega \bigcup \Gamma)$, is applied.

$$\int_{\Omega} (V\nabla^2 U - U\nabla^2 V)\, \mathrm{d}_{\xi}\, \Omega = \int_{\Gamma} (V \frac{\partial U}{\partial n} - U \frac{\partial V}{\partial n})\, \mathrm{d}_{\xi}\, \Gamma \tag{12}$$

From Eqs.(7), (8), (11)and (12), and noticing that $(1/2\pi)\ln r$ is the fundamental solution[10] of the Laplace operator, then the following integral equations are obtained.

$$w(\mathbf{x}) = -\frac{1}{D} \int_{\Omega} G(\mathbf{x}, \boldsymbol{\xi}) M(\boldsymbol{\xi})\, \mathrm{d}_{\xi}\, \Omega + \int_{\Omega} w(\boldsymbol{\xi}) K(\mathbf{x}, \boldsymbol{\xi})\, \mathrm{d}_{\xi}\, \Omega \tag{13}$$

$$M(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}, \boldsymbol{\xi})[-P_Z(\boldsymbol{\xi}) + \frac{Eh}{R^2} w(\boldsymbol{\xi}) + k w(\boldsymbol{\xi})]\, \mathrm{d}_{\xi}\, \Omega + \int_{\Omega} M(\boldsymbol{\xi}) K(\mathbf{x}, \boldsymbol{\xi})\, \mathrm{d}_{\xi}\, \Omega \tag{14}$$

where

$$K(\mathbf{x}, \boldsymbol{\xi}) = [R_1^2 \omega(\mathbf{x}) \nabla^2 \omega + 4\omega(\mathbf{x})\omega - 4(\mathbf{r} \cdot \nabla \omega)\omega(\mathbf{x}) - 4\omega^2(\mathbf{x})(\nabla \omega)^2] / \pi R_1^4 \tag{15}$$

here $\omega = \omega(\boldsymbol{\xi})$, $\nabla = \nabla_{\xi}$; and $\mathbf{r} = (\xi_1 - x_1)\mathbf{i} + (\xi_2 - x_2)\mathbf{j}$, in which $\mathbf{i}$ and $\mathbf{j}$ denote unit vectors in x_1 and x_2 directions, respectively.

$K(\mathbf{x}, \boldsymbol{\xi})$ in Eq.(15) appears discontinuous only if $R = 0$, i.e., both $\mathbf{x} = \boldsymbol{\xi}$ and $\omega = 0$ come into existence. Actually, when $\mathbf{x} = \boldsymbol{\xi}$, Eq.(15) can be reduced to.

$$K(\mathbf{x}, \boldsymbol{\xi})\big|_{\mathbf{x}=\boldsymbol{\xi}} = [1 + \omega \nabla^2 \omega - (\nabla \omega)^2] / 4\pi\omega^2 \tag{16}$$

To make the kernel of the integral equations $K(\mathbf{x},\xi)\in C(\Omega\bigcup\partial\Omega)$, A normalized boundary equation will be constructed to ensure the continuity of $K(\mathbf{x},\xi)$ in the following. It can be easily testified that.

$$\omega=[3\omega_0+\omega_0^2\nabla^2\omega_0-\omega_0(\nabla\omega_0)^2]/2 \tag{17}$$

where $\omega_0=0$ is the normalized equation on the boundary Γ, i.e., ω_0 satisfies Eq.(9). Obviously, equation ω is also a normalized boundary equation of the first-order.

Based on a suitably chosen normalized boundary equation $\omega_0=0$, a new normalized boundary equation $\omega=0$ can be constructed by using Eq.(17), which ensure the continuity of the integral kernel $K(\mathbf{x},\xi)$ in the integral domain.

To obtain the numerical results of the boundary problem, the integral domain Ω is divided into several subdomains $\Omega_i\,(i=1,2,......,N)$, and in each subdomain, a rectangular quadrature formula is applied. Thus, the integral equations (13) and (14) can be discretized into the linear algebraic equations. Then, the radial deflection $w(\mathbf{x})$ can be obtained by solving the algebraic equations.

Numerical example

We investigate a simply supported dodecagon shallow spherical shell on Winkler foundation with the planform shown in Fig.1. Take a =60, $b=80$, $c=40$ and d =30. The following reference parameters are used: the radius of curvature of the shell R =200, the thickness of the shell h =2, Poisson's ratio $\nu=0.3$, Young's modulus $E=2.1\times10^6$, the elastic coefficient of the foundation k =200, and the radial load P_Z =70. According to the R-function theory[4], a normalized boundary equation of the first rank $\omega_0=0$ can be constructed from the following equation:

$$\omega_0=(\omega_1\wedge_0\omega_2)\wedge_0(\omega_1\vee_0\omega_2) \tag{18}$$

where $\omega_1=(a^2-x_1^2)/2a\geq0$ is the vertical band limited by straight lines $x_1=\pm a$;

$\omega_2=(b^2-x_2^2)/2b\geq0$ is the horizontal band limited by straight lines $x_2=\pm b$;

$\omega_3=(c^2-x_1^2)/2c\geq0$ is the vertical band limited by straight lines $x_1=\pm c$;

and $\omega_4=(x_2^2-d^2)/2d\geq0$ is the outer part of the band limited by straight lines $x_2=\pm d$.

The Boolean operations $\vee_\alpha$, $\wedge_\alpha$ (disjunction and conjunction), which correspond to the union $\bigcup$ and intersection $\bigcap$. These R-operations are defined as follows[1]:

$$X\wedge_\alpha Y=\frac{1}{1+\alpha}(X+Y+\sqrt{X^2+Y^2-2\alpha XY}),\ X\vee_\alpha Y=\frac{1}{1+\alpha}(X+Y-\sqrt{X^2+Y^2-2\alpha XY}),$$

where the parameter α varies within $-1<\alpha\leq1$. For example, if the value α is equal to zero,then the whole domain can be presented as Eq.(18). $\omega_1,\omega_2,\omega_3$ and ω_4 are a normalized equation of the first rank. $\omega_1=0$, $\omega_2=0$, $\omega_3=0$ and $\omega_4=0$ denote various parts of the boundary of the dodecagon shallow spherical shell on Winkler foundation, respectively. The radial deflection curves of line $x_2=0$ and line $x_1=0$ for different k and different R by the QGFM and by the ANSYS finite element method (FEM) are shown in Figs.2-5 for a comparison, respectively; a good agreement is observed between the two methods.

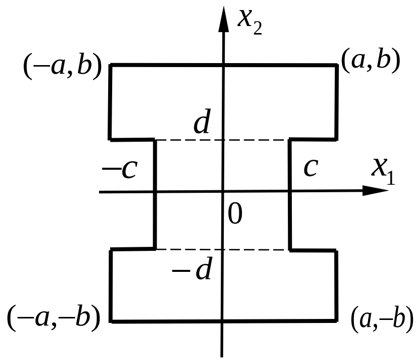

Fig.1 Dodecagon planform

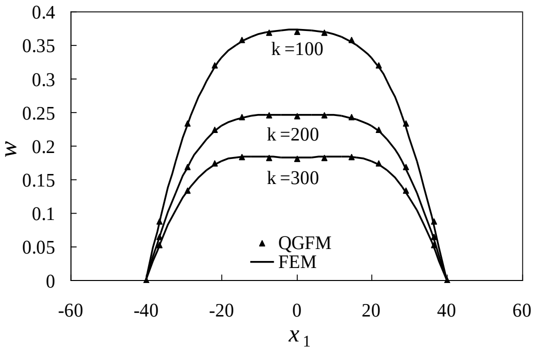

Fig.2 The deflection curve of line $x_2=0$ in Fig.1 for different k

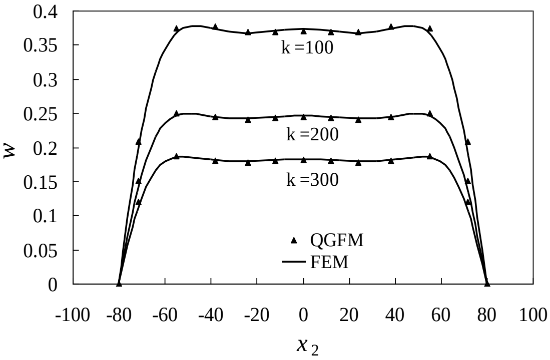

Fig.3 The deflection curve of line $x_1=0$ in Fig.1 for different k

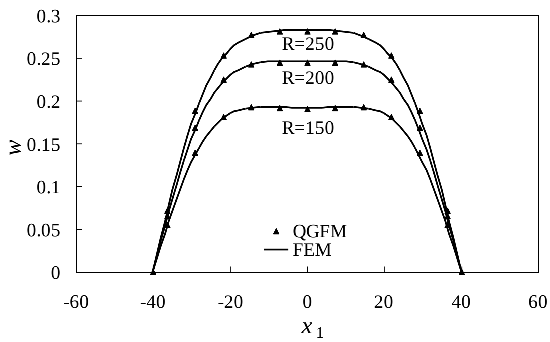

Fig.4 The deflection curve of line $x_2=0$ in Fig.1 for different R

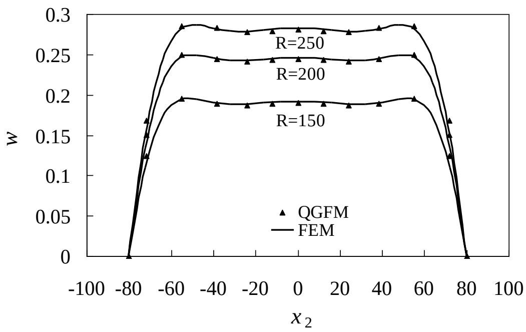

Fig.5 The deflection curve of line $x_1=0$ in Fig.1 for different R

Conclusions

In the present paper, the R-function theory is applied to describe a shallow spherical shells on Winkler foundation with concave boundary, and it is used to construct a quasi-Green's function. Compared with the FEM solution, the numerical results of the QGFM demonstrate its feasibility, efficiency and rationality. The R-function theory can also be used to effectively solve various boundary value problems of the plates and shells by constructing a trial function that satisfies the complex boundary shape and by combining with the other method of weighted residuals such as the variational method[11] and the spline-approximation[12].

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