Influence of Surface and Couple Stresses on Response of Surface-loaded Elastic Half-plane

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Abstract

A mathematical model integrating both Gurtin-Murdoch surface elasticity and consistent couple stress theories is proposed to simulate the simultaneous effects of the surface energy and the material microscopic structure on the mechanical response of an elastic half-plane under arbitrarily surface normal loadings. The displacement-based governing equations for the bulk material and the top material layer are established and then solved, via the method of Fourier integral transform together with the prescribed boundary conditions, to obtain the closed form solution of the elastic field in the transform space. To obtain solutions in the physical space, an efficient quadrature is adopted to evaluate all involved integrals associated with the Fourier transform inversion. A selected set of results is reported and they have indicated that both the surface and couple stresses significantly influence the elastic field within the bulk when the size of the loading region is comparable to the internal length scales of the surface material and the bulk.

Keywords: Elastic half-plane, Surface stresses, Couple stresses, Surface elasticity, Couple stress elasticity.

Introduction

In past several decades, micro- and nano-technologies have received increasingly growing attention due to their vast applications in various disciplines. In the field of material sciences and engineering, understanding the fundamental characteristics and mechanical behavior of materials at those tiny scales is considered essential in the design procedure and fabrication of micro- and nano-scale devices and systems such as MEMS and NEMS. Unfortunately, many difficulties and challenges arise in the study of small-size objects since the response at those scales is significantly complex, generally size-dependent, and mostly influenced by various actions such as the surface free energy [1]-[3], existing defects and flaws, and material microscopic structures [4]. Experimental-based approaches are ones of the most popular candidates extensively and successfully employed to investigate the physical phenomena in a tiny scale (e.g., [5]-[8]). While results gained from those approaches have been found reliable and closely resembling the actual response, tests can only be performed in fully equipped laboratories and significant amount of resources associated with sophisticated testing setups and procedures and preparations of specimens must also be paid. This therefore renders purely experimental-based approaches less cost efficient in comparison with those combined with theoretical-based simulations. The latter, once properly equipped with physically admissible and sufficiently validated mathematical models, can be used not only to obtain the first-order prediction of the actual phenomena but also to assist the reduction of the number of cases to be considered in the experiments.

It has been well recognized that classical size-independent theories in continuum mechanics adopted specifically for simulating mechanical response of macro-scale problems have failed to simulate situations when the external length scale (e.g., size of loading region, crack length, contact length, etc.) is comparable to the internal length scale of materials (e.g., granular distance, lattice parameters). Attempts have been devoted to modify/enhance existing continuum-based mathematical models by integrating the influences observed in a small-scale before used in the simulations. Several continuum-based models have been proposed and successfully employed to capture the size-dependent behavior due to the presence of both surface-free energy and microstructures of constituting materials. For instance, the model established by Gurtin and his colleagues (e.g., [9][10]), called the theory of surface elasticity, has been successfully utilized to capture the surface stress effects. Due to its mathematical simplicity and capability in handling small-scale influence, the Gurtin-Murdoch model has become popular and extensively applied to investigate various problems in mechanics, e.g., thin films [11][12], thin plates [13], dislocations [14], nano-scaled elastic layer [15]-[17], half-space [18][19], and layered elastic half-space [20]. To handle the influence of material microstructures, various theories have been considered including the Cosserat theory [21], the couple stress theory [22]-[24], the strain gradient elasticity theory [25]-[27], the modified couple stress theory [28], and the consistent couple stress theory [29]. During the past decades, these theories have been extensively employed in the simulations and modeling of nano/micro-structured systems, especially for small-scaled beams and plates [30]-[32] and the size-dependent contact problems of elastic solids [33]-[35].

Towards the modeling of micro-/nano-scale layered media, results from an extensive literature survey have indicated that most of existing studies considered separately either the effect of the surface-free energy or the influence of the microscopic structures of constituting materials (e.g., [13]-[20], [33]-[38]). Applications of both Gurtin-Murdoch surface elasticity and the couple stress theory to simultaneously handling those small-scale influences, especially within the framework of surface and contact mechanics, have not been well recognized and this, as a consequence, leaves a significant gap of knowledge for further investigations. The idea of integrating both the surface-free energy and the micro/nano-structure of the bulk material in the modeling has been found in the study of size-dependent responses of nano-scale structures such as nano-beams (e.g., [39]) and nano-plates (e.g., [40]-[42]). This set of investigations not only confirms the applicability of the two theories but also provides the useful basis and ingredients essential for the extension to treat nano-scale problems of interest.

The present study aims to investigate the mechanical response of an elastic half-plane loaded on its surface by taking the influence of both surface and couple stresses into account. Gurtin-Murdoch surface elasticity theory is employed together with the consistent couple stress theory to form the underlying mathematical model and the analytical solution of elastic fields is obtained via the method of Fourier integral transform and a selected efficient numerical quadrature. The complete elastic fields under the simultaneous effects of surface stresses and couple stresses within the half-plane are thoroughly studied.

Problem Formulation

Consider a linearly elastic half-plane loaded by an arbitrarily distributed normal traction p over the length $2a$ on the top surface as shown schematically in Fig. 1. For convenience, a reference Cartesian coordinate system $\{x, y, z; O\}$ is chosen such that the origin is at the center of the loading region and the x, y, and z-axes direct rightward, downward, and normal to the half-plane, respectively. The bulk material is assumed homogeneous and its response is described by the consistent couple stress theory (e.g., [29]) whereas the material layer at the top surface of the half-plane is governed by the theory of surface elasticity (e.g., [9][10]). In the present study, it is assumed that the body force and the body couple are negligible and the plane-strain deformation prevails.

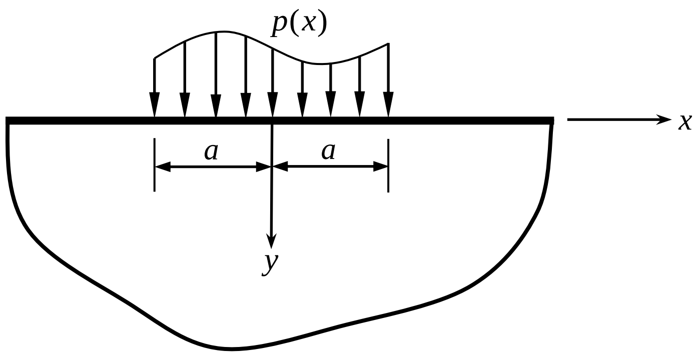

Figure 1. Schematic of an elastic half-plane loaded on its surface by normal traction

For the bulk medium, basic field equations (i.e., equilibrium equations, constitutive laws, and kinematics) from the consistent couple stress theory (e.g., [29]) when specialized to the two-dimensional body under the plane strain condition are given by

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0, \quad \frac{\partial \mu_{xz}}{\partial x} + \frac{\partial \mu_{yz}}{\partial y} + \sigma_{xy} - \sigma_{yx} = 0 \tag{1}$$

$$\begin{aligned} &\sigma_{xx} = \frac{2\mu}{1-2v}\left[(1-v)\varepsilon_{xx} + v\varepsilon_{yy}\right], \ \sigma_{yy} = \frac{2\mu}{1-2v}\left[v\varepsilon_{xx} + (1-v)\varepsilon_{yy}\right], \\ &\sigma_{xy} = 2\mu\varepsilon_{xy} - 2\mu\ell^2\Delta\omega, \ \sigma_{yx} = 2\mu\varepsilon_{xy} + 2\mu\ell^2\Delta\omega, \\ &\mu_{xz} = -8\mu\ell^2\kappa_{xz}, \ \mu_{yz} = -8\mu\ell^2\kappa_{yz} \end{aligned} \tag{2}$$

$$\begin{aligned} &\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \ \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \ \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \\ &\omega = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right), \ \kappa_{xz} = -\frac{1}{2}\frac{\partial \omega}{\partial x}, \ \kappa_{yz} = -\frac{1}{2}\frac{\partial \omega}{\partial y} \end{aligned} \tag{3}$$

where $\{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{yx}\}$ are the force-stress components; $\{\mu_{xz}, \mu_{yz}\}$ are the couple-stress components; $\{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}\}$ and ω are the infinitesimal strain components and the rotation about the z-axis, respectively; $\{\kappa_{xz}, \kappa_{yz}\}$ represent the curvature components; $\{u_x, u_y\}$ are the in-plane displacement components; μ and v are the elastic shear modulus and Poisson's ratio of the bulk material, respectively; ℓ is a length-scale parameter in the couple stress elasticity; and Δ denotes two-dimensional Laplacian operator. It is worth noting that the presence of the couple stresses renders the force-stress tensor non-symmetric and this is in contrast to the classical linear elasticity. Note also that by setting the parameter $\ell = 0$, one can readily recover the classical case.

For the material layer at the top surface of the half-plane, basic equations governing its response can be established from Gurtin-Murdoch surface elasticity theory [9][10] and, when specialized to this particular case, they are given by

$$\frac{\partial \sigma^s{}_{xx}}{\partial x} + t_x^s = 0, \; \frac{\partial \sigma^s{}_{xy}}{\partial x} + t_y^s + p = 0 \tag{4}$$

$$\sigma^s{}_{xx} = \tau^s + \kappa^s \varepsilon_{xx}^s, \; \sigma^s{}_{xy} = \tau^s \frac{\partial u_y^s}{\partial x} \tag{5}$$

$$\varepsilon_{xx}^s = \frac{\partial u_x^s}{\partial x} \tag{6}$$

where the superscript 's' is utilized to designate the surface quantities; $\{\mu^s, \lambda^s\}$ and τ^s denote surface Lamé constants and the residual surface tension, respectively; $\kappa^s = 2\mu^s + \lambda^s$ denote the surface elastic modulus; and $\{t_x^s, t_y^s\}$ are tractions acting to the bottom side of the material layer induced by the underlying bulk material.

Solution Procedure

A set of displacement-based governing equations for the bulk material can be readily obtained by properly combining all basic field equations, Eq. (1)-(3), and its general solution can be established in a closed-form via the method of Fourier integral transform (e.g., [43]). The final expression for the in-plane displacements $\{u_x, u_y\}$ is given by

$$u_x(x,y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[Ai|\xi| e^{-|\xi| y} - Bi(\kappa - |\xi| y) e^{-|\xi| y} + Ci\zeta e^{\frac{-\zeta}{\ell} y} \right] e^{-i\xi x} d\xi \tag{7}$$

$$u_y(x,y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[A\xi e^{-|\xi| y} + B\xi y e^{-|\xi| y} + C\ell\xi e^{\frac{-\zeta}{\ell} y} \right] e^{-i\xi x} d\xi \tag{8}$$

where $i = \sqrt{-1}$ denotes an imaginary number; ξ is a transform parameter; $\zeta = \sqrt{1+\xi^2\ell^2}$; $\kappa = 3-4\nu$; and A, B, C are arbitrary unknown functions of ξ to be determined from boundary conditions. The general solution for the rotation ω, the force stresses $\{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{yx}\}$, and the couple stresses $\{\mu_{xz}, \mu_{yz}\}$ can be obtained from Eq. (7)-(8) together with Eq. (2)-(3).

To form sufficient conditions for determining the unknown functions A, B, C, the continuity of the displacements and tractions at the interface of the bulk and the material layer is enforced along with the surface equations, Eq. (4)-(6). For the case of a constant residual surface tension, the following set of three boundary conditions is obtained:

$$\tau^s \left. \frac{\partial^2 u_y}{\partial x^2} \right|_{y=0} + \left. \sigma_{yy} \right|_{y=0} + p(x) = 0 \tag{9}$$

$$\kappa^s \left.\frac{\partial^2 u_x}{\partial x^2}\right|_{y=0} + \left.\sigma_{yx}\right|_{y=0} = 0 \tag{10}$$

$$\left.\mu_{yz}\right|_{y=0} = 0 \tag{11}$$

It is worth noting that the couple traction boundary condition Eq. (11) results directly from that the material layer cannot resist the bending moment. By substituting Eq. (7)-(8) and the general solution for the force and couple stresses into Eq. (9)-(11), it gives rise to a system of three linear algebraic equations:

$$\begin{bmatrix} \dfrac{\tau^s \xi^3}{\mu} + 2\xi|\xi| & 2\xi(2v-1) & \dfrac{\tau^s \ell \xi^3}{\mu} + 2\xi\zeta \\ i\xi^2\left(\Lambda|\xi|+2\right) & -i\left[\Lambda\xi^2(3-4v)+4|\xi|(1-v)\right] & i\xi^2\left(\Lambda\zeta+2\ell\right) \\ 0 & 8i\ell^2\xi^2(v-1) & 2i\zeta \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \frac{1}{\mu} \begin{Bmatrix} \hat{p}(\xi) \\ 0 \\ 0 \end{Bmatrix} \tag{12}$$

where $\Lambda = \kappa^s / \mu$ is defined as the material length-scale parameter corresponding to the presence of the surface stresses and $\hat{p}(\xi)$ is the Fourier transform of the prescribed normal traction $p(x)$ given by

$$\hat{p}(\xi) = \int_{-\infty}^{+\infty} p(x) e^{i\xi x} dx \tag{13}$$

By solving the system of linear equations, Eq. (12), the unknown functions *A*, *B*, *C* are obtained in a closed-form as functions of ξ, two length-scale parameters Λ and ℓ, and transformed traction $\hat{p}(\xi)$. Substituting these functions into the general solutions gives the integral expressions for the displacements, rotation, force-stress and couple-stress fields at any location within the half-plane. Interestingly, it can be observed that all elastic field quantities contain both Λ and ℓ representing the length-scale parameters corresponding to the presence of the surface stresses and couple stresses, respectively. Therefore, it is anticipated that the influence of simultaneous effects of the surface and couple stresses can be captured in predicted solutions as well as the size-dependent behavior.

Numerical Results and Discussion

To obtain numerical results for elastic field within the bulk material, standard Gaussian quadrature is adopted to efficiently evaluate all involved integrals resulting from Fourier integral inversion. In the numerical study, following material properties associated with Silicon [44] (i.e., $v =$ 0.33, $\mu = 40.23$ GPa, $\lambda = 78.08$ GPa, $\mu^s = 2.78$ N/m, $\lambda^s = 4.49$ N/m, $\tau^s = 0.61$ N/m) and the length-scale parameter associated with the presence of couple stresses $\ell = 50$ nm [45] are chosen. A representative surface load, corresponding to a uniformly distributed normal pressure $p(x) = p_0$ over the length $2a$, is chosen and its Fourier transform is given explicitly by

$$\hat{p}(\xi) = \frac{2\sin(a\xi)}{\xi} p_0 \tag{14}$$

Numerical results for the force stresses and couple stresses of an elastic half-plane under the uniformly distributed normal pressure $p(x) = p_0$ and the simultaneous influence of both surface and couple stresses are reported in Fig. 2 and Fig. 3, respectively. The stress distributions along the positive *x*-direction of the surface-loaded half-plane are considered at different normalized depths, $y/a \in \{0.2; 0.4; 0.8\}$ with $a = 0.5\Lambda$ and $\Lambda = 0.25$ nm. Results represented by the dash lines denote the classical solutions corresponding to $\ell = \Lambda = 0$. It is worth noting that while the classical solutions are independent of the length scale Λ, the use of Λ in the normalization is only for the purpose of comparison.

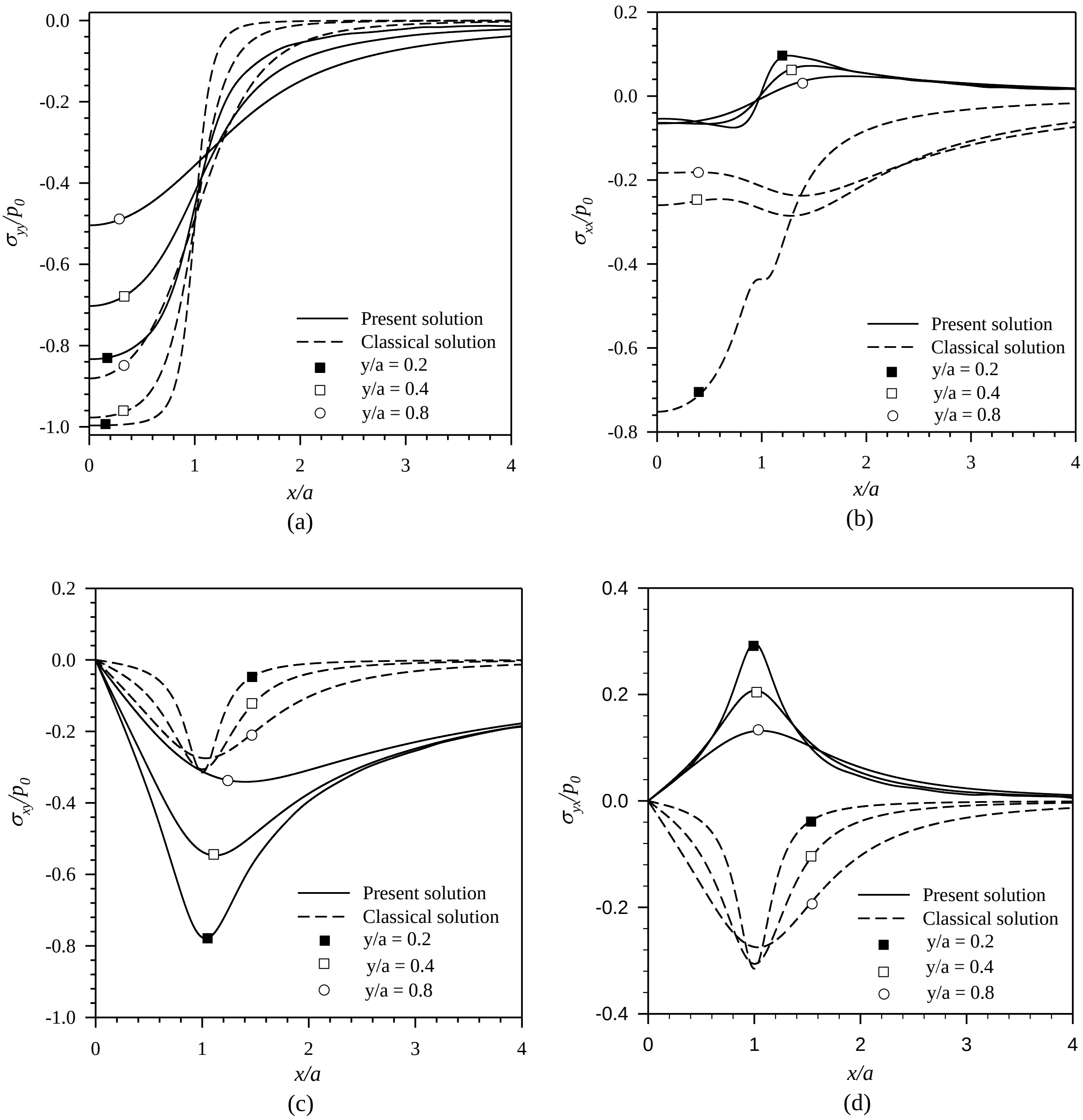

Figure 2. Normalized force stresses of elastic half-plane under uniformly distributed normal traction at various depths

Although the normalized vertical stress σ_{yy}/p_0 and the shear stress σ_{xy}/p_0 from the proposed model and the classical solution display similar trends for all values of the normalized coordinate x/a as displayed in Fig. 2(a) and Fig. 2(c), the magnitude of the normalized stresses σ_{yy}/p_0 and σ_{xy}/p_0 with the influence of the surface and couple stresses are lower and higher, respectively, than those of the classical case. In contrast, the normalized horizontal stress σ_{xx}/p_0 and the shear stress σ_{yx}/p_0 possess different characteristic in comparison with the classical solutions. In addition, the non-symmetric character of the force-stress tensor is confirmed by results shown in Fig. 2(c) and Fig. 2(d). Variations of the couple stresses μ_{xz}, μ_{yz} within the elastic half-plane at various depths are also displayed in Fig. 3. For the classical case, the couple stresses within the bulk vanish identically. It is also evident from this set of results that solutions predicted by the proposed model exhibit the significant departure from the classical solutions.

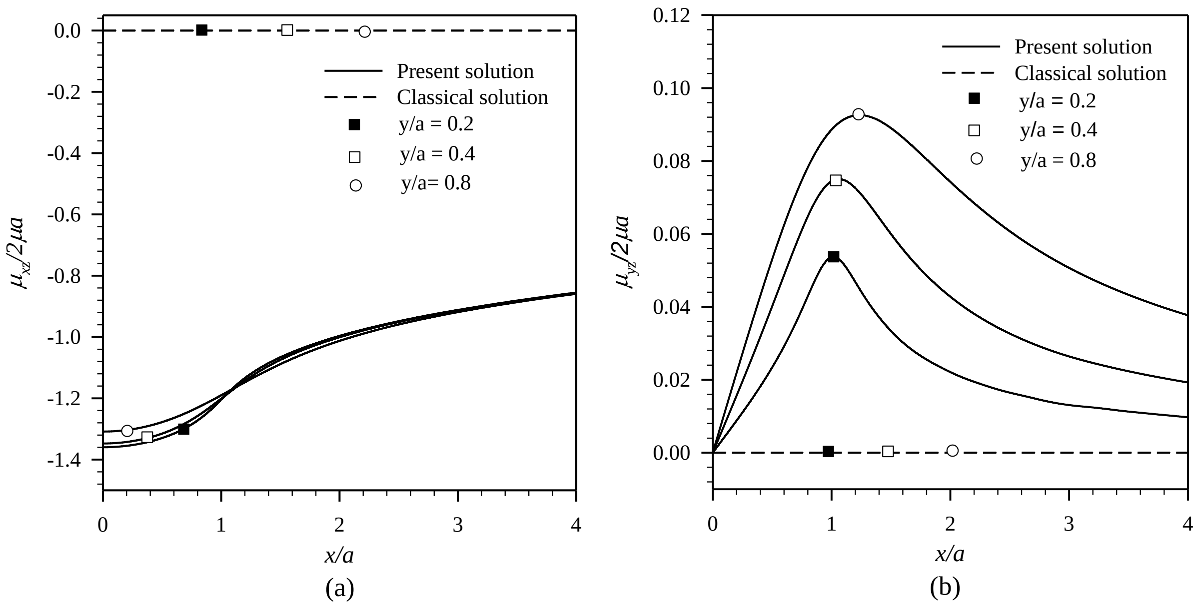

Figure 3. Normalized couple stresses of elastic half-plane under uniformly distributed normal traction at various depths

Conclusion and Remarks

An elastic half-plane under the plane-strain deformation and loaded on its surface by the normal traction has been studied by integrating the influence of both surface stresses and couple stresses. Gurtin-Murdoch surface elasticity theory and the consistent couple stress theory have been used in the problem formulation and the closed-form integral expressions of the elastic field have been derived via the method of Fourier integral transform. An efficient quadrature has been adopted to evaluate all involved integrals resulting from the Fourier transform inversion. A set of preliminary results has indicated the significant influence of the surface and couple stresses on the behavior of predicted solutions; in particular, the obvious deviation from the classical solutions has been observed. The size-dependence behavior of predicted responses and the effects of the two material length scales are also of key interest and still under investigation. It should be remarked that the analytical solutions established in the present study can be used either as benchmark results in the verification process or as the fundamental solutions in the formulation of related problems such as contact and indentation problems.

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References

[1] Dingreville, R., Qu, J. and Cherkaoui, M. (2005) Surface free energy and its effect on the elastic behavior of nano-sized particles, wires and films, *Journal of the Mechanics and Physics of Solids* **53**(8), 1827-1854.

[2] Cammarata, R. C. (1994) Surface and interface stress effects in thin films, *Progress in Surface Science* **46**(1), 1-38.

[3] Cammarata, R. C. (1997) Surface and interface stress effects on interfacial and nanostructured materials, *Materials Science and Engineering: A* **237**(2), 180-184.

[4] Maranganti, R. and Sharma, P. (2007) A novel atomistic approach to determine strain-gradient elasticity constants: Tabulation and comparison for various metals, semiconductors, silica, polymers and the (Ir) relevance for nanotechnologies, *Journal of the Mechanics and Physics of Solids* **55**(9), 1823-1852.

[5] Wong, E. W., Sheehan, P. E. and Lieber, C. M. (1997) Nanobeam mechanics: elasticity, strength, and toughness of nanorods and nanotubes, *Science* **277**(5334), 1971-1975.

[6] Peng, B., Locascio, M., Zapol, P., Li, S., Mielke, S. L., Schatz, G. C. and Espinosa, H. D. (2008) Measurements of near-ultimate strength for multiwalled carbon nanotubes and irradiation-induced crosslinking improvements, *Nature Nanotechnology* **3**, 626-631.

[7] Mao, S. X., Zhao, M. and Wang, Z. L. (2003) Nanoscale mechanical behavior of individual semiconducting nanobelts, *Applied Physics Letters* **83**(5), 993-995.

[8] Li, X., Gao, H., Murphy, C. J. and Caswell, K. K. (2003) Nanoindentation of silver nanowires, *Nano Letters* **3**(11), 1495-1498.

[9] Gurtin, M. E. and Murdoch, A. I. (1975) A continuum theory of elastic material surfaces, *Archive for Rational Mechanics and Analysis* **57**(4), 291-323.

[10] Gurtin, M. E. and Murdoch, A. I. (1978) Surface stress in solids, *International Journal of Solids and Structures* **14**(6), 431-440.

[11] He, L. H., Lim, C. W. and Wu, B. S. (2004) A continuum model for size-dependent deformation of elastic films of nano-scale thickness, *International Journal of Solids and Structures* **41**(3-4), 847-857.

[12] Huang, D. W. (2008) Size-dependent response of ultra-thin films with surface effects, *International Journal of Solids and Structures* **45**(2), 568-579.

[13] Lu, P., He, L. H., Lee, H. P. and Lu, C. (2006) Thin plate theory including surface effects, *International Journal of Solids and Structures* **43**(16), 4631-4647.

[14] Intarit, P., Senjuntichai, T. and Rajapakse, R. K. N. D. (2010) Dislocations and internal loading in a semi-infinite elastic medium with surface stresses, *Engineering Fracture Mechanics* **77**(18), 3592-3603.

[15] Intarit, P., Senjuntichai, T., Rungamornrat, J. and Rajapakse, R. K. N. D. (2011), Surface elasticity and residual stress effect on the elastic field of a nanoscale elastic layer, *Interaction and Multiscale Mechanics* **4**(2), 85-105.

[16] Zhao, X. J., *Surface loading and rigid indentation of an elastic layer with surface energy effects*, Master Thesis, The University of British Columbia, Canada, 2009.

[17] Rungamornrat, J., Tuttipongsawat, P. and Senjuntichai, T. (2016) Elastic layer under axisymmetric surface loads and influence of surface stresses, *Applied Mathematical Modelling* **40**(2), 1532-1553.

[18] Pinyochotiwong, Y., Rungamornrat, J. and Senjuntichai, T. (2013) Rigid frictionless indentation on elastic half space with influence of surface stresses, *International Journal of Engineering Science* **71**, 15-35.

[19] Zhou, S. and Gao, X.-L. (2013) Solutions of half-space and half-plane contact problems based on surface elasticity, *Zeitschrift für angewandte Mathematik und Physik ZAMP* **64**(1), 145-166.

[20] Tirapat, S., Senjuntichai, T. and Rungamornrat, J. (2017) Influence of surface energy effects on elastic fields of a layered elastic medium under surface loading, *Advances in Materials Science and Engineering* **2017**, 7530936.

[21] Cosserat, E. and Cosserat, F. (1909) *Théorie des corps déformables,* A. Herman et Fils, Paris.

[22] Mindlin, R. D. and Tiersten, H. F. (1962) Effects of couple-stresses in linear elasticity, *Archive for Rational Mechanics and Analysis* **11**(1), 415-448.
[23] Mindlin, R. D. (1963) Influence of couple-stresses on stress concentrations, *Experimental Mechanics* **3**(1), 1-7.
[24] Toupin, R. A. (1962) Elastic materials with couple-stresses, *Archive for Rational Mechanics and Analysis* **11**(1), 385-414.
[25] Mindlin, R. D. (1965) Second gradient of strain and surface-tension in linear elasticity, *International Journal of Solids and Structures* **1**(4), 417-438.
[26] Mindlin, R. D. and Eshel, N. N. (1968) On first strain-gradient theories in linear elasticity, *International Journal of Solids and Structures* **4**(1), 109-124.
[27] Chen, S. and Wang, T. (2001) Strain gradient theory with couple stress for crystalline solids, *European Journal of Mechanics - A/Solids* **20**(5), 739-756.
[28] Yang, F., Chong, A. C. M., Lam, D. C. C. and Tong, P. (2002) Couple stress based strain gradient theory for elasticity, *International Journal of Solids and Structures* **39**(10), 2731-2743.
[29] Hadjesfandiari, A. R. and Dargush, G. F. (2011) Couple stress theory for solids, *International Journal of Solids and Structures* **48**(18), 2496-2510.
[30] Ma, H. M., Gao, X. -L. and Reddy, J. N. (2011) A non-classical Mindlin plate model based on a modified couple stress theory, *Acta Mechanica* **220**(1-4), 217-235.
[31] Ma, H. M., Gao, X. -L. and Reddy, J. N. (2008) A microstructure-dependent Timoshenko beam model based on a modified couple stress theory, *Journal of the Mechanics and Physics of Solids* **56**(12), 3379-3391.
[32] Beni, Y. T., Mehralian, F. and Razavi, H. (2015) Free vibration analysis of size-dependent shear deformable functionally graded cylindrical shell on the basis of modified couple stress theory, *Composite Structures* **120,** 65-78.
[33] Gourgiotis, P. and Zisis, T. (2016) Two-dimensional indentation of microstructured solids characterized by couple-stress elasticity, *The Journal of Strain Analysis for Engineering Design* **51**(4), 318-331.
[34] Song, H. X., Ke, L. L. and Wang, Y. S. (2017) Sliding frictional contact analysis of an elastic solid with couple stresses, *International Journal of Mechanical Sciences* **133**, 804-816.
[35] Song, H., Ke, L., Wang, Y., Yang, J. and Jiang, H. (2018) Two-dimensional frictionless contact of a coated half-plane based on couple stress theory, *International Journal of Applied Mechanics* **10**(5), 1850049.
[36] Muki, R. and Sternberg. E. (1965) The influence of couple-stresses on singular stress concentrations in elastic solids, *Zeitschrift für angewandte Mathematik und Physik ZAMP* **16**(5), 611-648.
[37] Zisis, T., Gourgiotis, P. A., Baxevanakis, K. P. and Georgiadis, H. G. (2014) Some basic contact problems in couple stress elasticity, *International Journal of Solids and Structures* **51**(11-12), 2084-2095.
[38] Wang, Y., Shen, H., Zhang, X., Zhang, B., Liu, J. and Li, X. (2018) Semi-analytical study of microscopic two-dimensional partial slip contact problem within the framework of couple stress elasticity: Cylindrical indenter, *International Journal of Solids and Structures* **138**, 76-86.
[39] Gao, X. -L. and Mahmoud, F. F. (2014) A new Bernoulli–Euler beam model incorporating microstructure and surface energy effects, *Zeitschrift für angewandte Mathematik und Physik ZAMP* **65**(2), 393-404.
[40] Shaat, M., Mahmoud, F. F., Gao, X. -L. and Faheem, A. F. (2014) Size-dependent bending analysis of Kirchhoff nano-plates based on a modified couple-stress theory including surface effects, *International Journal of Mechanical Sciences* **79**, 31-37.
[41] Gao, X. -L. and Zhang, G. Y. (2016) A non-classical Kirchhoff plate model incorporating microstructure, surface energy and foundation effects, *Continuum Mechanics and Thermodynamics* **28**, 195-213.
[42] Liu, S., Yu, T., Lich, L. V., Yin, S. and Bui, T. Q. (2019) Size and surface effects on mechanical behavior of thin nanoplates incorporating microstructures using isogeometric analysis, *Computers and Structures* **212**, 173-187.
[43] Sneddon, I. N. (1951) *Fourier Transforms*, McGraw-Hill, New York.
[44] Miller, R. E. and Shenoy, V. B. (2000) Size-dependent elastic properties of nanosized structural elements, *Nanotechnology* **11**(3), 139-147.
[45] Rahimi, Z., Rezazadeh, G. and Sadeghian, H. (2018) Study on the size dependent effective Young modulus by EPI method based on modified couple stress theory, *Microsystem Technologies* **24**(7), 2983–2989.