On the heat conduction in micro-periodic laminate: uncertainty of material properties

***Piotr Ostrowski and Jarosław Jędrysiak**

Department of Structural Mechanics, Lodz University of Technology, Poland

*Presenting and corresponding author: piotr.ostrowski@p.lodz.pl

Abstract

An object of our consideration is a two-phase laminate of micro-periodic structure in one of directions. That periodicity is represented by distinguished unit cell of diameter l much smaller when compared with other composite dimensions. In the remaining directions laminate has uniform structure. Each component has isotropic material properties, however theirs values are uncertain from assumption, e.g. ratios of conductivities $\omega = k_2 / k_1$ and specific heats $\chi = c_2\rho_2 / c_1\rho_1$ are random variables of known probabilistic distribution.

By any heat flow, transversally to the laminae, there can be two phenomena observed: jump of the gradient of temperature field on interfaces, and apparent temperature oscillations around averaged temperature, with (local) maximum value on interfaces. That function of temperature oscillations depend strongly on parameters ω and χ, what makes it new random variable, cf. [1]. In this presentation we are going to answer if the maximum of these oscillations is also Gaussian random variable. Steady state as well as transient state will be considered, and all obtained results will be presented.

The govern equation of heat transfer is described by well-known Fourier's law. Easy to see that for considered structure coefficients in the PDE are discontinuous and highly oscillating. Therefore, a tolerance averaging technique (cf. Woźniak et al. [2]) is used in order to get averaged model equations of constant coefficients. These new equations will be used in further numerical (Monte-Carlo) simulations.

Keywords: Heat transfer, laminate, uncertain material properties, tolerance averaging technique

Introduction

The object of our consideration is a multi-layered heat conductor, build of two different materials distributed alternately along $x := x_1$ axis, Fig. 1. Composite layout is l-periodic in such a way that each interval $\Delta \subset \mathbb{R}$ of length $l > 0$ consists of two components, and the first one is of volume ratio $\gamma \in [0,1]$. These components, numbered by $i = 1$ or 2, are called sub-laminae and have isotropic material properties: k_i - conductivity, c_i - specific heat, and ρ_i - density.

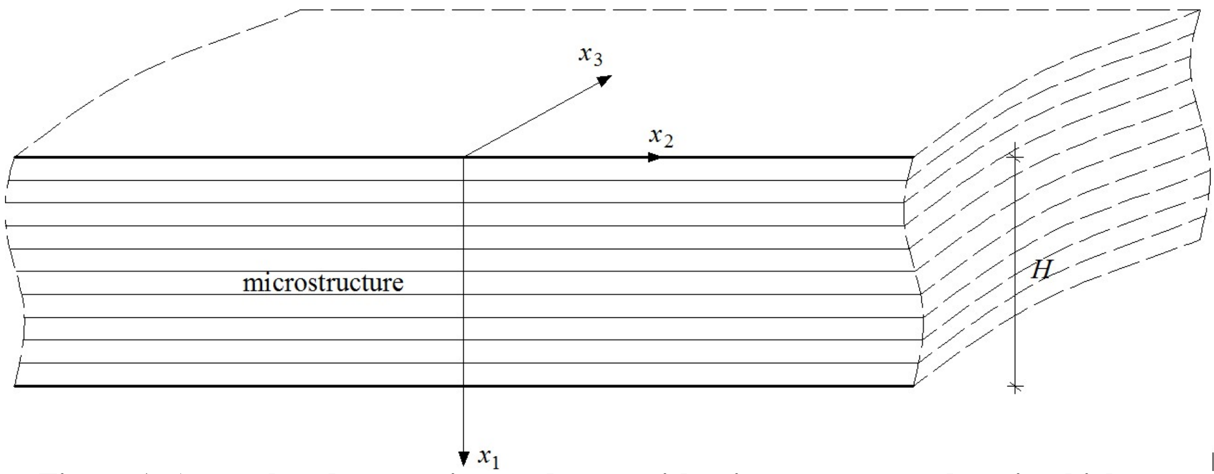

Figure 1. An analysed composite conductor with micro-structure along its thickness

The govern equation of the heat transfer in a conductor under consideration is a partial differential equation (PDE) having, due to micro-structure, highly oscillating and discontinuous coefficients. The exact solution to the steady or transient state of heat transfer is reachable for uniform conductor, but this is not the case. Naturally, one can build a system of PDEs, written for every single sub-layer, satisfying appropriate continuity and boundary conditions, but still, one may find a number of problems in resolvability. Even if we narrow our restrictions to one-directional heat flow, perpendicular to the layers, it would not relax our problem entirely. Therefore, we shall use in this paper the tolerance averaging technique, cf. Woźniak et al. [2]-[3], which leads to the system of differential equations with continuous coefficients and, what is crucial, for special cases (steady state) to the same solution as the exact one.

Number of publications have been devoted to the problem of heat transfer in micro-heterogeneous structures and its modelling, for which differential equations are of discontinuous coefficients. The most popular averaging approaches are based on the asymptotic homogenisation cf. Jikov et al. [4]. For instance, homogenisation theory was realised by Matysiak and Yevtushenko [5] or Matysiak and Perkowski [6] by using a concept of micro-local parameters. To another approaches, but still popular, belong RVE methods. We can mention here for example works of Han et al. [7] or Bayat and Gaitanaros [8].

A new class of problems is recently investigated. Namely the uncertainty effect of physical or geometrical properties on the overall composite behaviour. For instance, probabilistic homogenization with Monte-Carlo simulation in fiber composites is investigated by Kamiński in [9]. Dynamics of micro-periodic composite rod with uncertain parameters under a moving random load is presented by Mazur-Śniady et al. [10]. A refined averaged theory of a rigid heat conductor with a micro-periodic structure is used by Ignaczak and Baczyński [11] to solve a one-dimensional heat conduction problem in a periodically layered plate.

Averaged model equations used in presented paper describe heat transfer in considered composite, and theirs origin we find in tolerance averaging technique. This particular method is commonly used to other then heat conduction problems: e.g. to dynamics of shells [12] and plates [13]-[15], to thermoelasticity [16]-[17].

There are two general goals we would like to present. The first one deals with application of tolerance averaging technique in order to have equations of constant coefficients describing well the heat conduction problem in considered laminate. The second one is to investigate the measure of magnitude of temperature oscillations and how these oscillations depend on uncertainty of material properties.

Preliminaries

Let $\Xi \subset \mathbb{R}^3$ be a bounded region in Euclidean space occupied by the conductor under consideration, wherein Cartesian coordinate system $Ox_1x_2x_3$ is introduced. Denote by $\Omega := (0, H)$, $H > 0$, a bounded and regular region in $\mathbb{R}$ assigned to the micro-structure. Hence, the region of the composite can be expressed as $\Xi = \Omega \times \Pi$, where $\Pi := (0, L) \times (0, B)$, for $B, L > 0$, is a bounded subspace of $\mathbb{R}^2$. In other words, the conductor has for every $x_1 \in [0, H]$ invariant homogeneous structure and material properties, see Fig. 2.

Through this paper points from Ω are denoted by Latin letters $x \equiv x_1$ or $y \equiv y_1$, while from Ξ by $\mathbf{x} = (x_1, x_2, x_3)$. Points from Π are denoted by $\mathbf{z} = (x_2, x_3)$, and the time coordinate by *t*. Gradient operators used in this contribution are $\partial = (\partial_1, 0, 0)$, $\overline{\nabla} = (0, \partial_2, \partial_3)$ and $\nabla = \partial + \overline{\nabla}$, where $\partial_i = \partial / \partial x_i$, $i = 1, 2, 3$, stand for partial derivatives. Dots over the function name stand for the time derivatives.

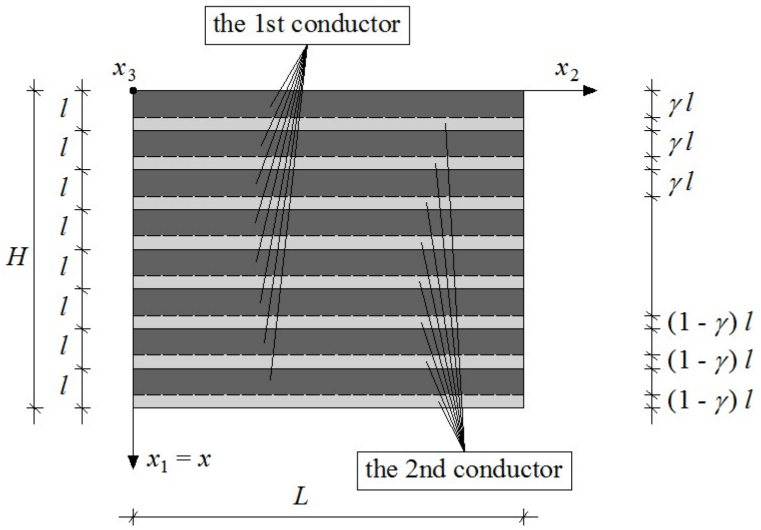

Figure 2. Laminar layout of micro-structure

The heat conduction equation is based in this contribution on the well-known Fourier's law

$$\nabla \cdot (\mathbf{K} \cdot \nabla \Theta) = c\rho\, \dot{\Theta}, \tag{1}$$

where $\mathbf{K} = [k_{ij}]$, $k_{ij} : \Xi \to \mathbb{R}$, stands for the second order conductivity tensor, $c : \Xi \to \mathbb{R}$ is a specific heat and $\rho : \Xi \to \mathbb{R}$ is the density. Since material properties are isotropic we have $k_{ij} = k \cdot \delta_{ij}$ for some $k > 0$, where δ_{ij} is Kronecker's delta.

Equation (1) has to be satisfied for every $\mathbf{x} \in \Xi$ and $t \in (t_0, t_1)$ by continuous function of temperature $\Theta : \Xi \times (t_0, t_1) \to \mathbb{R}$. Easy to see that direct description leads to the system of PDEs with highly oscillating coefficients, and it might be far to complicated to solve it in engineering applications. Even for a unidirectional problem

$$\begin{aligned}
&\partial \cdot (\mathbf{K} \cdot \partial \Theta) = c\rho\, \dot{\Theta}, \\
&\Theta(0, t) = \Theta_1, \quad \Theta(H, t) = \Theta_2 \quad \text{for } t \in (t_0, t_1), \\
&\Theta(x, t_0) = \Theta_0 \quad \text{for } x \in (0, H),
\end{aligned} \tag{2}$$

where the number of independent variables decreases to two, x and t.

Denote the ratios

$$\omega = \frac{k_2}{k_1} \text{ and } \chi = \frac{c_2}{c_1}\frac{\rho_2}{\rho_1} \tag{3}$$

as inhomogeneity parameters. They take only positive values, and equal one for homogeneous material.

In order to give appropriate motivation to our studies, consider a following problem of heat transfer in the laminate build of n two-component layers: find continuous function $\Theta:[0,H]\to\mathbb{R}$ satisfying boundary conditions, $\Theta=\Theta_1$ at $x=0$ and $\Theta=\Theta_2$ at $x=H$. Easy to prove that general solution to Eq. (2) for the case of steady state (time derivative vanishes and the problem is independent of time t) can be decomposed into the sum of averaged temperature

$$\Theta_{avg}(x)=\Theta_1+(\Theta_2-\Theta_1)\cdot\frac{x}{H},\ x\in[0,H], \tag{4}$$

and oscillating temperature Θ_{osc}. The last one is a "saw-type" function, oscillating around zero value, having local extrema on interfaces and depending explicitly on parameter ω, e.g. for a special distribution of sub-laminae we have

$$\sup_{x\in[0,H]}\left|\Theta_{osc}(x)\right|=|\Delta\Theta|\cdot\frac{\gamma\cdot(1-\gamma)}{2n}\cdot\frac{|\omega-1|}{1+\gamma\cdot(\omega-1)}, \tag{5}$$

where $\Delta\Theta=\Theta_2-\Theta_1$. Part of this function, $h(\omega)=\frac{\gamma\cdot(1-\gamma)}{2}\cdot\frac{|\omega-1|}{1+\gamma\cdot(\omega-1)}$, is depicted in Fig. 3 where we can see how strong values of Θ_{osc} depend on ω.

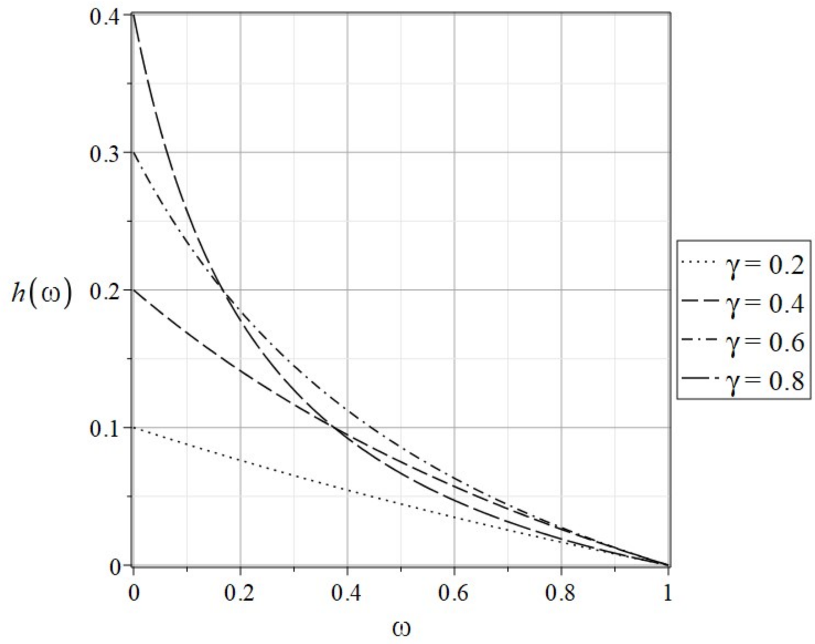

Figure 3. Plot of function $h(\omega)$

One can also observe that limits

$$\lim_{\omega\to 0^+} h(\omega)=\frac{\gamma}{2} \text{ and } \lim_{\omega\to +\infty} h(\omega)=\frac{1-\gamma}{2} \tag{6}$$

exist and are finite.

We can imagine now that the magnitude of Θ_{osc} is sensitive, in transient state as well as in steady state, to differences of material properties. The bigger differences we have, the bigger magnitude of oscillations we get. In other words, function Θ_{osc} on interfaces depends on parameters ω and χ.

In this paper we assume that parameters ω and χ are uncertain, i.e. they are random variables of known probability distribution, lognormal to be precise. Hence, there exist $\mu_\omega, \mu_\chi > 0$ and σ_ω, $\sigma_\chi \ge 0$ such that $\omega \sim \Lambda\left(\mu_\omega, \sigma_\omega^2\right)$ and $\chi \sim \Lambda\left(\mu_\chi, \sigma_\chi^2\right)$. This denotation means that median values of ω, χ are respectively $\exp(\mu_\omega)$, $\exp(\mu_\chi)$, while expected values are $\exp(\mu_\omega + \sigma_\omega^2/2)$, $\exp(\mu_\chi + \sigma_\chi^2/2)$. For the sake of simplicity, initial-boundary conditions to Eq. (1) are imposed in such a way to assure unidirectional (along x_1 axis) heat transfer problem.

Modelling concepts

Throughout this paper, by $H^r(\Xi)$, $r \ge 0$, we shall understand a Sobolev space of functions which are, together with theirs weak derivatives to the *r*th order, L^2-measurable on Ξ. Function space $H^r(\Delta)$ denotes the space of all Δ-periodic functions which are $H^r(V)$ on any arbitrary compact subset $V \subset \mathbb{R}$. Easy to see that H^0 means the same what L^2, however there is no equilibrium between $H^r(\Xi)$ and $H^r(V)$ for $V = \Xi$. All essentially bounded functions on $X \subset \mathbb{R}^m$, $m = 1,2,3$, are denoted by $L^\infty(X)$.

Let $n \in \mathbb{N}$ be the number of two-component layers of common width $l = H/n$. Each layer (cf. Fig. 2) consists of two sub-layers made of different material. The first one, called „conductor 1", is of width $d_1 = \gamma \cdot l$, where $\gamma \in [0,1]$ is fixed. Second sub-layer, called „conductor 2", has therefore width $d_2 = (1-\gamma)\cdot l$. Easy to see that uniform conductor provides $\gamma \in \{0,1\}$.

Fix for a moment $x \in \bar{\Omega}$. Since the composite is periodic, cf. Fig. 2, representative volume element $\Delta = (-l/2, l/2) \subset \mathbb{R}$, called further *a unit cell*, can be simply distinguished. To every cell Δ we can assign a local coordinate system Oy, and the cell with a centre at *x* is denoted by $\Delta(x) = x + \Delta$. Note, that above representation of cell Δ is not the only one, it is only an example. It can be any *l*-length surrounding of 0, i.e. $\Delta = (a, a+l)$ for $a \in [-l/2, 0]$.

In order to derive averaged model equations we apply the tolerance averaging technique, which is based mainly on the concept of tolerance and in-discernibility relation. Its definition is given below.

Definition 1. Let ε stands for an arbitrary positive number. We say that numbers $a,b \in \mathbb{R}$ are in tolerance relation $a \approx b$ if and only if $|a-b| \leq \varepsilon$. Parameter ε is called *the tolerance parameter*.

The general modelling procedures, basic definitions and theorems of this technique can be found in the book by Woźniak and Wierzbicki [2] or by Ostrowski [18]. We will mention here some basic concepts of this technique, but first we introduce a notion of maximum oscillation of continuous function *f* on $\Delta(x)$ as follows

$$[f]_x = \Omega_\Delta \ni x \mapsto \sup_{y\in\Delta(x)} f(y) - \inf_{y\in\Delta(x)} f(y) \in \mathbb{R}, \tag{7}$$

which is very helpful in constructing definition of slowly-varying function.

Definition 2. Function $F \in C^r(\Xi)$ is called *the slowly varying function of* r*th order*, with respect to cell Δ and tolerance parameter ε, if for every $p = 0,1,...,r$ following conditions hold

$$\forall (x,\mathbf{z}) \in \Xi \quad \left[\partial_1^p F(\cdot,\mathbf{z})\right]_x \approx 0. \tag{8}$$

Set of all *r*th order slowly varying functions with respect to the cell Δ and tolerance parameter ε is denoted by $SV_\varepsilon^r(\Xi,\Delta)$.

Another important definition in tolerance modelling is the definition of the mean value operator

$$\langle f \rangle(x,\cdot) := \frac{1}{l} \cdot \int_{\Delta(x)} f(y,\cdot)\, dy. \tag{9}$$

which can be applied to any locally integrable function $f \in L_{loc}^1(\Xi)$. In applications we restrict ourselves to essentially bounded functions, i.e. $f \in L^\infty(\Xi)$.

The last definition related to tolerance averaging technique is a *periodic-like function* [2], whose name was after years changed into *tolerance periodic function*.

Definition 3. Function $f \in L^\infty(\Xi) \cap H^r(\Xi)$ will be called *the tolerance periodic function of* r*th order*, with respect to the cell Δ and tolerance parameter ε, if for every $p = 0,1,...,r$ and every $x \in \Omega_\Delta$ there exists periodic approximation $f_x \in L_{per}^\infty(\Xi) \cap H^r(\Xi)$ of function *f* such that

$$\forall \mathbf{z} \in \Pi \quad \left[\left\langle \partial_1^p f - \partial_1^p f_x \right\rangle(\cdot,\mathbf{z})\right]_x \approx 0. \tag{10}$$

Set of all *r*th order tolerance periodic functions with respect to the cell Δ and tolerance parameter ε will be denoted by $TP_\varepsilon^r(\Xi,\Delta)$.

The space $L_{per}^\infty(\Xi)$ mentioned above is a set of all essentially bounded functions defined on Ξ which are periodic, in particular Δ-periodic. By the notion of $\partial_1^0 f$ we shall understand *f*, and by $O(\cdot)$ Landau's symbol is denoted. The tolerance parameter ε related to any tolerance periodic function can be determined only a posteriori.

Another class of functions possessing special properties is a class of fluctuation shape functions.

Definition 4. Function $g \in C^0(\Xi)$ is called *the fluctuation shape function* of weight $\rho \in L^\infty(\Xi)$, if following conditions hold

(a) $\langle \rho g \rangle \approx 0$ on Ξ,

(b) $\partial_1 g$ is piecewise continuous,

(c) $g \in O(l)$.

Set of all fluctuation shape functions of weight ρ is denoted by $FS_\varepsilon^\rho(\Xi,\Delta)$.

For further simplifications we consider only these functions $g \in FS_\varepsilon^1(\Xi,\Delta)$ which satisfy $\langle g \rangle \equiv 0$. An example of such function is shown in Fig. 4.

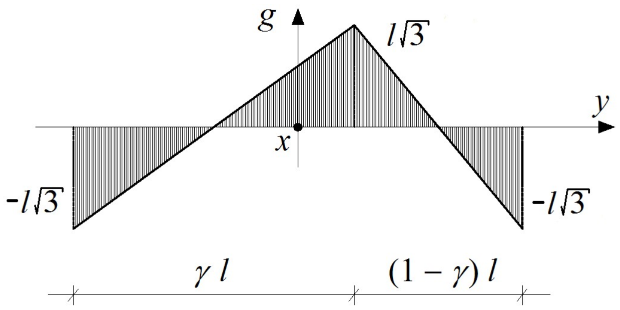

Figure 4. Fluctuation shape function

The last part of this section deals with theorem as a conclusion implied from all presented definitions and theirs properties.

Theorem 1 (Tolerance Averaging Approximations). For every $g \in FS_\varepsilon^1(\Xi,\Delta)$, $F \in SV_\varepsilon^1(\Xi,\Delta)$, $f \in TP_\varepsilon^0(\Xi,\Delta)$, $\varphi \in L^1(\Omega)$ and $k \in L_{per}^\infty(\Xi)$ the following proposition hold

(a) $\nabla\langle\varphi\rangle = \langle\nabla\varphi\rangle$,

(b) $\langle k\,\nabla(gF)\rangle \approx \langle k\,\partial g\rangle F + \langle kg\rangle \bar{\nabla} F$,

(c) $\langle kF\rangle \approx \langle k\rangle F$,

(d) $\langle g\,\nabla(kf)\rangle \approx \bar{\nabla}\langle gkf\rangle - \langle kf\,\partial g\rangle$.

Proof of this theorem is given inter alia by Ostrowski in monograph [18].

Model equations

In this section we derive averaged model equations describing heat conduction in micro-structured laminate. We must assume first that temperature field Θ and its time derivative $\dot{\Theta}$ are tolerance periodic functions, i.e. $\Theta \in TP_\varepsilon^1(\Xi,\Delta)$ and $\dot{\Theta} \in TP_\varepsilon^0(\Xi,\Delta)$. Secondly, by tolerance averaging technique we impose on temperature field *micro-macro decomposition* as follows

$$\Theta(\mathbf{x},t)=\theta(\mathbf{x},t)+g(\mathbf{x})\cdot\psi(\mathbf{x},t) \tag{11}$$

for every $\mathbf{x}\in\bar{\Xi}$ and $t\in[t_0,t_1]$, where $\theta(\cdot,t),\psi(\cdot,t)\in SV_\varepsilon^1(\Xi,\Delta)$. This assumption states that θ and ψ are unknown slowly varying functions for every $t\in[t_0,t_1]$. Function $g\in FS_\varepsilon^1(\Xi,\Delta)$ is *a priori* given, dependent on the micro-structure size parameter *l*, fluctuation shape function as depicted in Fig. 4. Function θ is called *the averaged temperature* in a medium, while ψ stands for *the temperature oscillation amplitude*.

About functions of material properties we assume that each component has isotropic properties, i.e. $k_{ij}=k\cdot\delta_{ij}$ for some $k>0$, where δ_{ij} stands for Kronecker's delta. Moreover, let $k,c,\rho\in L_{per}^\infty(\Xi)$.

It is obvious that (1) may not be satisfied by decomposition (11) everywhere on Ξ and for every $t\in[t_0,t_1]$. Nevertheless, we expect from residuum function

$$\Re=\nabla\cdot(\mathbf{K}\cdot\nabla\Theta)-c\rho\,\dot{\Theta} \tag{12}$$

to satisfy on its domain some orthogonal conditions, namely

$$\langle\Re\rangle=0 \text{ and } \langle\Re g\rangle=0. \tag{13}$$

Bearing in mind all properties from Theorem 1 and by omission all terms $O(\varepsilon)$, $O(l)$, we conclude to the final averaged model equations

$$\begin{aligned}&\nabla\cdot\left(\langle\mathbf{K}\rangle\cdot\nabla\theta+\langle\mathbf{K}\cdot\partial g\rangle\cdot\psi\right)=\langle c\rho\rangle\,\dot{\theta},\\&\bar{\nabla}\cdot\left(\langle\mathbf{K}gg\rangle\cdot\bar{\nabla}\psi\right)-\langle\partial g\cdot\mathbf{K}\cdot\partial g\rangle\cdot\psi-\langle\mathbf{K}\cdot\partial g\rangle\cdot\nabla\theta=\langle c\rho gg\rangle\,\dot{\psi}.\end{aligned} \tag{14}$$

The above system has continuous, for periodic structure even constant, coefficients in contrast to equations from the direct description (1) which has discontinuous and highly oscillating ones. System (14) represents equations for the averaged temperature θ and the temperature fluctuation amplitude ψ, and together with micro-macro decomposition (11) constitutes the tolerance model (TM) of the heat conduction in considered laminated conductor.

Along with micro-macro decomposition (11) came two unknown functions, θ and ψ, instead of one Θ. Thus, we need to formulate somehow, based on the known, conditions for these new functions. Let $\Theta_0(\cdot)=\Theta(\cdot,t_0)$ on Ξ be the initial temperature, while $\Theta_1(\cdot)=\Theta(0,\cdot)$ and $\Theta_2(\cdot)=\Theta(H,\cdot)$ be on $\bar{\Pi}\times[t_0,t_1]$ the temperature on the top and bottom surface, respectively. On the remaining boundary surfaces we assume that they are subjected to thermal isolation. Conditions for averaged temperature and temperature oscillation amplitude we can evaluate as

$$\theta_i=\langle\Theta_i\rangle \text{ and } \psi_i=\frac{\langle\Theta_i g\rangle}{\langle gg\rangle}, \tag{15}$$

for every $i=0,1,2$. Easy to see that $\theta_i(\cdot,t)$ and $\psi_i(\cdot,t)$ are for every $t\in[t_0,t_1]$ constant functions if $\Theta_i(\cdot,t)$ is *l*-periodic. In particular, $\psi_i(\cdot,t)\equiv 0$ iff Θ_i is constant in *x*.

Suppose now Θ_0, Θ_1 and Θ_2 are constant functions in theirs domain and material properties are isotropic for each of component, i.e. k_1, c_1, ρ_1 for the first phase and k_2, c_2, ρ_2 for the second phase. The rest of boundary surfaces are thermally isolated. So formulated Cauchy's problem assures unidirectional heat flow in a media (along *x*-axis) and the tolerance model equations (14) reduce to a simpler form

$$\begin{aligned} &\langle k \rangle \cdot \partial_1^2 \theta + \langle k\, \partial_1 g \rangle \cdot \partial_1 \psi - \langle c\rho \rangle\, \dot{\theta} = 0, \\ &\langle k\, \partial_1 g \rangle \cdot \partial_1 \theta + \langle k\, \partial_1 g\, \partial_1 g \rangle \cdot \psi + \langle c\rho g g \rangle\, \dot{\psi} = 0. \end{aligned} \tag{16}$$

and the initial-boundary conditions

$$\theta_i = \Theta_i \text{ and } \psi_i = 0, \tag{17}$$

for every $i = 0,1,2$. Averaged coefficients

$$\begin{aligned} &\langle k \rangle = k_1 \cdot \left(\omega + (1-\omega)\cdot\gamma\right), && \langle c\rho \rangle = c_1\rho_1 \cdot \left(\chi + (1-\chi)\cdot\gamma\right), \\ &\langle k\, \partial_1 g \rangle = k_1\sqrt{12}\cdot(1-\omega), && \langle c\rho g g \rangle = l^2 \langle c\rho \rangle, \\ &\langle k\, \partial_1 g\, \partial_1 g \rangle = \frac{12 k_1}{\gamma\cdot(1-\gamma)}\cdot\left(1 + (\omega - 1)\cdot\gamma\right), \end{aligned} \tag{18}$$

are in this case constant and depend explicitly on parameters ω or χ, cf. (18), that play the role of random variables. If $\omega = \chi = 1$ then we deal with uniform conductor. For further applications we introduce the dimensionless spatial $\xi = x / H$ and time $\tau = t / 3600[s]$ coordinates.

If we neglect time derivatives in (16) and make all functions as time independent, then we obtain description to the steady state of heat conduction. Temperature oscillation amplitude ψ depends then explicitly on averaged temperature

$$\psi = -\frac{\langle k\, \partial_1 g \rangle}{\langle k\, \partial_1 g\, \partial_1 g \rangle}\cdot \partial_1 \theta, \tag{19}$$

while θ must satisfy

$$\partial_1^2 \theta = 0 \tag{20}$$

under already known boundary conditions. What is most interesting, the obtained solution satisfies all continuity conditions, including heat flux across interfaces. Ergo, by tolerance averaging technique we get the exact solution, without solving large system of equations, and without solving any eigenvalue problem.

Monte-Carlo simulation

The exact solution for the direct description of the heat transfer problem (1) in micro-periodic laminate exists, however it usually needs complex algebraic calculations. For example, by n cells in our two-phase laminate we have $2n-1$ interfaces where continuity of temperature and heat flux field should be assured. That makes $4n-2$ equations plus two boundary conditions. By the use of tolerance averaging technique we obtain system of PDEs (or ODEs) but of constant coefficients, wherein the number of equations depends on the number of terms in micro-macro decomposition (11). But that is much smaller then $4n$, usually it is only two.

Table 1. Reference material properties

	Component 1 (steel)	Component 2 (aluminium)
$k\left[W\,m^{-1}K^{-1}\right]$	58	200
$c\left[J\,kg^{-1}K^{-1}\right]$	500	920
$\rho\left[kg\,m^{-3}\right]$	7800	2700

Example 1. Let us consider periodic laminate of thickness $H = 1[m]$, consisting of $n = 20$ two-component layers. Hence, the thickness of a single layer is $l = 5[cm]$. Volume fraction of the first component is fixed at $\gamma = 0.25$. Let $\Theta_0 = 0[°C]$ for $t_0 = 0[s]$ and $\Theta_1 = 1[°C]$, $\Theta_2 = 0[°C]$ for $x = 0[m]$ and $x = H$, respectively. These imply $\theta_0 = 0[°C]$ for $\tau_0 = 0$ and $\theta_1 = 1[°C]$, $\theta_2 = 0[°C]$ for $\xi = 0$ and $\xi = 1$, respectively. Material properties are given in Tab. 1, and thus parameters $\omega = 3.448$ and $\chi = 0.637$ are fixed. The considered time range for this example is one hour, $t_1 = 3600[s]$, and it provides range of $[0,1]$ for τ.

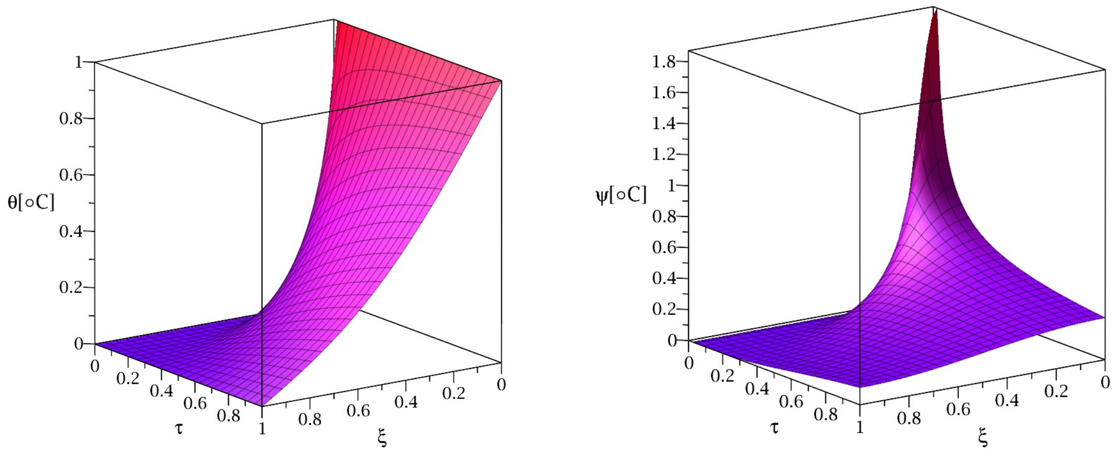

Figure 5. The averaged and the temperature oscillation amplitude varying in time during one hour for fixed material properties and $\gamma = 0.25$

System (16) were solved under assumption that time derivative of ψ can be omitted (asymptotic model, cf. [1]), i.e. instead of $(16)_2$ we have (19). Fig. 5 depicts changes in time of averaged temperature θ and temperature oscillation amplitude ψ. There is apparent convergence to the steady state, and the maximum oscillation amplitude appears close to top surface ($\xi = 0$).

These results were prepared only for certain values of material properties. Suppose now that k_1, c_1, ρ_1 are fixed while ω and $\chi = \omega - 2.811$ are uncertain parameters. To be precise, we will investigate the impact of randomness of parameter ω on the magnitude of temperature oscillations

$$\Psi(x,t) = \sup_{y\in\Delta(x)} \left|g(y)\cdot\psi(y,t)\right|,\ x\in\Omega_\Delta,\ t\in[t_0,t_1]. \qquad (21)$$

All following simulations are restricted to the case of fixed spatial coordinate $\xi = 1/n$, but the whole analysis could be simply transferred for any value of ξ.

In the next example we expect to find and investigate an effect of variables ω and $\chi = \omega - 2.811$ on the function $\Psi = \Psi(\omega, \xi, \tau)$, under given above geometry and initial-boundary conditions, but for various values of γ.

Example 2. We postulate that $\Psi(\omega, \xi, \tau)$, given by Eq. (21), is for $\xi = 1/n$ and every $\tau \in [0,1]$ a new random variable with unknown probability distribution. To variable ω, we say that it is of lognormal distribution with parameters: mean value $\mu = \mu_\omega$ and standard deviation $\sigma = \sigma_\omega$. Moreover, we assume that $\sigma = \nu \cdot \mu$ for $\mu = 3.448$ and $\nu = \nu(\omega) > 0$. All statistical characteristics, like expected value $E(\Psi)$, standard deviation $\sigma(\Psi)$, skewness $\beta(\Psi)$, kurtosis $\kappa(\Psi)$ and coefficient of variation $\nu(\Psi)$, will be determined in order to qualify Ψ to Gaussian distributed variable. Subsequent numerical experiment is based on the Monte Carlo simulation for $N = 1000$ probe values.

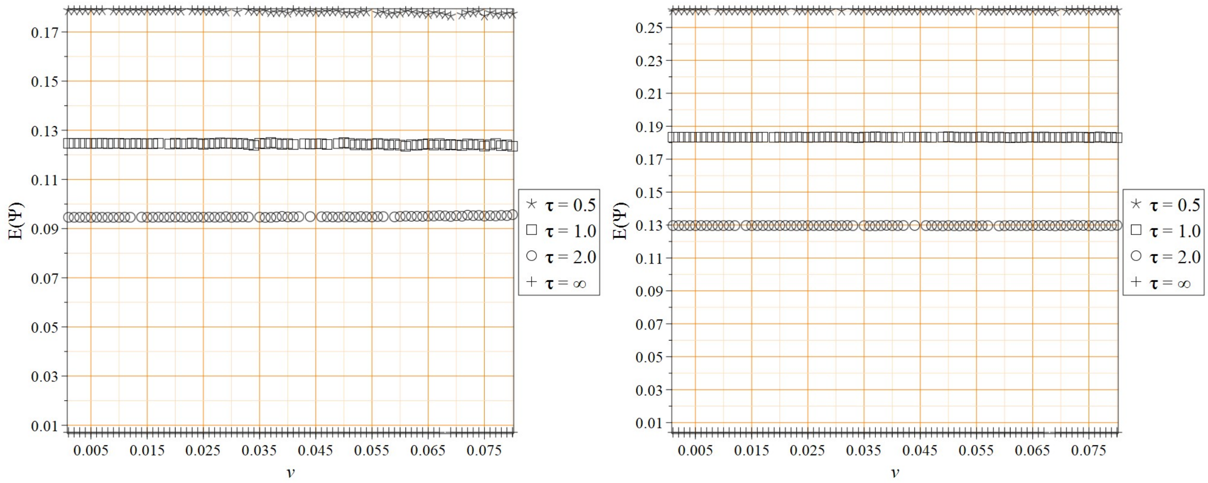

Figure 6. The expected values of Ψ against ν varying in time for $\gamma = 0.25$ and $\gamma = 0.75$

In Fig. 6 we see how the magnitude of temperature oscillation vary in time, and theirs values are not necessary negligibly small when compared with total temperature. It seems that they are not affected with parameter ν.

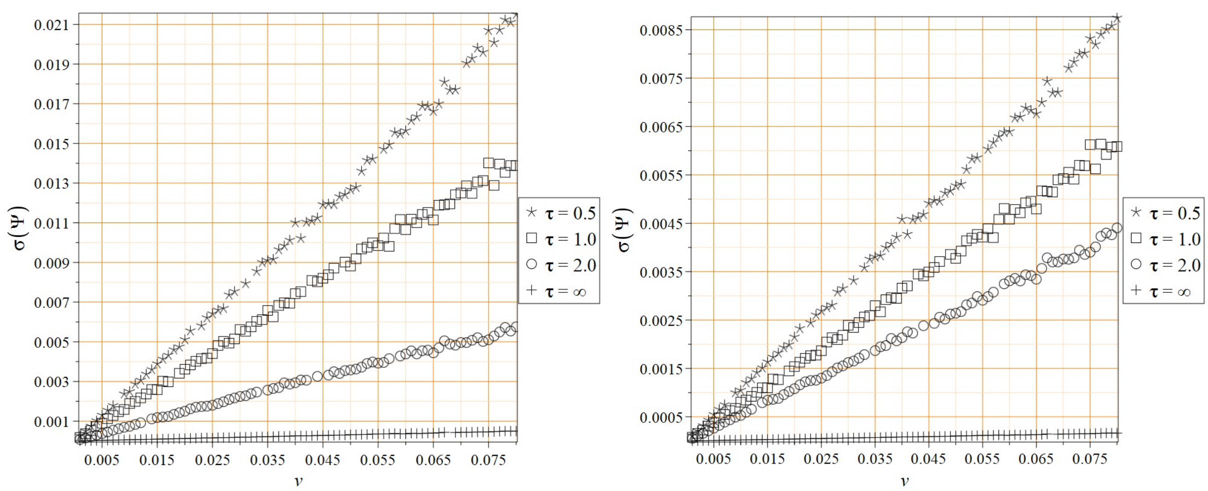

Figure 7. The standard variations of Ψ against ν varying in time for $\gamma = 0.25$ and $\gamma = 0.75$

Standard deviations depend on parameter ν for sure, but that dependence is almost linear. The highest standard deviations we get for first moment of time, then they drop down.

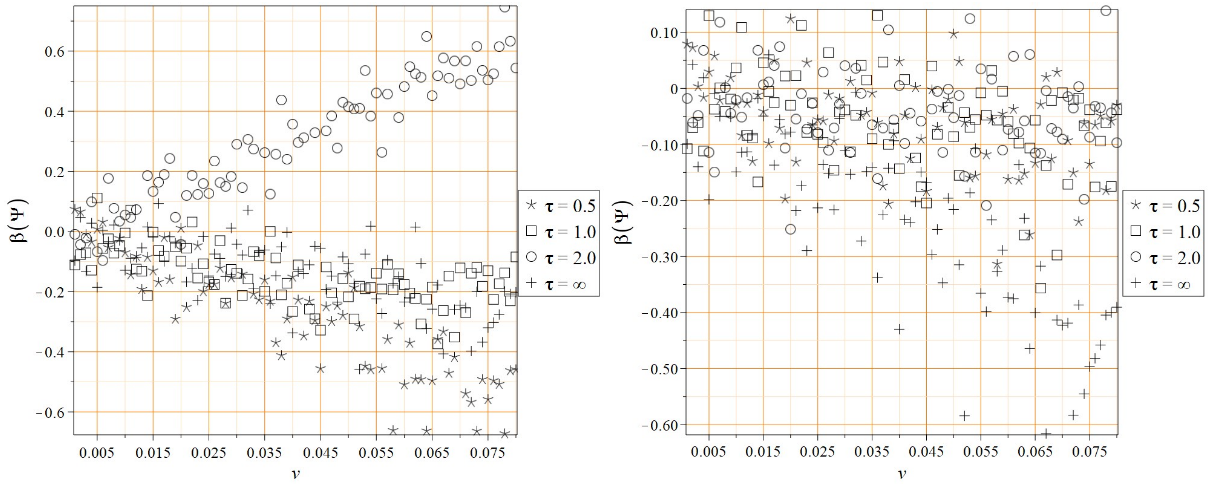

Figure 8. The skewness's of Ψ against ν varying in time for $\gamma = 0.25$ and $\gamma = 0.75$

Skewness is first statistical parameter that says a lot about character of randomness. If its value is close to zero then we may suspect that this is Gaussian distribution. In Fig. 8 we can see that there is no typical pattern for skewness's but theirs values are sufficiently small.

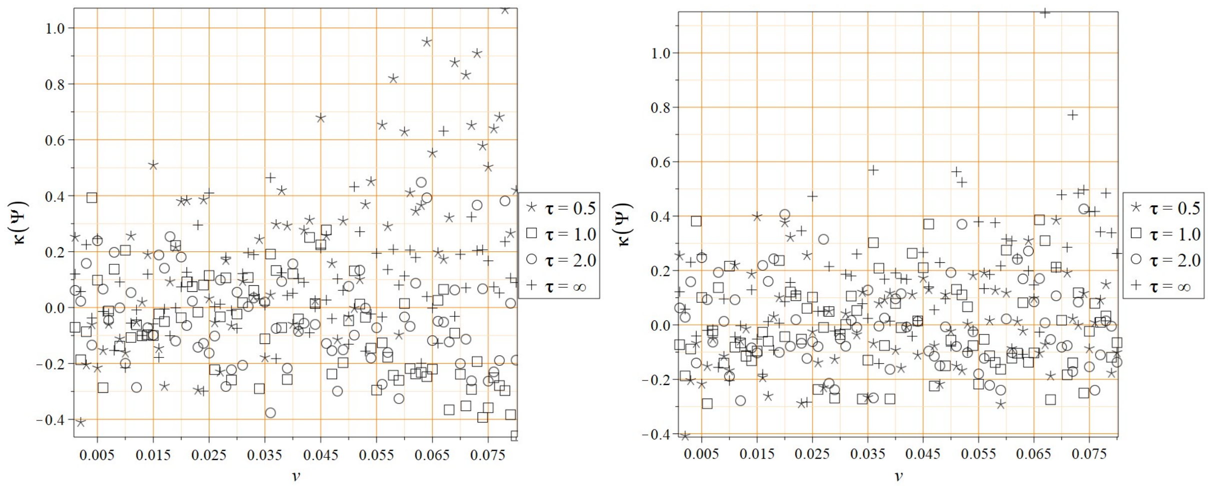

Figure 9. The kurtoses of Ψ against ν varying in time for $\gamma = 0.25$ and $\gamma = 0.75$

The second statistical parameter is kurtosis, Fig. 9, determining whether Ψ is Gaussian variable. The closer value to zero the better alignment we get. As well as skewness's, results are kind of chaotic but also sufficiently small.

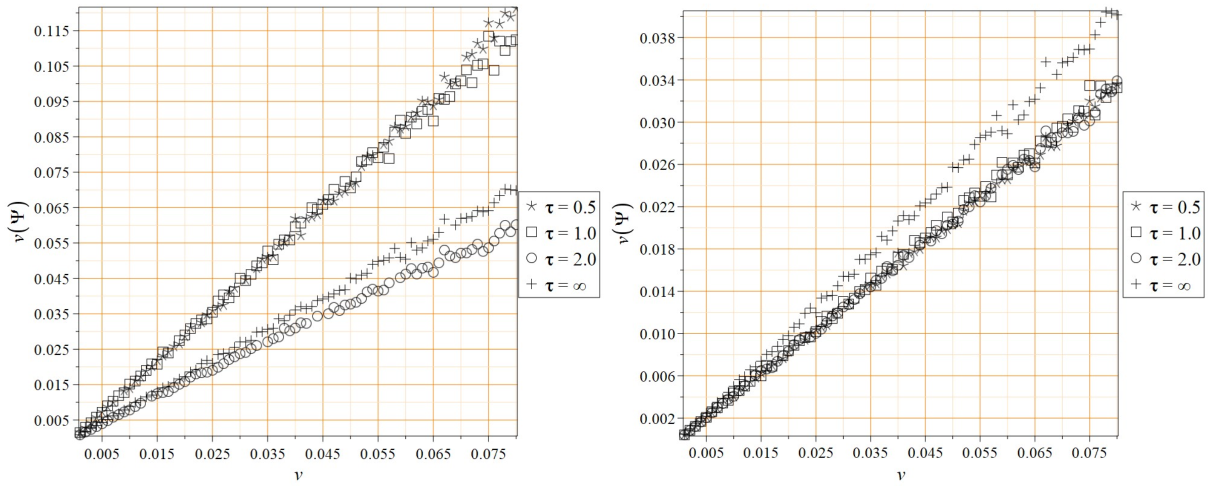

Figure 10. The coefficients of variation of Ψ against ν varying in time for $\gamma = 0.25$ and $\gamma = 0.75$

The most interesting parameter is the ratio of standard deviation and expected value, called coefficient of variation, Fig. 10, because it says a lot about relative dispersion of expected values.

Summary

We assumed that ratio of conductivities ω (ratio of specific heats χ depends explicitly on ω) is a random variable of lognormal distribution. That distribution was considered in many variations, i.e. for different values of ν. Its change affects naturally the function of magnitude of temperature oscillation Ψ, which is also a random variable. Statistical characteristics were calculated and Shapiro-Wilk's test for normality was made for each case. Only those results, for which the test gave positive answer, were presented in Figs 6-10. Plots of coefficient of variation reveal interesting conclusion: estimation of Ψ might be of higher probability then made for ratio ω. But this is only from specific moment of time: $\nu(\Psi) \leq \nu$ for $\tau > 2$ regardless from γ.

References

[1] Ostrowski, P. and Jędrysiak, J. (2017) Heat conduction in periodic laminates with probabilistic distribution of material properties, *Heat and Mass Transfer* **53**(4), 1425–1437.

[2] Woźniak, Cz and Wierzbicki, E. (2000) *Averaging techniques in thermomechanics of composite solids*, Wydawnictwo Politechniki Częstochowskiej, Częstochowa, Poland.

[3] Woźniak, Cz. et al. [Eds] (2008) *Thermomechanics of microheterogeneous solids and structures. Tolerance averaging approach*, Lodz University of Technology Press, Łódź, Poland.

[4] Jikov, V. V, Kozlov, S. M. and Oleinik, O. A. (1994) *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, Heidelberg.

[5] Matysiak, S. J. and Yevtushenko, A. A. (2008) Influence of microstructures on temperature distribution for some mixed problem of composite half-space, *International Communications in Heat and Mass Transfer* **35**, 704–709.

[6] Matysiak, S. J. and Perkowski, D. M. (2014) Temperature distributions in a periodically stratified layer with slant lamination, *Heat and Mass Transfer* **50**(1), 75–83.

[7] Han, X., Xu, Ch., Xie, W. and Meng, S. (2019) Multiscale computational homogenization of woven composites from microscale to mesoscale using data-driven self-consistent clustering analysis, *Composite Structures* **220**, 760–768.

[8] Bayat, A. and Gaitanros, S. (2019) Elastic wave propagation in open-cell foams, *Journal of Applied Mechanics* **86**(5), 051008 (11 pages).

[9] Kamiński, M. (1999) Monte-Carlo simulation of effective conductivity for fiber composites, *International Communications in Heat and Mass Transfer* **26**(6), 791–800.

[10] Mazur-Śniady, K., Śniady, P. and Zielichowski-Haber, W. (2009) Dynamic response of micro-periodic composite rods with uncertain parameters under moving random load, *Journal of Sound and Vibrations* **320**(1-2), 273–288.
[11] Ignaczak, J. and Baczyński, Z. F. (1997) On a refined heat conduction theory for microperiodic layered solids, *Journal of Thermal Stresses* **20**(7), 749–771.
[12] Tomczyk, B. and Szczerba, P. (2018) A new asymptotic-tolerance model of dynamic and stability problems for longitudinally graded cylindrical shells, *Composite Structures* **202**, 473–481.
[13] Marczak, J. and Jędrysiak, J. (2018) Some remarks on modelling of vibrations of periodic sandwich structures with inert core, *Composite Structures* **202**, 752–758.
[14] Wirowski, A., Michalak, B. and Gajdzicki, M. (2015) Dynamic modelling of annular plates of functionally graded structure resting on elastic heterogeneous foundation with two modules, *Journal of Mechanics* **31**(5), 493–504.
[15] Świątek, Ma., Domagalski, Ł. and Jędrysiak, J. (2019) Free vibrations spectrum of periodically inhomogeneous Rayleigh beams using the tolerance averaging technique, *Journal of Theoretical and Applied Mechanics* **57**(1), 141–154.
[16] Pazera, E. and Jędrysiak, J. (2018) Effect of microstructure in thermoelasticity problems of functionally graded laminates, *Composite Structures* **202**, 296–303.
[17] Ostrowski, P., Thermoelasticity in a two-phase hollow cylinder with longitudinally graded material properties, *Shell Structures: Theory and Applications, Vol. 3*, Gdańsk, Poland, 2014, Pietraszkiewicz, W. and Górski, J., Eds, 133–136, CRC Press Taylor & Francis Group.
[18] Ostrowski, P. (2017) *Tolerance modelling of thermomechanics in microstructured media*, Lodz University of Technology Press, Łódź, Poland.