A Non-parametric Form-Finding Method for Designing Membrane Structure

*** Masatoshi Shimoda[1] and Koichi Yamane[2]**

[1]Department of Advanced Science and Technology, Toyota Technological Institute, 2-12-1 Hisakata, Tenpaku-ku, Nagoya, Japan.
[2]Graduate School of Toyota Technological Institute, Japan

*Corresponding author: shimoda@toyota-ti.ac.jp

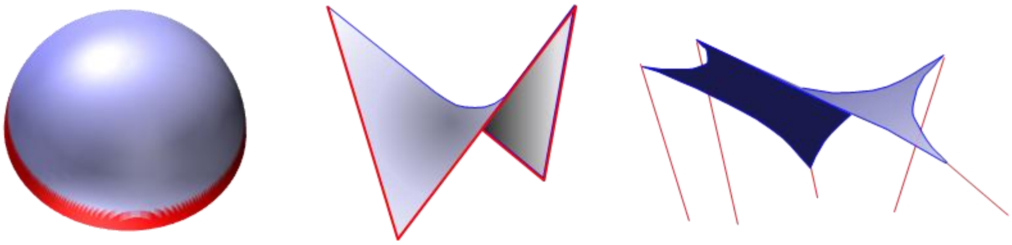
Abstract

In this paper, we present a non-parametric form-finding method for designing the minimal surface, or the uniformly tensioned surface of membrane structures with arbitrary specified boundaries. The area minimization problems are formulated as distributed-parameter shape optimization problems, and solved numerically. The internal volume or the perimeter is added as the constraints according to the type of a structure such as pneumatic or suspended membranes. It is assumed that the membrane is varied in the out-of-plane and/or the in-plane direction to the surface. The shape gradient function for each problem is derived using the material derivative method. The minimal surface is numerically determined without the shape parameterization by the free-form optimization method, a gradient method in a Hilbert space, where the shape is varied by the traction force in proportion to the sensitivity function. The calculated results show the effectiveness of the proposed method for finding the optimal form of membrane structures.

Keywords: Membrane structure, Form finding, Shape optimization, Minimal surface, Free form

Introduction

Membrane structures have many advantages: they contribute to safety and economy, they are lightweight and not bulky, and they have good aesthetic aspects due to their curved surface and translucency. By taking advantage of these characteristics, various membrane structures have been developed and widely used for industrial products such as roofs, yacht sails, balloons and air-bags. Membranes must maintain their shapes by mainly in-plane tension due to their negligible bending stiffness, which makes it difficult for designers to create the required shapes. In addition, in order to maintain the designed shape and to secure sufficient stiffness and strength against self-weight and external force, initial tensions must be appropriately applied to membranes. Therefore, form-finding is vitally important in the design process.
Membrane structures are classified into the pneumatic (air-support) membrane type in which tensions are generated by differential pressure and the non-pneumatic membrane type to which tensions are applied by mechanical force. Non-pneumatic structures are also classified into the frame membrane type and the suspension membrane type. Non-pneumatic structures must have non-positive Gaussian curvature over the whole surface to maintain the shape. Fig. 1 shows the classification of membrane structures. Regardless of the structure type, membranes cannot be expected to have much bearing capacity due to their thinness even if they are made of strong materials. Therefore, designers need to determine the membrane shape so that there is a uniform stress field across the entire surface.



(a) Pneumatic type (b) Frame type (c) Suspension type

Figure 1. Classification of membrane structures

It is well known that a shape with a uniform stress field conforms to the minimal surface which has zero mean curvature across the surface if the deformation due to the self-weight is negligible. If a constraint condition is given, it has a certain amount of curvature. Such a surface with a constant curvature is also regarded as the minimal surface under the constraint condition. Physical experiments using a soap film or a hanging cloth can easily find minimal surfaces (Otto, 1973). It was also mathematically studied as a variational problem and many minimal surface functions were found (Gray, 1998). However, it is difficult to find minimal surfaces taking account of complicated boundary conditions or mechanical characteristics. In order to solve these disadvantages, many versatile numerical solutions have been studied (e.g., Monterde, 2004, Bletzinger *et al*., 2005, Pan and Xu, 2011). The solutions can be classified into node-based and parametric surface-based methods, or geometry-based and experiment simulation-based methods.

The authors also have proposed a numerical solution for finding a minimal surface, i.e., an equally tensioned surface, with an arbitrarily specified boundary (Shimoda and Yamane, 2013). This is a node-based and geometry-based method. In this paper we introduce the method and present application examples by the method. This method finds a minimal surface by formulating the form-finding problem as a distributed-parameter shape optimization problem, and applying the sensitivity function derived by the material derivative method to the proposed method, which was based the free-form optimization methods for shells (Shimoda and Tsuji, 2006, Shimoda *et al*, 2009). The advantages of this method include efficiency for treating large-scale problems and the ability to obtain a smooth shape without any shape parameterization. With this method, numerical form-finding can be performed for a pneumatic membrane structure, a frame membrane structure and a suspension membrane structure, where the shape could vary in the in-plane direction and/or out-of-plane direction according to the type of membrane structures to be solved.

In the following sections, we will first show the formulations of minimal surface problems as distributed-parameter optimization problems and derive each sensitivity function, which is called the shape gradient function. Then, the free-form optimization method for membrane structures will be introduced. Finally, we will show examples of each type of membrane structure.

Domain variation of membrane structure

Definition of shape variation for free-form design

As shown in Fig. 2, consider that a membrane having an initial domain A with the boundary ∂A is varied into one having domain A_s with the boundary ∂A_s by the shape variation (the design velocity field) $\boldsymbol{V}$ distributed across the surface. It is assumed that the boundary ∂A is included in the domain A ($\partial A \subset A$) and that the thickness h is constant during the deformation. The shape variation $\boldsymbol{V}$ consists of the out-of-plane variation $\boldsymbol{V}_n$ which deforms in the normal direction to the surface and the in-plane variation $\boldsymbol{V}_t$ which deforms in the tangential direction to the surface. The membrane shape is varied by $\boldsymbol{V}_n(A)$ distributed on A and $\boldsymbol{V}_t(\partial A)$ distributed on ∂A since $\boldsymbol{V}_t(A)$ does not affect the shape variation except on ∂A. The shape variation is expressed by the piecewise smooth mapping $T_S : \boldsymbol{X} \in A \mapsto \boldsymbol{X}_s(\boldsymbol{X}) \in A_s,\ 0 \le s \le \varepsilon$, where ε and $(\cdot)_s$ indicate a small integer and the iteration history of the shape variation equivalent to time. Using the relations $\boldsymbol{X}_s = T_s(\boldsymbol{X}),\ A_s = T_s(A)$, the small shape variation around the s-th variation is expressed as

$$T_{s+\Delta s}(\boldsymbol{X}) = T_s(\boldsymbol{X}) + \Delta s \boldsymbol{V} + O(|\Delta s|^2) \ , \tag{1}$$

where the design velocity field $\boldsymbol{V}(\boldsymbol{X}_s) = \partial T_s(\boldsymbol{X}) / \partial s$ is given as the Euler derivative of the mapping $T_s(\boldsymbol{X})$, and $O(|\Delta s|^2)$ is assumed to be neglected as a high-order term. The optimal design velocity

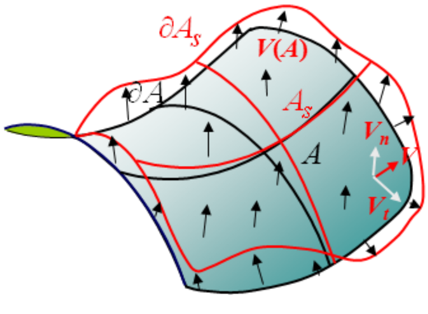


Figure 2. Shape variation of membrane by *V*

field $\boldsymbol{V}(\boldsymbol{X}_s)$ is determined by the free-form optimization method proposed, which will be explained later.

Form-finding problems of membrane structures and derivation of shape gradient function

In order to find the minimal surface, the area of a membrane is defined as an objective functional. In addition to the boundary shape, the internal volume or perimeter is set as another constraint condition according to the structure type. In this section, we will formulate a distributed-parameter shape optimization problem for each type of membrane structure, so as to determine the design velocity field that leads to the minimal surface, and then the shape gradient function will be derived.

Frame membrane structure problem

Consider a shape optimization problem for minimizing the area of a frame membrane structure like that shown in Fig. 1(b). When an initial membrane shape A_0 and a specified boundary shape formed by the frame, which may be an open boundary, are given, this problem is expressed as

$$\text{Given} \quad A_0, \tag{2}$$

$$\text{find} \quad \boldsymbol{V} \ (\text{or } A_s), \tag{3}$$

$$\text{that minimizes} \quad A \ (=\int_A dA). \tag{4}$$

The Lagrange functional L for this problem is expressed as

$$L(A) = \int_A dA. \tag{5}$$

The material derivative (Choi and Kim, 2005) $\dot{L}$ of the Lagrange functional L with respect to shape variation is expressed as

$$\dot{L} = < G_A \boldsymbol{n}, \boldsymbol{V} >_A + < G_{\partial A} \boldsymbol{t}, \boldsymbol{V} >_{\partial A}, \quad \boldsymbol{V} \in C_\Theta, \tag{6}$$

where the notations of $< \boldsymbol{G}_A n, \boldsymbol{V} >_A$ and $< \boldsymbol{G}_{\partial A} t, \boldsymbol{V} >_{\partial A}$ are defined as

$$< G_A \boldsymbol{n}, \boldsymbol{V} >_A \equiv \int_A G_A \boldsymbol{n} \cdot \boldsymbol{V} dA = \int_A G_A V_n dA, \tag{7}$$

$$< G_{\partial A} \boldsymbol{t}, \boldsymbol{V} >_{\partial A} \equiv \int_{\partial A} G_{\partial A} \boldsymbol{t} \cdot \boldsymbol{V} d\Gamma = \int_{\partial A} G_{\partial A} V_t d\Gamma, \tag{8}$$

$$G_A = H_A, \tag{9}$$

$$G_{\partial A} = H_{\partial A}, \tag{10}$$

where C_Θ indicates the admissible function space that satisfies the specified geometric boundary condition. The coefficient functions G_A and $G_{\partial A}$ are called the shape gradient function and are distributed on the surface and on the boundary, respectively. The notation $V_n (\equiv \boldsymbol{n} \cdot \boldsymbol{V})$ is the normal component of $\boldsymbol{V}$ and the vector $\boldsymbol{n}$ is an outward unit normal vector to the surface. $V_t (\equiv \boldsymbol{t} \cdot \boldsymbol{V})$ is the tangential component of $\boldsymbol{V}$ and the vector $\boldsymbol{t}$ is an unit tangential vector to the surface. H_A and $H_{\partial A}$ indicate twice the mean curvature on the surface A and the curvature on the boundary ∂A.
If the arbitrary boundary is closed, the second term on the right-hand side in Eq. (6) is omitted.

Pneumatic membrane structure problem

Consider a problem for minimizing the area of a pneumatic membrane structure subjected to differential pressure, which is shown in Fig. 1(a). Defining a specified boundary as the geometric constraint condition and an internal volume Ω (i.e., a space bounded by the membrane) as the equality constraint condition (the constraint value is represented as $\hat{\Omega}$), this problem is expressed as

$$\text{Given} \quad A_0, \tag{11}$$

$$\text{find} \quad \boldsymbol{V} \ (\text{or } A_s), \tag{12}$$

$$\text{that minimizes} \quad A \ (= \int_A dA) \, , \tag{13}$$

$$\text{subject to} \quad \Omega \ (= \int_\Omega d\Omega) = \hat{\Omega} \, . \tag{14}$$

The Lagrange functional L for this problem is expressed as

$$L(A, \Omega) = \int_A dA + \Lambda_\Omega (\int_\Omega d\Omega - \hat{\Omega}) \, . \tag{15}$$

The material derivative $\dot{L}$ of the Lagrange functional L with respect to shape variation is expressed as

$$\dot{L} = < G_A \boldsymbol{n}, \boldsymbol{V} >_A + \Lambda'_\Omega (\int_\Omega d\Omega - \hat{\Omega}), \quad \boldsymbol{V} \in C_\Theta, \tag{16}$$

$$G_A = H_A + \Lambda_\Omega. \tag{17}$$

When the constraint condition with regard to the internal volume is met, Eq. (16) can be written as

$$\dot{L} = < G_A \boldsymbol{n}, \boldsymbol{V} >_A, \quad \boldsymbol{V} \in C_\Theta. \tag{18}$$

Suspension membrane structure problem

Consider the problem for minimizing the area of a suspension membrane structure like that shown in Fig. 1(c). Defining specified fixed points on the boundary as the geometric constraint condition and a perimeter Γ of the boundary as the equality constraint condition (the constraint value is represented as $\hat{\Gamma}$), this problem is expressed as

$$\text{Given} \quad A_0 \, , \tag{19}$$

$$\text{find} \quad \boldsymbol{V} \ (\text{or} \ A_s) \, , \tag{20}$$

$$\text{that minimizes} \quad A \ (= \int_A dA) \, , \tag{21}$$

$$\text{subject to} \quad \Gamma \ (= \int_{\partial A} d\Gamma) = \hat{\Gamma} \, . \tag{22}$$

The Lagrange functional L for this problem is represented as

$$L(A, \Gamma) = \int_A dA + \Lambda_\Gamma (\int_{\partial A} d\Gamma - \hat{\Gamma}) \, . \tag{23}$$

If the constrained perimeter condition is met, the material derivative $\dot{L}$ of the Lagrange functional L with respect to shape variation is represented as

$$\dot{L} = < G_A \boldsymbol{n}, \boldsymbol{V} >_A + < G_{\partial A} \boldsymbol{t}, \boldsymbol{V} >_{\partial A}, \quad \boldsymbol{V} \in C_\Theta, \tag{24}$$

$$G_A = H_A, \tag{25}$$

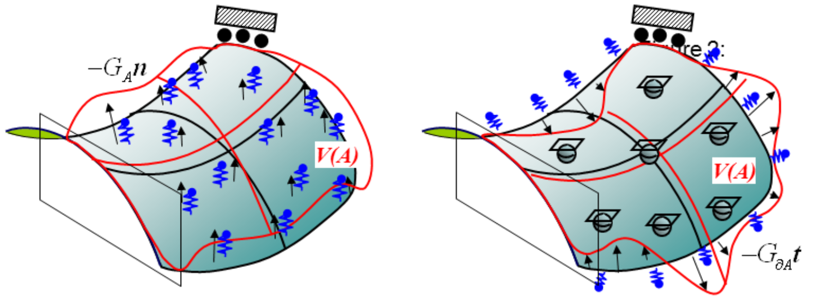
$$G_{\partial A} = 1 + \Lambda_\Gamma H_{\partial A}. \tag{26}$$

The shape gradient functions derived here are used for determining the minimal surface (or the optimal design velocity field or the optimal shape variation). We will explain the method in the next section.

Free-form optimization method for form-finding of membranes

The free-form optimization method for form-finding, i.e., minimal surface of membrane structures, were developed by combining the free-form optimization methods for shells with respect to the in-plane variation (Shimoda and Tuji, 2006) and the out-of-plane variation (Shimoda *et al*, 2009). The free-form optimization method is a node-based shape optimization method based on the traction method (Azegami, 1994, Shimoda *et al*, 1998) which is a gradient method in a Hilbert space, and can treat all nodes as design variables without parameterization.
In order to determine the optimal design velocity field that minimizes the objective functional using both the derived shape gradient functions and the gradient method in a Hilbert space, a tensor with

positive definitiveness must be introduced. A unit tensor cannot maintain the smoothness of the shape since it leads to a jagged shape problem. A stiffness tensor of an elastic shell under the Robin boundary condition similar in shape to a membrane is used in this method to make the computation simple and linear. This stiffness tensor serves not only to reduce the objective functional, but also to maintain the mesh smoothness. The Robin condition is employed to stabilize the convergence. The reference shape is updated with the optimal design velocity field $\boldsymbol{V}$ obtained by applying the distributed external forces in proportion to the negative shape gradient function to this pseudo-elastic shell. Consider the design velocity field $\boldsymbol{V}=\{V_i\}_{i=1,2,3}$ divided between the in-plane component $\boldsymbol{V}_0=\{V_{0_\beta}\}_{\beta=1,2}$ and the out-of-plane component V_3 on the local coordinate systems. Using Kirchhoff's theorem as a plate bending theory, each governing equation of the design velocity field is explained as Eq. (27) and Eq. (29), respectively. In the case of the in-plane variation, the out-of-plane velocity field needs to be constrained (i.e., $V_3=0$). Therefore, after each design field is determined separately, they are synthesized as required with the relation $\boldsymbol{V}=\boldsymbol{V}_n+\boldsymbol{V}_t$. We call this analysis for $\boldsymbol{V}$ "velocity analysis". Fig. 3 shows schematics of the velocity analysis for (a) out-of-plane shape variation and (b) in-plane shape variation.



(a) Out-of-plane shape variation (b) In-plane shape variation

Figure 3. Schematics of the free-form optimization method for membranes

$$a((\boldsymbol{V}_{0\beta},V_3,\boldsymbol{\theta}),(\bar{\boldsymbol{u}}_0,\bar{w},\bar{\boldsymbol{\theta}}))+\alpha\left\langle(\boldsymbol{V}\cdot\boldsymbol{n})\boldsymbol{n},(\bar{\boldsymbol{u}}_0,\bar{w},\bar{\boldsymbol{\theta}})\right\rangle_A=-<\boldsymbol{G}_A n,(\bar{\boldsymbol{u}}_0,\bar{w},\bar{\boldsymbol{\theta}})>_A,$$
$$(\boldsymbol{V}_{0\beta},V_3,\boldsymbol{\theta})\in C_\Theta,\quad \forall(\bar{\boldsymbol{u}}_0,\bar{w},\bar{\boldsymbol{\theta}})\in C_\Theta, \tag{27}$$

$$C_\Theta=\{(V_{0_1},V_{0_2},V_3,\theta_1,\theta_2)\in(H^1(A))^5 \mid (\boldsymbol{V}_0,V_3,\boldsymbol{\theta})\ \textit{satisfy the constraints of shape variation on } A\}. \tag{28}$$

$$a((\boldsymbol{V}_{0\beta},V_3,\boldsymbol{\theta}),(\bar{\boldsymbol{u}}_0,\bar{w},\bar{\boldsymbol{\theta}}))+\alpha\left\langle(\boldsymbol{V}\cdot\boldsymbol{t})\boldsymbol{t},(\bar{\boldsymbol{u}}_0,\bar{w},\bar{\boldsymbol{\theta}})\right\rangle_{\partial A}=-<\boldsymbol{G}_{\partial A} t,(\bar{\boldsymbol{u}}_0,\bar{w},\bar{\boldsymbol{\theta}})>_{\partial A},$$
$$(\boldsymbol{V}_{0\beta},V_3,\boldsymbol{\theta})\in C_\Theta,\quad \forall(\bar{\boldsymbol{u}}_0,\bar{w},\bar{\boldsymbol{\theta}})\in C_\Theta\ , \tag{29}$$

$$C_\Theta=\{(V_{0_1},V_{0_2},V_3,\theta_1,\theta_2)\in(H^1(A))^5 \mid (\boldsymbol{V}_0,V_3,\boldsymbol{\theta})\ \textit{satisfy the constraints of shape}\ \text{variation on}\ S\ \textit{and}\ V_3=0\ \textit{on}\ A\}. \tag{30}$$

Here, the bilinear form $a(\cdot\ ,\ \cdot)$, which represents virtual work related to internal force, and the linear forms $<\cdot\ ,\ \cdot>_A$, $<\cdot\ ,\ \cdot>_{\partial A}$ are expressed as Eq. (31), Eq. (32) and Eq. (33) , respectively. $\bar{(\ )}$ expresses the variation. The tensor subscript notation uses Einstein's summation convention and a partial differential notation for the spatial coordinates $(\cdot)_{,i}=\partial(\cdot)/\partial x_i$.

$$a((\boldsymbol{u}_0,w,\boldsymbol{\theta}),(\bar{\boldsymbol{u}}_0,\bar{w},\bar{\boldsymbol{\theta}}))=\int_A\{c^B_{\alpha\beta\gamma\delta}\kappa_{\gamma\delta}\bar{\kappa}_{\alpha\beta}+c^M_{\alpha\beta\gamma\delta}\varepsilon_{0\gamma,\delta}\bar{\varepsilon}_{0\alpha,\beta}\}dA, \tag{31}$$

$$<G_A\boldsymbol{n},(\bar{\boldsymbol{u}}_0,\bar{w},\bar{\boldsymbol{\theta}})>_A=\int_A G_A\bar{w}dA, \tag{32}$$

$$<G_{\partial A}\boldsymbol{t},(\bar{\boldsymbol{u}}_0,\bar{w},\bar{\boldsymbol{\theta}})>_{\partial A}=\int_{\partial A}(G_{\partial A_\beta}\bar{u}_{0\beta})d\Gamma\ , \tag{33}$$

where w and $\boldsymbol{u}_0=\{u_{0\alpha}\}_{\alpha=1,2}$ represent the out-of-plane displacement and the in-plane displacement vector at the mid-plane, respectively. $\{\kappa_{\alpha\beta}\}_{\alpha,\beta=1,2}$ and $\{\varepsilon_{0\alpha\beta}\}_{\alpha,\beta=1,2}$ represent the curvature tensor and the strain tensor at the mid-plane, respectively, which are defined as

$$\kappa_{\alpha\beta}\equiv\frac{1}{2}(w_{,\alpha\beta}+w_{,\beta\alpha}),\ . \tag{34}$$

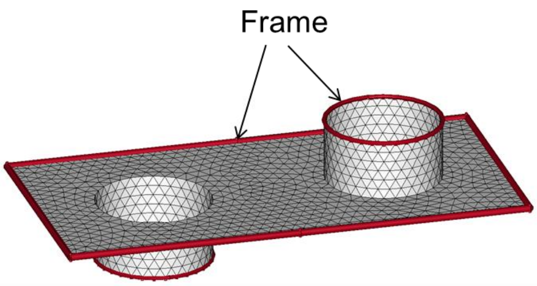
$$\varepsilon_{0\alpha\beta} \equiv \frac{1}{2}(u_{0\alpha,\beta} + u_{0\beta,\alpha}) . \tag{35}$$

H_A and $H_{\partial A}$ in the shape gradient functions are approximately calculated at all points of a finite element model by a discrete method proposed by Meyer *et al*., (2002).
The fact that the shape variation due to the design velocity field ***V*** obtained in the velocity analysis, i.e., Eq. (27) and/or Eq. (29) decreases the objective functional was verified in the previous papers (Shimoda and Yamane, 2013, Shimoda *et al*., 2009).
The minimal surface can be obtained by repeating the three processes: (i) computation of the shape gradient function, (ii) velocity analysis and (iii) shape updating. In this study, a general-purpose FEM code was used in the velocity analysis.

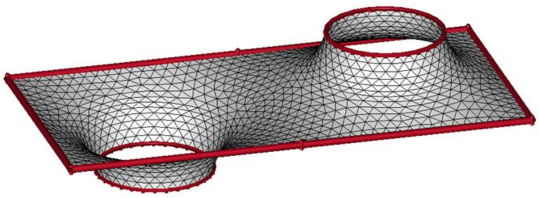
Calculated results for three types of membrane structures

Frame membrane structure problem

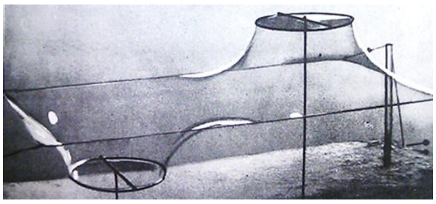
The initial shape of a frame membrane structure is shown in Fig. 4(a). The red lines show the fixed frames, or the specified boundaries. In the velocity analysis their boundaries were simply supported. The minimal surface obtained is shown in Fig. 4(b), which was determined by the out-of-plane variation according to the shape gradient function, i.e., Eq. (9). Fig. 4(c) shows the result of soap-film experiment by Otto (1973).



(a) Initial



(b) Obtained



(c) Soap-film experiment

Figure 4. Optimization result of frame membrane structure

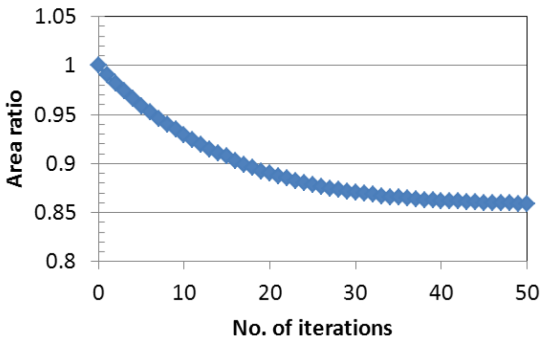
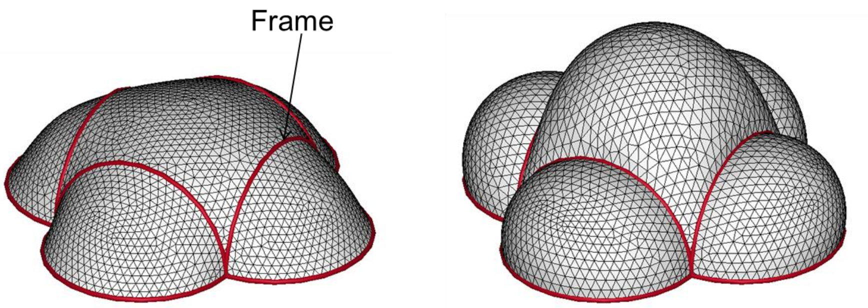


Figure 5. Iteration histories for frame membrane structure

Fig. 5 shows the iteration convergence histories of the area. The results show that the shape obtained is smooth and well-approximated to the soap-film experiment. The area was decreased by about 14% and converged steadily.

Pneumatic membrane structure problem

The initial shape having internal volume, i.e., the space bounded by the membrane with frames, was designed as shown in Fig. 6(a). Under the internal volume constraint condition (i.e., 150% of the initial shape), analysis of the minimal surface was conducted. In the velocity analysis, the specified boundaries by the frames were simply supported. The minimal shapes were determined by the out-of-plane variation according to the shape gradient function, i.e., Eq. (17). The internal volume was computed by space discretization using tetra elements. Fig. 6(b) shows the minimal surface obtained, and Fig. 7 shows the iteration convergence histories of the area and the internal volume. The graph shows that the area was increased by about 40%, while satisfying the internal volume constraint.



(a) Initial
(Initial internal volume: V_0)

(b) Obtained
(Constraint volume: V=1.5V_0)

Figure 6. Optimization result of pneumatic membrane structure

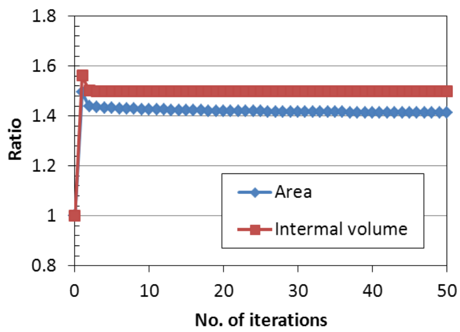
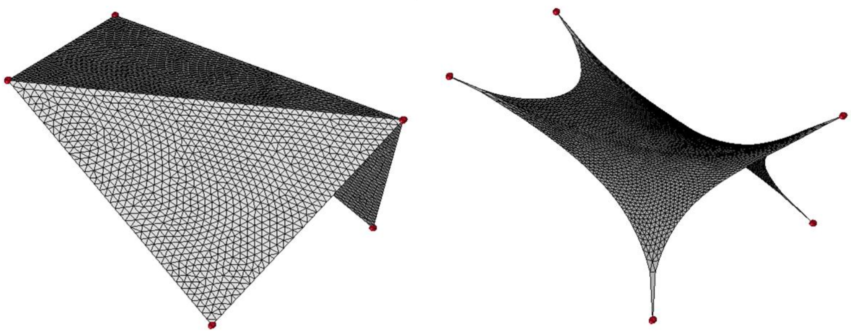


Figure 7. Iteration histories for pneumatic membrane structure

Suspension membrane structure problem

As a problem of suspension membrane structures, an area minimization analysis was conducted under a perimeter constraint condition. The initial shape of a tarpaulin-like structure, the five vertices of which were fixed, is shown in Fig. 8(a). In the velocity analysis, the vertices were simply supported. The perimeter constraint was set as 102% of the initial shape. The minimal surface was determined by the out-of-plane and in-plane variations according to the shape gradient functions, i.e., Eq. (25) and Eq. (26). Fig. 8 (b) shows the minimal surface obtained. The area was minimized and reduced by around 58%, while the perimeter constraint was satisfied.



(a) Initial (b) Obtained

Figure 8. Optimization result of 5-points spatial suspension membrane structure

Conclusion

In this paper, we presented a numerical non-parametric form-finding method for designing the minimal surface of a membrane structure. Design problems according to the type of membrane structure (i.e., frame, pneumatic and suspension type) are formulated as distributed-parameter shape optimization problems, and the shape gradient functions are derived, where the in-plane shape variation and/or the out-of-plane shape variation was defined as the variable for form-finding. By applying the derived sensitivity function to the gradient method in a Hilbert space, the minimal surfaces were determined by iterative computations. With this method, the optimal and smooth free-form of membranes can be found without any shape parameterization. Design examples were illustrated for each type of membrane structure.

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