Dynamic reliability based structural optimization

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Abstract

Structural dimension and shape optimization based on the structural dynamic reliability is investigated in this paper. Structural gross mass is taken as the objective function and the structural dynamic reliability is incorporated into the constraints. The dynamic reliability constraints are transferred and simplified, and the normalization of design variables is discussed to avoid some variables being drowned by others during optimization due to their different dimensions and orders of magnitude. The optimal models of dimension and shape with dynamic reliability constraints are then presented. Numerical examples are used to illustrate the results of different optimal designs, which demonstrate that the efficiency to solve the structural optimization with dynamic reliability constraints can be significantly improved if the design variables and their initial values are selected properly.

Keywords: Dimension and shape optimization, normalization of design variables, dynamic reliability constraints, random process excitation.

Introduction

Structural reliability optimization conducted from 1960s and has been extensively investigated (Chen et al., 1997; Gao et al., 2003; Kang and Luo, 2010; Li et al., 2011; Jiang et al., 2010). Probabilistic approaches have been developed to account for the uncertainties in structural parameters, such as stochastic finite element method, perturbation method, response surface method and Monte-Carlo simulation based methods. First-order and second-order reliability methods have been proposed and improved to assess the reliability/safety of structures with uncertain properties. The randomness of forces has been also considered in the reliability assessments. Structural optimization with the reliability constraints accounting for the uncertainties in structural parameters and inputs has been investigated by many researchers and has been widely applied to the design of different types of structures (Gao, 2006). However, dynamic reliabilities or random process excitations were rarely adopted in the structural optimal design. In reality, the external loads are often random process excitations such as winds, earthquake motions, waves and explosions.

In this paper, a framework is presented to optimize the dimension or shape of the truss structures in the context of the element or system dynamic reliability constraints. Central to the construction of this framework is the reasonable mathematic models of dimension and shape optimization of truss structures where the minimization of structural gross mass is taken to be the optimization goal, with a particular emphasis on the discussion of the simplification of element and system dynamic reliability constraints, as well as on the normalization of design variables in order to facilitate the optimization. Finally, the feasibility and rationality of models and method given are demonstrated by the implementation of examples and some important conclusions are obtained.

Construction of general optimization models with implicit dynamic reliability constraints

Due to the different optimization aims, choices of the optimal design variables are different too. In the topology optimization, topology variables are design variables whether structural elements with topology variables exist or not; in the shape optimization, the coordinates of structural nodes are design variables; in the dimensions optimization, cross-section areas of structural elements are design variables. Generally, optimization with only one kind of design variables is presented, but sometimes it is necessary to consider the optimization including several kinds of design variables simultaneously. For this purpose, both shape and dimension optimizations are considered at the same time in the following deduction.

Because external loads are random process excitations, all constraints like structural displacement and stress are then functions of random process in the dynamic optimization and they are given in the form of dynamic reliability. For the two-side boundary constraints (Ma et al., 2010), the optimal model of structural dimensions and shape for the minimal gross mass is constructed in Eq. (1) where design variables are bars' cross-section areas and nodal coordinates and the dynamic reliability constraints are nodal displacement and element's stress

$$\begin{aligned}
&find: \ \{\vec{A}\} = (A_1, A_2, \ldots, A_n)^T, \ \{\vec{Z}\} = (Z_1, Z_2, \ldots, Z_m)^T \\
&\min: \ W(\vec{A}) = \sum_{i=1}^{ne} \rho_i A_i l_i \\
&s.t.: \ R^*_{x_j} - P_r\{(\max_{t\in T} x_j(t) \le \lambda_{xju}) \cap (\min_{t\in T} x_j(t) \ge -\lambda_{xjl})\} \le 0 \quad (j = 1,2,\ldots,N) \\
&R^*_{S_e} - P_r\{(\max_{t\in T} S_e(t) \le \lambda_{Seu}) \cap (\min_{t\in T} S_e(t) \ge -\lambda_{Sel})\} \le 0 \quad (e = 1,2,\ldots,M) \\
&A_{li} \le A_i \le A_{ui} \quad (i = 1,2,\ldots,ne) \\
&Z_{li} \le Z_i \le Z_{ui} \quad (i = 1,2,\ldots,me)
\end{aligned} \tag{1}$$

where elemental cross-section areas $\vec{A}$ and nodal coordinates $\vec{Z}$ are design vectors; ρ is mass density of bars; A_i and l_i are cross-section area and bar's length corresponding to the *ith* type of design variables; $W(\vec{A})$ is the gross mass; x_j and S_e are the displacement response of the *jth* degree of freedom and the stress response of the *eth* element under random process excitation respectively; λ_{xju}, λ_{xjl}, λ_{Seu} and λ_{Sel} are the upper and lower transcending bounds of displacement of the *jth* degree of freedom and stress of *eth* element respectively; $R^*_{x_j}$ and $R^*_{S_e}$ are dynamic reliabilities required by design; $\Pr(\cdot)$ is the dynamic reliability solved; A_{ui} and A_{li} are upper and lower bounds of the *ith* type of cross-section area design variables; Z_{ui} and Z_{li} are upper and lower bounds of the *ith* type of nodal coordinate design variables; *ne* and *me* are dimensions of cross-section area design vector and nodal coordinate design vector; N is the number of displacement constraints and M is the number of structural elements.

If the dynamic reliability is one-side transcending bound (Ma et al., 2010), constraints in Eq. (1) can then be replaced by

$$\begin{aligned}
&s.t.: \ R^*_{x_j} - P_r\{x_j(t) \le \lambda_{xj}, \ 0 \le t \le T\} \le 0 \quad (j = 1,2,\ldots,N) \\
&R^*_{s_e} - P_r\{s_e(t) \le \lambda_{se}, \ 0 \le t \le T\} \le 0 \quad (e = 1,2,\ldots,M) \\
&A_{li} \le A_i \le A_{ui} \quad (i = 1,2,\ldots,ne) \\
&Z_{li} \le Z_i \le Z_{ui} \quad (i = 1,2,\ldots,me)
\end{aligned} \tag{2}$$

where λ_{xj} and λ_{Se} are transcending bounds of displacement of the *jth* degree of freedom and stress of *eth* element respectively.

Optimization models of truss structures under stationary stochastic process excitation

Considering the structural determinate parameters, structural dynamic responses are stationary stochastic process too when external excitation is a stationary random process. From the first-passage failure theory, Poisson equations to compute the dynamic reliability of single degree of freedom with two-side boundary, symmetric boundary and one-side boundary are respectively

$$R_i(t) = \exp\{-\frac{1}{2\pi}\frac{\dot{\psi}_{Y_i(t)}}{\psi_{Y_i(t)}}T[\exp(-\frac{\lambda_{Yiu}^2}{2\psi_{Y_i(t)}^2}) + \exp(-\frac{\lambda_{Yil}^2}{2\psi_{Y_i(t)}^2})]\} \tag{3}$$

$$R_i(t) = \exp\{-\frac{1}{\pi}\frac{\dot{\psi}_{Y_i(t)}}{\psi_{Y_i(t)}}T\exp(-\frac{\lambda_{Yi}^2}{2\psi_{Y_i(t)}^2})\} \tag{4}$$

$$R_i(t) = \exp\{-\frac{1}{2\pi}\frac{\dot{\psi}_{Y_i(t)}}{\psi_{Y_i(t)}}T\exp(-\frac{\lambda_{Yi}^2}{2\psi_{Y_i(t)}^2})\} \tag{5}$$

In Eq. (3), $\psi_{Y_i(t)}$ and $\dot{\psi}_{Y_i(t)}$ are roots of mean square values of stationary response $Y_i(t)$ (displacement, stress or strain) and its derivation response $\dot{Y}_i(t)$, respectively. λ_{Yil} and λ_{Yiu} are lower and upper safe bounds of $Y_i(t)$, and λ_{Yi} is the safe bound too. The two-side boundary dynamic reliability is most common among Eqs. (3)-(5).

Then the structural displacement and stress dynamic reliability constraints with the two-side boundary are

$$R_{x_j}^* - \exp\{-\frac{1}{2\pi}\frac{\dot{\psi}_{x_j}}{\psi_{x_j}}T[\exp(-\frac{\lambda_{xju}^2}{2\psi_{x_j}^2}) + \exp(-\frac{\lambda_{xjl}^2}{2\psi_{x_j}^2})]\} \le 0 \tag{6}$$

$$R_{s_e}^* - \exp\{-\frac{1}{2\pi}\frac{\dot{\psi}_{s_e}}{\psi_{s_e}}T[\exp(-\frac{\lambda_{Seu}^2}{2\psi_{s_e}^2}) + \exp(-\frac{\lambda_{Sel}^2}{2\psi_{s_e}^2})]\} \le 0 \tag{7}$$

where ψ_{x_j} , ψ_{S_e} , $\dot{\psi}_{x_j}$ and $\dot{\psi}_{S_e}$ are roots of mean square values of the *jth* nodal displacement response $x_j(t)$ and the *eth*-element's stress response $S_e(t)$ and their corresponding derivation response respectively. λ_{xju} , λ_{xjl} , λ_{Seu} , λ_{Sel} , $R_{x_j}^*$ and $R_{s_e}^*$ are the same as those in Eq. (1).

The structural dimension optimization model with dynamic reliability constraints of the two-side boundary is

$$\begin{aligned}
&find: \quad \vec{A} = (A_1, A_2, \ldots, A_n)^T \\
&\min: \quad W(\vec{A}) = \sum_{i=1}^{ne} \rho_i A_i l_i \\
&s.t.: \quad R_{x_j}^* - \exp\{-\frac{1}{2\pi}\frac{\dot{\psi}_{x_j}}{\psi_{x_j}}T[\exp(-\frac{\lambda_{xju}^2}{2\psi_{x_j}^2}) + \exp(-\frac{\lambda_{xjl}^2}{2\psi_{x_j}^2})]\} \le 0 \quad (j = 1,2,\ldots,N) \\
&\qquad R_{s_e}^* - \exp\{-\frac{1}{2\pi}\frac{\dot{\psi}_{s_e}}{\psi_{s_e}}T[\exp(-\frac{\lambda_{Seu}^2}{2\psi_{s_e}^2}) + \exp(-\frac{\lambda_{Sel}^2}{2\psi_{s_e}^2})]\} \le 0 \quad (e = 1,2,\ldots,M) \\
&\qquad A^l \le A_i \le A^u \quad (i = 1,2,\ldots,ne)
\end{aligned} \tag{8}$$

Similarly, optimization models with symmetric boundary dynamic reliability constraints and one-side boundary dynamic reliability constraints can also be obtained. Here only dynamic reliability constraints of symmetric boundary are given in Eq. (9)

$$s.t.: \quad R_{x_j}^* - \exp\{-\frac{1}{\pi}\frac{\dot{\psi}_{x_j}}{\psi_{x_j}} T \exp(-\frac{\lambda_{xj}^2}{2\psi_{x_j}^2})\} \le 0 \qquad (j = 1,2,\ldots,N)$$

$$R_{s_e}^* - \exp\{-\frac{1}{\pi}\frac{\dot{\psi}_{s_e}}{\psi_{s_e}} T \exp(-\frac{\lambda_{Se}^2}{2\psi_{s_e}^2})\} \le 0 \qquad (e = 1,2,\ldots,M) \tag{9}$$

When bar's cross-section areas are design variables and the minimization of structural gross mass is the objective function, the dimension optimization model based on the system reliability is

$$\begin{aligned} find: \quad & \vec{A} = (A_1, A_2, \ldots, A_n)^T \\ \min: \quad & W(\vec{A}) = \sum_{i=1}^{ne} \rho_i A_i l_i \\ s.t.: \quad & P_j(\vec{A}) \le P_j^* \qquad (j = 1,2,\ldots,m) \\ & P_f(\vec{A}) \le P_f^* \end{aligned} \tag{10}$$

where $P_j(\vec{A})$ and $P_f(\vec{A})$ are firs-passage failure probability of the *jth* displacement response and first-passage failure probability of the whole system respectively, and both of them are implicit complex random-process function; P_j^* and P_f^* are the failure probabilities of the *jth* displacement and the system respectively; m is the number of displacement constraints.

In Eq. (10), the solution to system dynamic reliability constraint is very difficult. Especially for hyperstatic structures of higher degree or complicated structures, it is impossible to accurately compute the system failure probability. For multi-degree of freedom system, due to the correlation of responses which leads to the correlation of dynamic damage modes, one can not obtain the precise value of system dynamic reliability at all. So two kinds of extreme cases (complete correlation $\rho_{ij} = 1$ and complete no correlation $\rho_{ij} = 0$) among every damage modes are considered. From the reliability theory, the structural system dynamic reliability (Chen et al., 1997) is

$$\prod_i^n P_{ri}(t) \le P_r(t) \le \min\{P_{ri}(t)\} \qquad (0 \le t \le T) \tag{11}$$

where the dynamic reliability function of the *ith* element $P_{ri}(t) = P_r\{-\lambda_{li} \le Y_i(t) \le \lambda_{ui},\ 0 \le t \le T\}$, $Y_i(t)$ is the dynamic response of the *ith* element, λ_{ui} and λ_{li} are upper and lower bounds given for $Y_i(t)$. When $\rho_{ij} = 1$, the right equal sign holds, when $\rho_{ij} = 0$, the left equal sign holds, when $0 \prec \rho_{ij} \prec 1$, inequality holds.

Eq. (11) shows that the dynamic reliability of structural weakest element is the upper bound of $P_r(t)$. Suppose that there are $m (m \le n)$ structural elements whose dynamic reliabilities are less than 1 in $[0, T]$, to obtain the lower bounds of $P_r(t)$ by a smallest amount of computation, only the elements with $P_{ri}(t) \prec 1$ ($i = 1,2,\cdots,m,\ m \le n$) are searched, the elements with $P_{ri}(t) = 1$ are rejected. Then it is necessary to determine a weakest element with the smallest dynamic reliability and $m-1$ weaker elements. The searching sequence is as follows

$$\min_{i=1,n}\{P_{ri}(t)\} \longrightarrow \text{seek for the weakest among } n \text{ structural elements}$$

$$\min_{i=1,n-1}\{P_{ri}(t)\} \longrightarrow \text{seek for the weakest among the } n-1 \text{ elements left}$$

$$\vdots \qquad\qquad \vdots$$

$$\min_{i=1,n-(m-1)} \{P_{ri}(t)\} \longrightarrow \text{seek for the } (m\text{-}1)th \text{ weakest among the left } n-(m-1) \text{ elements}$$

Taking the lower bound $\prod_{i}^{n} P_{ft}(t)$ of $P_r(t)$ as system dynamic reliability, the constraints of the new optimization model with the same design variables and objective function same as those in Eq. (10) are

$$s.t.: \quad P_j(\bar{A}) \le P_j^* \qquad (j=1,2,\ldots,m)$$
$$\prod_{i}^{n} P_{ft}(t) \le P_f^* \qquad (12)$$

Because the lower bound of system dynamic reliability is smaller than practical system dynamic reliability, the optimal results under the constraints in Eq. (12) tend to conservative, which is acceptable in the practical engineering.

Example

In Figure 1, elastic module $E = 210Gpa$, mass density $\rho = 7800kg/m^3$. A random load P acts on the node 3 in the vertical direction, excitation source is the Gaussian white noise process with zero mean value and its power spectrum density is $S_P = 100N^2s$, the load's durative time $t = 1000s$. The two-side symmetric boundary is $\lambda = 100Mpa$, the stress dynamic reliability of every bar is 0.99. The objective function is the structural gross mass W. When the design variables are bars' cross-section areas with initial values $0.005m^2$, the dimension optimal results are listed in Table 1.

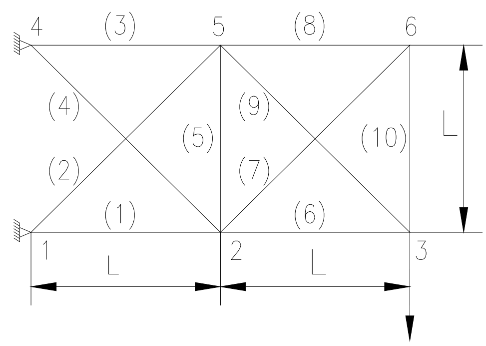

Figure 1. 10-bar truss

Table 1. Dimensions optimal results under dynamic reliability constraint

parameters	$A1(cm^2)$	$A2(cm^2)$	$A3(cm^2)$	$A4(cm^2)$	$A5(cm^2)$	$A6(cm^2)$
initial	50	50	50	50	50	50
optimum	23.7908	20.9263	30.1170	11.4981	2.0000	10.5384
parameters	$A7(cm^2)$	$A8(cm^2)$	$A9(cm^2)$	$A10(cm^2)$	$W(kg)$	$R_{\min}$
initial	50	50	50	50	909.2346	1
optimum	5.9143	4.8669	13.7567	20.4613	258.0997	0.9900

From results of Table 1, the structural gross mass reduces to 28.4 percent of the original mass on the basis of ensuring the stress reliability. Moreover, among optimal results of $Ai(i=1,2,\cdots,10)$, $A3$ is the greatest and $A5$ is the smallest, this is because bar 3 is the an important workload bearing element while bar 5 is the less important one according to the analysis of theoretical mechanics.

In the following, structural shape and dimension are optimized simultaneously under dynamic reliability constraint. Taking $Ai(i=1,2,\cdots,10)$ and ordinates of node 4, 5 and 6 as design variables simultaneously, initial values of Ai are $0.005m^2$, initial values of three ordinates are $2m$. The optimal results are given in Table 2.

Table 2. Shape and dimension optimum results under dynamic reliability constraint

parameters	$A1(cm^2)$	$A2(cm^2)$	$A3(cm^2)$	$A4(cm^2)$	$A5(cm^2)$
initial design	50	50	50	50	50
optimum	14.1104	14.8950	16.6942	3.3414	3.6604
parameters	$A6(cm^2)$	$A7(cm^2)$	$A8(cm^2)$	$A9(cm^2)$	$A10(cm^2)$
initial design	50	50	50	50	50
optimum	7.8812	3.4494	4.9532	13.5123	65.7645
parameters	$Y4(m)$	$Y5(m)$	$Y6(m)$	$W(kg)$	$R_{\min}$
initial design	2	2	2	909.2346	1
optimum	4.94243	3.38567	0.22864	209.5853	0.9900

From Table 2, it can be observed that

1) the gross mass reduces further under the premise of dynamic reliability satisfying the constraint condition.

2) the optimal results of dimension variables are different from those in Tab.11, and $A10$ is the greatest while $A8$ is the smallest, $A3$ and $A1$ are still comparatively greater as the connection elements of structural root, which is similar to the conclusion of Tab.11.

The structural shape after optimization is showed in Figure 5, and it tends to a triangle as a whole which enables structure to be more stable and is a better load-carrying shape.

Conclusion

The structural optimal design based on dynamic reliability is more complicated than conventional structural static optimal design, and optimal design with system dynamic reliability constraint is more complicated than that with element dynamic reliability constraint. How to quickly finish structural dynamic analysis and further improve optimization methods are critical to the widespread application of dynamic optimal design based on the dynamic reliability.

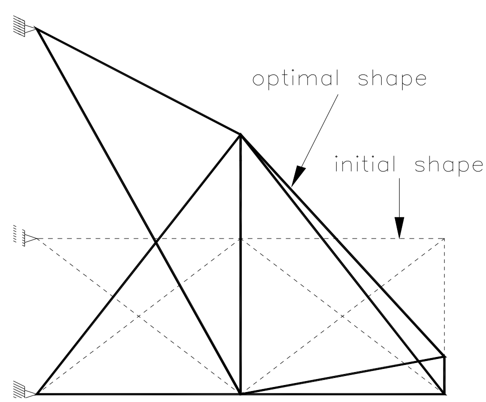

Figure 2. shape optimal result

Because it's very hard to obtain system dynamic reliability according to Eq. (10) and sometimes it is impossible at all. So one can only consider two extreme cases to approximately evaluate it. Moreover, its solution is based on the simplified dynamic reliability which is suitable for many engineering structures because their responses are often narrow band processes. The dynamic reliability obtained by the simplified method does not greatly differ from its true value because the upper bound of the integral interval of power spectral density is not much greater than the intermediate value, thus the lower bound of system dynamic reliability will not much smaller than its true value. Hence the optimization results are just comparatively conservative and still applicable in practical engineering when lower bound of system dynamic reliability is used

Some measures such as reasonably choosing design variables (dimensions or shape parameters) or evaluating initial values of the design variables can effectively enhance the efficiency of dynamic optimal design based on dynamic reliability.

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