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A Linear Response Surface based on SVM for Structural Reliability Analysis

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Abstract

The Structural Reliability theory allows the rational treatment of the uncertainties and gives the methods for the evaluation of the safety of structures in presence of uncertain parameters. The main challenge is the computational cost, since the failure probability with respect to an assigned limit state is given as the solution of a very complicated multidimensional integral. The most robust procedure is the Monte Carlo Simulation (MCS), but especially in its crude form is very demanding. For this reason, wide popularity has been gained by the First Order Reliability Method (FORM) by its simplicity and computational efficiency. However, for strongly nonlinear systems the FORM approximation is not very close to the exact one. To this aim, in this paper we introduce a novel Linear approximation of the limit state, based on the Support Vector Method (SVM), and which allows to improve the FORM solution, starting from the knowledge of the design point.

Introduction

Recently it has been largely recognized that a realistic analysis of the structural systems should take into account all the unavoidable uncertainties appearing in the problem at hand. In this context a powerful tool is represented from the structural reliability theory (Madsen et al. 1986, Ditlevsen & Madsen 1999, Melchers, 1999) which gives a rational treatment of the uncertainties and which allows the assessment of the evaluation of the safety of structures in presence of uncertain parameters.

The failure probability P_f with respect to an assigned limit state is defined as

$$P_f = \int_{g(\boldsymbol{x})\le 0} f_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x} \tag{1}$$

where $\boldsymbol{x}$ is an $n-$vector collecting the basic random variables, $y = g(\boldsymbol{x})$ is the Limit State Function (LSF), $g(\boldsymbol{x}) = 0$ is the Limit State Surface (LSS) separating the failure set $g(\boldsymbol{x}) \le 0$ from the safe set $g(\boldsymbol{x}) > 0$, $f_{\boldsymbol{X}}(\boldsymbol{x})$ is the joint probability density function of the random variables $x_1, x_2, \ldots, x_n$. The evaluation of the failure probability P_f is known in closed form only for a very restricted number of cases, in the most general case it is necessary to solve numerically a multidimensional integral, which is computationally demanding.

The most robust procedure for the evaluation of the failure probability is represented by the Monte Carlo Simulation (MCS), which however, especially in its crude form, requires an excessive computational effort for the evaluation of the very small failure probabilities.

For this reason, wide popularity has been gained by the First Order Reliability Method (FORM) by its simplicity and computational efficiency, moreover extensive numerical experimentation has shown that it gives good approximations of the failure probability for most practical problems. However, it is known that the FORM approximation is not adequate for limit state surfaces which depart significantly from linearity around the design point.

In this paper we overcome this shortcoming with a particular type of Response Surface Methodology (RSM) based on the Support Vector Method (SVM) and the theory of the statistical learning (Vapnik 1995, Burges 1998).

The basic idea of the RSM is the building of a surrogate model of the target limit state function, defined in a simple and explicit mathematical form; once the Response Surface (RS) is built, it is possible to substitute the RS with the target LSF, and then it is no longer necessary to run demanding finite element analyses; starting from this definition, FORM itself is a particular kind of RS, which approximates the LSS with the hyperplane passing through the design point $\boldsymbol{u}^*$ and normal to the design point direction, the latter being the ray joining the origin of the standard normal space with the design point.

The RS models can be built to find the design point with reduced computational cost (Bucher & Burgound 1990, Alibrandi & Der Kiureghian 2012); recently, many alternative response surface methodologies have been proposed, whose aim is the improvement of the FORM approximation (Bucher & Most, 2008; Alibrandi & Ricciardi 2005, Alibrandi & Ricciardi 2008, Alibrandi, Impollonia & Ricciardi 2010).

To the latter cathegory belong the RS approaches based on the SVM (Hurtado 2004; Alibrandi & Ricciardi 2011). Using the SVM the reliability problem is treated as a classification approach (Hurtado & Alvarez 2003), since we are not interested to the exact value of the LSF, but only to its sign. Therefore the samples are labelled with the value "+1" (safe sample) and "−1" (unsafe sample) and this requisite is less strong than approximating the exact value of the LSF.

In the existing approaches based on SVM, the improvement of the FORM solution is obtained by choosing non-linear models for approximating the limit state; conversely, in this paper, we adopt a simple linear model, where the constraints of correct classification are relaxed, accepting therefore that some points may be misclassified (Alibrandi 2012)

The starting model is built choosing a set of sample points along the design point direction, the latter being the direction of probabilistic interest. In this way, starting from the knowledge of the design point, it is possible to approximate the limit state with an hyperplane close to FORM but secant to the limit state, giving rise to an alternative Linear response surface based on SVM (LSVM), and giving a better approximation than FORM.

Structural Reliability Analysis as a classification approach

Usually the multidimensional integral (1) is very difficult to be evaluated, and then some approximate techniques are used. As a first step, a probabilistic coordinate transformation is done toward the standard normal space and the failure probability is given as

$$P_f = \int_{g(\boldsymbol{u})\leq 0} f_U(\boldsymbol{u})d\boldsymbol{u} \tag{2}$$

In (2) the integrand function is the multivariate normal standard probability density function (pdf), while the integration domain is the region failure $g(\boldsymbol{u}) \leq 0$. According to (2), the failure probability can be obtained using the Monte Carlo Simulation (MCS), considering a set of N samples $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N$ and evaluating the ratio between the number N_f of samples belonging to the failure region $g(\boldsymbol{u}) \leq 0$ and the total number of simulated samples N :

$$P_{f,MCS} = \frac{N_f}{N} = \frac{1}{N}\sum_{j=1}^{N} I\left[g(\boldsymbol{u}_j) \le 0 \right] \tag{3}$$

where $I[\cdot]$ is an indicator function, equal to 1 if $g(\boldsymbol{u}_j) \le 0$ and zero otherwise. The quality of the obtained approximation is determined evaluating the coefficient of variation of $P_{f,MCS}$; it is however well known that the MCS, especially in its crude form, requires an excessive computational effort.

Alternatively, a good estimate of (2) is obtained by approximating the target Limit State Function (LSF) $y = g(\boldsymbol{u})$, usually complicated and implicit, with an approximate model $\tilde{y} = \tilde{g}(\boldsymbol{u})$, called Response Surface (RS). Once the RS is built, it is no longer necessary to run demanding Finite Element Analyses, but we can use the surrogate model. In the following we will consider only problems where the limit state surface has neither peak or valleys, nor multiple design points.

From (3) it is seen that we are not interested to the value $y = g(\boldsymbol{u})$ of the LSF, but to its sign $z = \mathrm{sign}[g(\boldsymbol{u})]$, so that the points $\boldsymbol{u}_i$ belonging to the safe region have the value $z_i = +1$, and the points $\boldsymbol{u}_i$ belonging to the failure region have the value $z_i = -1$. It is easy to see that the building of a surrogate model $\tilde{y} = \tilde{g}(\boldsymbol{u})$ such that it satisfies only the sign constraints $z = \mathrm{sign}[g(\boldsymbol{u})]$, is equivalent to the building of a RS which models directly the LSS. Then, according to the RS Methodology, the estimated failure probability can be evaluated using (3), by substituting the indicator function $I[g(\boldsymbol{u}_j) \le 0]$ with $I[\tilde{g}(\boldsymbol{u}_j) \le 0]$.

A function which separates the points belonging to the safe set from the ones belonging to the failure set, is named classifier, since attributes a class ("safe" or "failure") to each point. The LSS $g(\boldsymbol{u}) = 0$ is the target classifier, while a RS which approximates the LSS, is able to classify correctly only a limited number of points. Clearly, because the RS works well, it is necessary that it classifies correctly the points at least in the region of probabilistic interest.

A Linear Response Surface based on SVM

Let be known a set of m sampling points $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$, while $y_1, y_2, \ldots, y_m$ and $z_1, z_2, \ldots, z_m$ be the corresponding values of the LSF $y_i = g(\boldsymbol{u}_i)$ and signs $z_i = \mathrm{sign}[g(\boldsymbol{u}_i)]$, respectively.

Suppose that the target LSS is linear, $g(\boldsymbol{u}) = \boldsymbol{a} \cdot \boldsymbol{u} - c$, then the sampling points $\boldsymbol{u}_i$ are linearly separable. Consider now the approximated LSS $\hat{g}(\boldsymbol{u}) = \boldsymbol{w} \cdot \boldsymbol{u} - b$, where $\boldsymbol{w}$ determines the orientation of the plane, while the scalar b determines the offset of the plane from the origin. Clearly, when the number of support points converges toward infinity $m \to \infty$, then the linear classification function becomes coinciding with the target LSS, i.e. $\boldsymbol{w} \to \boldsymbol{a}$, $b \to c$. Conversely, for a limited number of support points, there are infinite possible planes that classify the points correctly. Intuitively, a hyperplane that passes too close to the sampling points will be less likely to generalize well for the unseen data, while it seems reasonable to expect that a hyperplane that is farthest from all points will have better generalization capabilities. Given a set of m sampling points, the *margin* is defined as the minimum distance between points belonging to different classes. Therefore, the optimal separating hyperplane is the one maximizing the margin.

Recall from the elementary geometry that the distance δ_i of a point $\boldsymbol{u}_i$ from the hyperplane $\hat{g}(\boldsymbol{u}) = 0$ reads as $\delta_i = \left| \boldsymbol{w} \cdot \boldsymbol{u}_i - b \right| / \left\| \boldsymbol{w} \right\|$; noticing that $\hat{g}(\boldsymbol{u}) = \boldsymbol{w} \cdot \boldsymbol{u} - b = 0$ is invariant under a positive rescaling, we choose the solution for which the function $\hat{y} = \hat{g}(\boldsymbol{u})$ becomes one for the points closest to the boundary, i.e. $|\boldsymbol{w}\cdot\boldsymbol{u}_i - b| = 1$. The couple of hyperplanes $\hat{g}_s(\boldsymbol{u}) = \boldsymbol{w}\cdot\boldsymbol{u} - b = 1$, $\forall \boldsymbol{u} \in g(\boldsymbol{u}) > 0$ and $\hat{g}_f(\boldsymbol{u}) = \boldsymbol{w}\cdot\boldsymbol{u} - b = -1$, $\forall \boldsymbol{u} \in g(\boldsymbol{u}) \le 0$, are called *canonical hyperplanes* (or *support hyperplanes*). The distance from the closest points to the boundary $\hat{g}(\boldsymbol{u}) = 0$ is $\delta = 1/\|\boldsymbol{w}\|$, and the margin becomes $M = 2/\|\boldsymbol{w}\|$, as shown in Figure 1(a).

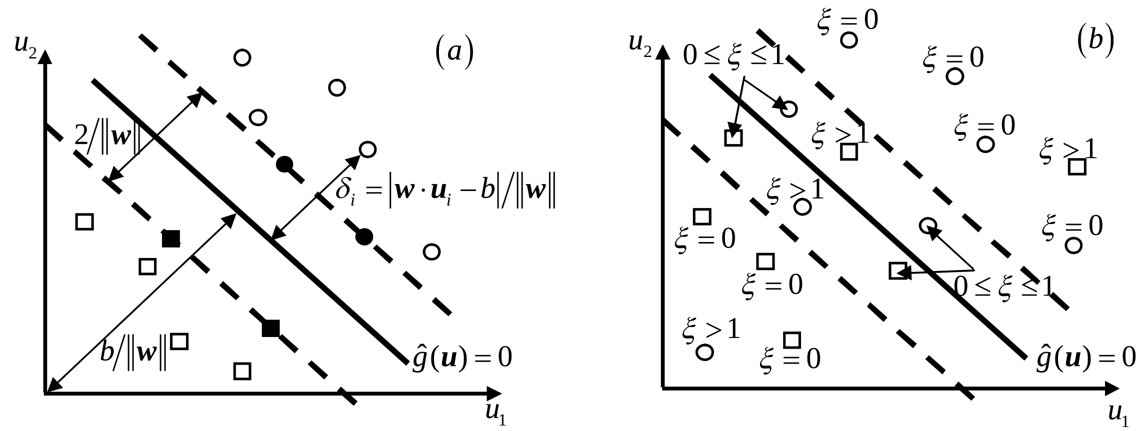

Figure 1. (a) Linear SVM, (b) Non-linearly separable SVM

Maximizing the margin is equivalent to minimize $\|\boldsymbol{w}\|/2$, giving rise to the following Quadratic Programming (QP) problem

$$\begin{cases} \min\limits_{\boldsymbol{w},b} \frac{1}{2}\|\boldsymbol{w}\|^2 \\ s.t. \quad z_i\left(\boldsymbol{w}\cdot\boldsymbol{u}_i - b\right) \ge 1, \quad i = 1,2,\ldots,m \end{cases} \tag{4}$$

where the inequality constraints are equivalent to $\boldsymbol{w}\cdot\boldsymbol{u}_i - b \ge 1$, $\forall \boldsymbol{u}_i \in g(\boldsymbol{u}_i) > 0$, and $\boldsymbol{w}\cdot\boldsymbol{u}_i - b \le -1$, $\forall \boldsymbol{u}_i \in g(\boldsymbol{u}_i) \le 0$. It is here noted that (4) is a standard convex optimization problem, so the uniqueness of the solution is guaranteed and moreover there are many robust algorithms that can effectively solve it.

Among the m sampling points, the *support vectors* are the m_{SV} points lying on the support hyperplanes $\boldsymbol{w}\cdot\boldsymbol{u}_i - b = \pm 1$; in Figure 1($a$) the support vectors are represented from the filled markers. It is seen that only the support vectors contribute to defining the optimal hyperplane, thus, the complete sampling set could be replaced by only the m_{SV} support vectors, and the separating hyperplane would be the same.

Suppose now that the LSS is non-linear, so that it is not possible to identify an hyperplane which correctly classifies all the sampling points. To this aim, we relax the constraints of (4) by introducing the slack variables $\xi_i \ge 0$, giving rise to $\boldsymbol{w}\cdot\boldsymbol{u}_i - b \ge 1 - \xi_i$, $\forall \boldsymbol{u}_i \in g(\boldsymbol{u}_i) > 0$ and $\boldsymbol{w}\cdot\boldsymbol{u}_i - b \le -1 + \xi_i$, $\forall \boldsymbol{u}_i \in g(\boldsymbol{u}_i) \le 0$. The variables ξ_i give a measure of the departure from the condition of correct classification. In particular, when $0 < \xi_i \le 1$ the point is well classified but falls inside the margin, while when $\xi_i > 1$ the point is not well classified. Finally, if $\xi_i = 0$ the point is correctly classified, see Figure 1(b). Under this hypothesis, the optimal separating hyperplane has maximum margin with minimum classification error. The optimization problem (4) becomes

$$
\begin{cases}
\min\limits_{\boldsymbol{w},b,\xi} & \frac{1}{2}\|\boldsymbol{w}\|^2 + \sum\limits_{i=1}^{m} \xi_i \\
s.t. & z_i\left(w \cdot u_i - b\right) \geq 1 - \xi_i, \quad i = 1, 2, \ldots, m \\
 & \xi_i \geq 0 \qquad\qquad\qquad\qquad i = 1, 2, \ldots, m
\end{cases} \tag{5}
$$

Outline of the procedure

It will be used the following iterative procedure:

1 Probabilistic transformation toward the *n*-dimensional standard normal space, as it is usually done in reliability analysis;

2 Evaluation of the design point $\boldsymbol{u}^*$ and of the corresponding reliability index $\beta = \|\boldsymbol{u}^*\|$, together with the FORM approximation $P_{f,FORM} = \Phi(-\beta)$, see Figure 2.

3 Choice of a set of sampling points $\boldsymbol{u}_k$, $k = 1, 2, \ldots$ along the design point direction $\boldsymbol{u}^* / \|\boldsymbol{u}^*\|$. Classification of failure and safe points belonging to the direction through $z_k = sign[g(\boldsymbol{u}_k)]$;

4 Building of a first LSVM model through (5), together with its margins;

5 Choice of a new point $\boldsymbol{u}_k$ inside the margin and its classification;

6 Building of the SVM model through (5) and evaluation of the estimated failure probability. Since the LSVM is a linear response surface we have $P_{f,LSVM} = \Phi(-\beta_{LSVM})$, β_{LSVM} being the distance of the LSVM from the origin of the standard normal space.

7 If the convergence on P_f is achieved stop, else go to step 5.

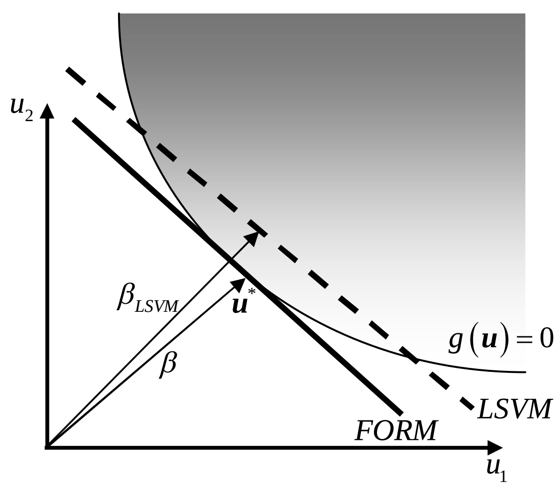

Figure 2. FORM and LSVM

Example 1 – Static Analysis of a 2-dof frame

As a first example, we applied the proposed approach to a shear type frame with two-stories, subjected to two deterministic concentrated loads, $F_1 = F_2 = \bar{F}$ with random stiffnesses $k_1 = \bar{k}_1(1+x_1)$ and $k_2 = \bar{k}_2(1+x_2)$, being $\bar{F}=1$, $\bar{k}_1 = \bar{k}_2 = 1$ and the fluctuations x_1, x_2 modelled as two independent Gaussian random variables with zero mean and standard deviation $\sigma = 0.20$ (Figure 3a). Be $X_2(x_1, x_2)$ the horizontal displacement of the second storey, and the considered limit state is $g(x_1,x_2) = X_2(x_1,x_2) - X_{2,\lim}$, with $X_{2,\lim} = 2.25$. The failure probability is $P_f = 9.427\times10^{-3}$ obtained using MCS, with an estimated coefficient of variation equal to $v_{P_f} = 1\%$, and required 1,050,848 analyses. The corresponding generalized reliability index is $\beta_G = \Phi^{-1}\left(1-P_f\right) = 2.348$.

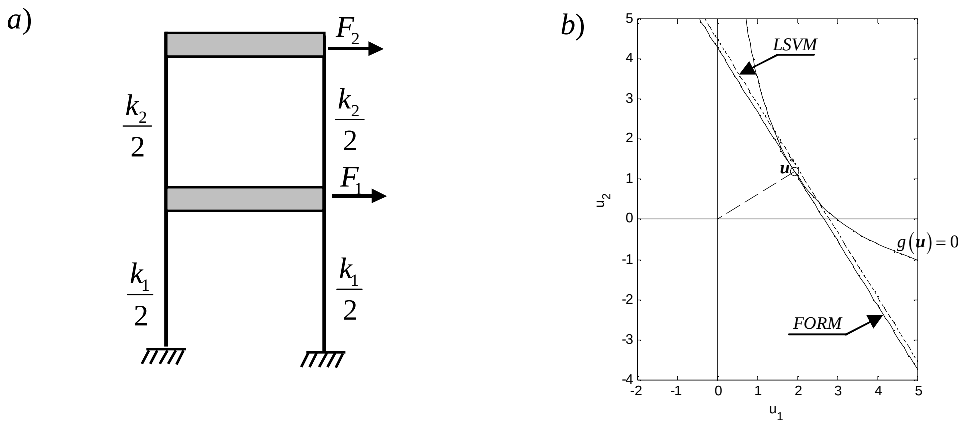

Figure 3. Example 1: (*a*) 2-dof shear-type frame, (*b*) LSS, FORM and LSVM

As shown in Figure 3(b), the LSS is nonlinear, and The FORM solution gives a failure probability $P_{f,FORM} = 11.593\times10^{-3}$, with a relative error $e_{FORM} = 22.97\%$. The LSVM achieves the convergence after 328 samples, obtaining $\beta = 2.344$ and corresponding failure probability $P_{f,LSVM} = 9.471\times10^{-3}$ (relative error $e_{LSVM} = 0.46\%$). In Figure 4(a) we represent the obtained failure probabilities in terms of the number of samples, while in Figure 4(b) the corresponding relative errors are shown.

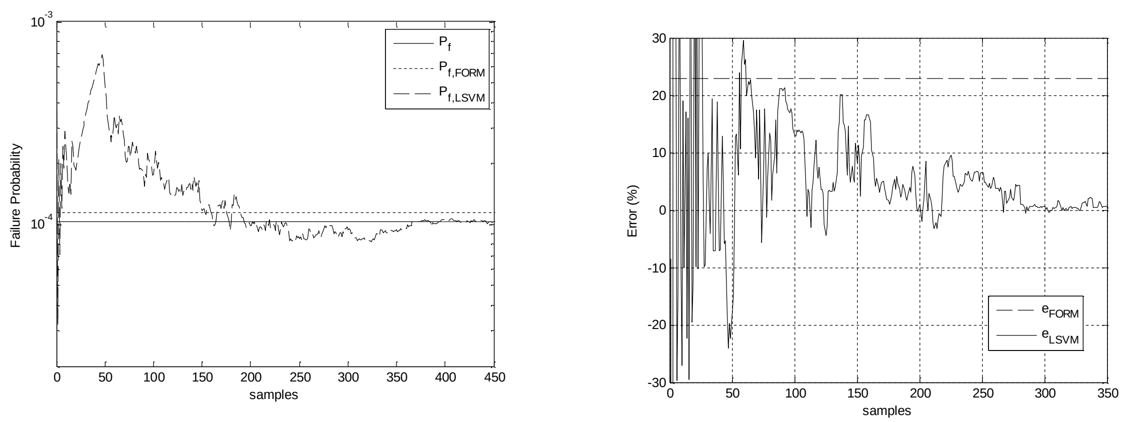

Figure 4. Example 1: (*a*) Failure Probability, (*b*) Relative error, in terms of the number of samples

Example 2 – Nonlinear random waves

Let $S_{XX}(\omega)$ a wave spectrum partitioned into N components of equal interval $\Delta\omega$. The second-order wave elevation is represented as (Longeut-Higgins 1963, Moarefzadeh & Melchers 2006)

$$\eta = \eta_1 + \eta_2 = \sum_{n=1}^{2N} \alpha_n u_n + \sum_{n=1}^{2N}\sum_{m=1}^{2N} \gamma_{nm} u_n u_m \tag{6}$$

where $u_1, u_2, \ldots, u_{2N}$ are normal standard random variables, while η_1 and η_2 are the first and second order terms of the nonlinear sea elevation,

$$\begin{gathered} \alpha_n = \sigma_n, \quad \alpha_{n+N} = 0, \\ \gamma_{nm} = \frac{1}{2g}\min\left(\omega_n^2, \omega_m^2\right)\sigma_n\sigma_m, \quad \gamma_{n+N,m+N} = -\frac{1}{2g}\max\left(\omega_n^2, \omega_m^2\right)\sigma_n\sigma_m, \quad \gamma_{n+N,m} = \gamma_{n,m+N} = 0 \\ n, m = 1, 2, \ldots, N \end{gathered} \tag{7}$$

being $\sigma_n = \sqrt{G_{\eta\eta}\left(\omega_n\right)\Delta\omega}$, with $G_{\eta\eta}\left(\omega\right)$ the spectral density function of the sea elevation. As a case study, we considered the JONSWAP spectrum with a shape factor of unity $G_{\eta\eta}\left(\omega\right) = a\omega^{-5}\exp[-1.25(\omega/\omega_p)^{-4}]$, where $\omega_p = 0.4$ rad/sec is the peak frequency, a is a scaling factor that is selected so that the area under the spectrum is 15 m^2 (Low 2013). The spectrum is divided into $N = 40$ harmonic components from 0.2 to 1.2 rad/sec. It is required to evaluate the probability that the wave elevation is greater than 15 *m*, that is $\mathrm{P}_f = \mathrm{Prob}[\eta \geq 15]$; the Limit State Function is $G\left(\boldsymbol{u}\right) = 15 - \eta\left(\boldsymbol{u}\right)$, while the number of basic random variables is $n = 2N = 80$. It is here noted that $\eta(\boldsymbol{u})$ is a quadratic function of the normal standard random variables, from which it follows the nonlinearity of the Limit State Surface.

The "exact" solution obtained with a MCS is $P_f = 1.035 \times 10^{-4}$ with a coefficient of variation $v_{P_f} = 5\%$ and it required 3,863,373 samples. The FORM approximation is $P_{f,FORM} = 1.15 \times 10^{-4}$ with a relative error $e_{FORM} = 11.11\%$, while the LSVM, after 355 samples, gives $P_{f,LSVM} = 1.021 \times 10^{-4}$ ($e_{LSVM} = 1.28\%$). In Figure 5 we represent the obtained failure probabilities and the corresponding relative errors in terms of the number of samples.

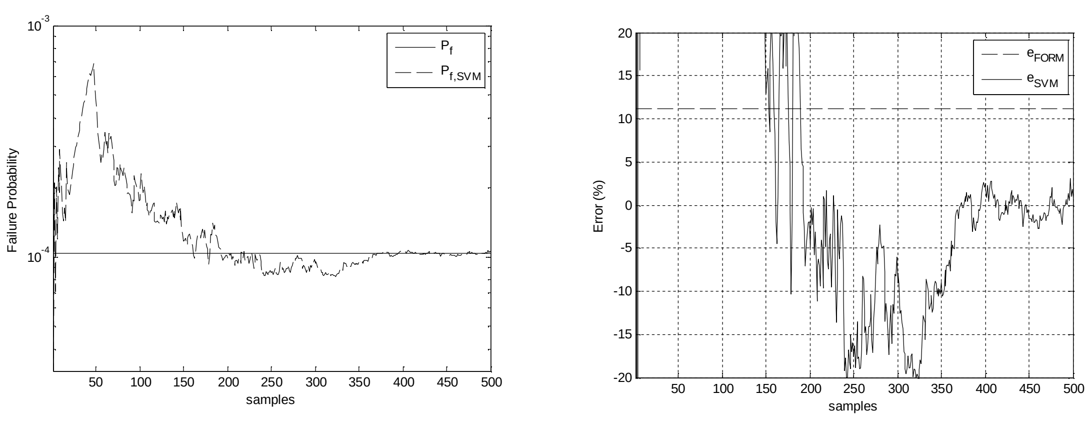

Figure 5. Example 2: (*a*) Failure Probability, (*b*) Relative error, in terms of the number of samples

Conclusions

FORM is a powerful tool for Structural Reliability Analysis, and in most cases of practical interest it gives a good approximation of the failure probability. However, for limit state surfaces which depart significantly from linearity around the design point, the FORM solution may be inadequate. In this paper we presented a Linear response surface based on SVM, called LSVM, which starting from the design point direction, allows an improvement of FORM, requiring a reduced number of sampling points. Two simple numerical examples showed the accuracy and effectiveness of the presented procedure.

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