Inverse scattering analysis of an elastic half space

by means of volume integral equation method

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Abstract

Inverse scattering analysis is dealt in this article by means of the fast volume integral equation method (Touhei, 2011). The purpose of this research is to reconstruct a fluctuation of the wave field in an elastic half space. Incident waves due to a point source and scattered waves observed at the free surface are used for reconstructing fluctuations of the wave field. The formulation uses the direct representation of the relationship between scattered waves and fluctuation of the wave field by the volume integral equation. The Tikhonov regularization method is also employed to improve the convergence properties of the solution. Several numerical examples are carried out to examine the effects of regularization parameter on the convergence properties of the iterative method.

Keywords: Volume integral equation, Elastic half space, Generalized Fourier transform, Tikhonov regularization method

Introduction

Inverse scattering analysis of an elastic half space is an important issue in the fields of earthquake engineering, site characterization and among others. This article presents a method for the inverse scattering analysis based on the fast volume integral equation method (Touhei, 2011). The proposed method in this article reconstructs the fluctuation of the wave field from the observed scattered waves at the free surface. The method is due to the advantage of the volume integral equation is in that it clarifies the relationship fluctuation of the medium and scattered wave field. During the formulation, the Tikhonov regularization method is also employed to the equation for the inverse scattering analysis to improve the convergence properties of the solution. Numerical calculations are carried out to examine the accuracy of the inverse scattering analysis as well as the effects of the regularization.

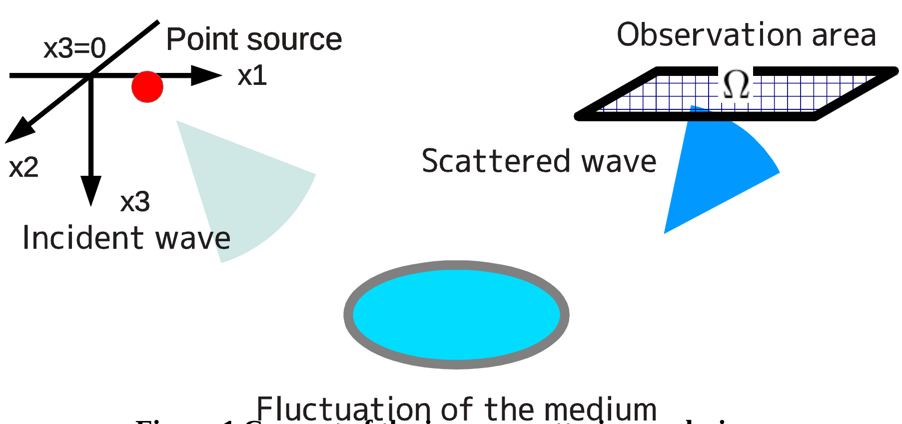

Figure 1 Concept of the inverse scattering analysis

Formulation

Figure 1 shows a concept of analyzed model dealt in this article. Fluctuations of the medium are embedded in an elastic half space. A point source is applied to a surface of an elastic half space to generate the incident wave. Scattered waves are caused due to the interaction between the incident wave and the fluctuations. Scattered waves are observed in a restricted area, which is denoted by Ω. The formulation in this article is carried out to reconstruct fluctuation of wave field from observed scattered wave. The wave field is characterized by Lamé constants as well as mass density as can be seen in the following equations

$$\lambda(\mathrm{x}) = \lambda_0 + \tilde{\lambda}(x) \tag{1}$$
$$\mu(x) = \mu_0 + \tilde{\mu}(x) \tag{2}$$
$$\rho(x) = \rho_0 + \tilde{\rho}\,(x) \tag{3}$$

where λ_0, μ_0 and ρ_0 are the background Lamé constants and mass density, and $\tilde{\lambda}$, $\tilde{\mu}$ and $\tilde{\rho}$ are their fluctuations, respectively. Let the time factor of the wave field is e^{-iwt}, where ω is the circular frequency and t is the time.

The equation of wave propagation is defined by

$$L_{ij}(\partial_1, \partial_2, \partial_3)\mathrm{u}_j\ (x) = N_{ij}(\partial_1, \partial_2, \partial_3, x)u_j(x) - F_i \tag{4}$$
$$P_{ij}(\partial_1, \partial_2, \partial_3, x)u_j(x) = 0, \quad (at\ \ x_3 = 0) \tag{5}$$

where δ_{ij} is Kronecker's delta, F is the amplitude of a point source, and L_{ij}, N_{ij} and P_{ij} are differential operators constructed by Lamé constants as well as mass density. The explicit forms of the operators L_{ij}, N_{ij} and P_{ij} are expressed as

$$L_{ij}(\partial_1, \partial_2, \partial_3) = (\lambda_0 + \mu_0)\partial_i\partial_j + \delta_{ij}\mu_0\partial_k^2 + \delta_{ij}\rho_0\omega^2 \tag{6}$$

$$\begin{aligned} N_{ij}(\partial_1, \partial_2, \partial_3, x) = -\left(\tilde{\lambda}(x) + \tilde{\mu}(x)\right)\partial_i\partial_j - \delta_{ij}\tilde{\mu}(x)\partial_k^2 - \partial_i\tilde{\lambda}(x)\partial_j \\ -\delta_{ij}\partial_k\tilde{\mu}(x)\partial_k - \partial_j\tilde{\mu}(x)\partial_i - \delta_{ij}\tilde{\rho}\,(x)\omega^2 \end{aligned} \tag{7}$$

$$P_{ij}(\partial_1, \partial_2, \partial_3, x) = \begin{bmatrix} \mu(x)\partial_3 & 0 & \mu(x)\partial_1 \\ 0 & \mu(x)\partial_3 & \mu(x)\partial_2 \\ \lambda(x)\partial_1 & \lambda(x)\partial_2 & (\lambda(\mathrm{x}) + 2\mu(x))\partial_3 \end{bmatrix} \tag{8}$$

The Green's function of the equation of wave propagation is defined as

$$L_{ij}(\partial_1, \partial_2, \partial_3)G_{jk}(x, y) = -\delta_{ik}\delta(x - y) \tag{9}$$
$$P_{ij}^{(0)}(\partial_1, \partial_2, \partial_3, x)G_{jk}(x, y) = 0, \quad (at\ \ x_3 = 0) \tag{10}$$

where G_{jk} is the Green's function, $\delta(\cdot)$ is the Dirac delta function, and $P_{ij}^{(0)}$ is as follows

$$P_{ij}^{(0)}(\partial_1, \partial_2, \partial_3, x) = \begin{bmatrix} \mu_0\partial_3 & 0 & \mu_0\partial_1 \\ 0 & \mu_0\partial_3 & \mu_0\partial_2 \\ \lambda_0\partial_1 & \lambda_0\partial_2 & (\lambda_0 + 2\mu_0)\partial_3 \end{bmatrix} \tag{11}$$

The volume integral equation for the present problem is defined by

$$u_i^S(x) = -\int_{\mathbb{R}_+^3} G_{ij}(x,y)N_{jk}(\partial_1,\partial_2,\partial_3,y)u_k^I(y)\,dy - \int_{\mathbb{R}_+^3} G_{ij}(x,y)N_{jk}(\partial_1,\partial_2,\partial_3,y)u_k^S(y)\,dy \tag{12}$$

where u_i^S and u_k^I are the scattered and incident wave field, respectively. In order to carry out the inverse scattering analysis, let Eq. (12) be modified into

$$u_i^S(x) = -\int_{\mathbb{R}_+^3} G_{ij}(x,y)M_{jk}(\partial_1,\partial_2,\partial_3,y)q_k(y)\,dy \tag{13}$$

where q_k is the states vector of the fluctuation whose components are

$$q_k(x) = (\tilde{\lambda}(x), \tilde{\mu}(x), \tilde{\rho}\,(x))^T \tag{14}$$

and M_{jk} is the differential operator constructed by incident wave field. Note that the Born approximation is applied to Eq. (12) to obtain Eq. (13) for linearizing equation. In addition, the generalized Fourier transform and inverse transforms (Touhei, 2011) are employed to Eq. (13) such as

$$u_i^S(x) = -U_{ij}^{-1}\hat{h}(\xi)U_{jk}M_{kl}(\partial_1,\partial_2,\partial_3,y)q_l(y) \tag{15}$$

where U , U^{-1} are the operators of the generalized Fourier transform and inverse transforms, respectively, and $\hat{h}$ is the Green's function in wavenumber domain. Equation (15) shows the relationship between the scattered wave field $u_i^S(x)$, ($x \in \mathbb{R}_+^3$) and the fluctuation of the medium. We have to, however, restrict the range of $x \in \mathbb{R}_+^3$ for $u_i^S(x)$, because we observe the scattered waves at the restricted area of Ω at the free surface. To restrict the range, let us introduce the definition function as follows

$$\chi_\Omega(x) = \begin{cases} 1 & (x \in \Omega) \\ 0 & (otherwise) \end{cases} \tag{16}$$

and apply it to Eq. (15) such that

$$\chi_\Omega(x)u_i^S(x) = -\chi_\Omega(x)U_{ij}^{-1}\hat{h}(\xi)U_{jk}M_{kl}(\partial_1,\partial_2,\partial_3,y)q_l(y) \tag{17}$$

The Tikhonov regularization method is also employed to stabilize the convergence property of solution. The result of the application of the Tikhonov regularization method is

$$A_{li}^*u_l^S(x) = [\alpha\delta_{il} + A_{ji}^*A_{jl}]q_l(y) \tag{18}$$

where A_{li}^* is the adjoint operator for A_{jl}, and α is the regularization parameter.

Numerical examples

Several numerical examples are shown in this section to examine the effects of regularization parameter on convergence properties of the solution. Setting an adequate Tikhonov parameter is very important to have accurate result of the inverse scattering analysis. In general, a larger Tikhonov parameter is known to improve the convergence properties of iterative procedures, while that makes the accuracy of the reconstructed results worse. There are two cases for numerical examples, one is that the analysis is carried out without regularization, namely $\alpha = 0.0$, and the other is that the regularization parameter is set at $\alpha = 1.0 \times 10^{-8}$. Figure 2 shows the target model of the inverse scattering analysis for numerical examples. The fluctuation of the wave field is embedded in the form of a cube at depth 5km from the free surface whose size is 3km on a side. The amplitude and excitation frequency of the point source are 10^7kN and 1.0Hz, respectively, and direction of the force is vertical. Figure 3 shows the observed data of the scattered waves which are obtained from the forward analysis. The background and fluctuation Lamé constants are (λ_0, μ_0)=(4.0,2.0)[GPa] and $(\tilde{\lambda}, \tilde{\mu})$=(0.1,0.1)[GPa], and their mass density ρ=2.0[g/cm^3] and $\tilde{\rho}$ =0.0[g/cm^3], respectively. Figure 4 shows the target of the fluctuation of the wave field.

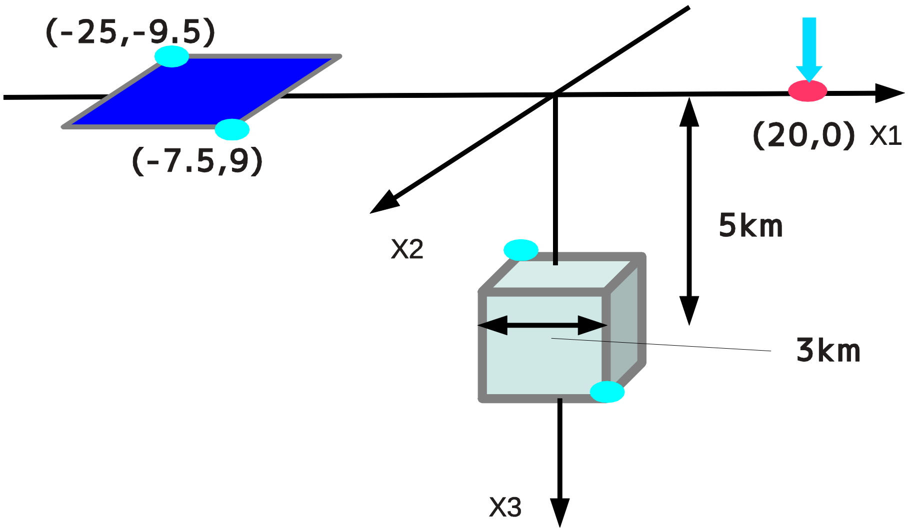

Figure 2 Target model

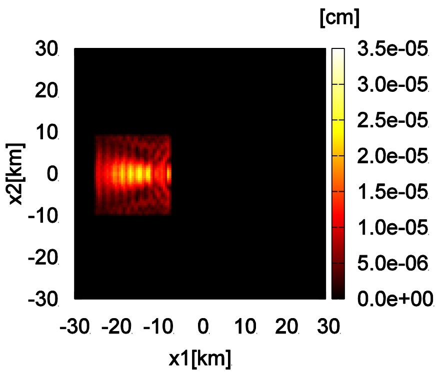

Figure 3 Observed data

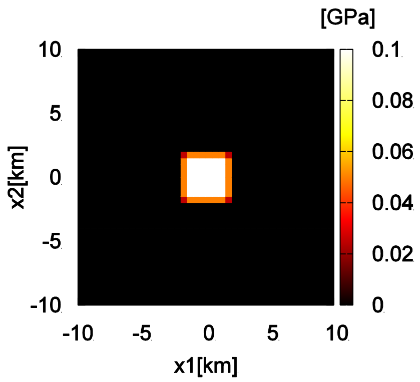

Figure 4 Target of the fluctuation of the wave field

Figures 5 and 6 show the convergence properties of the solution in case that Bi-CGSTAB method is employed for the solver. In the figures, horizontal axis denotes the number of iterations and vertical axis denotes the relative errors. According to Fig. 5, relative errors don't decrease even if the numbers of iterations increase. On the other hand, according to Fig. 6, the relative errors decrease as the numbers of the iterations increase. Therefore, the Tikhonov regularization method is effective for improving the convergence of the solution. Figures 7 and 8 show the results of the reconstruction of $\tilde{\mu}$ at depth $x_3 = 5.0$[km]. It is found from Fig. 7 that the reconstruction results overestimate the target model of the fluctuation shown in Fig. 4. However, comparing between the target model and Fig. 8, the fluctuation is found to be close to that of the target model. From the above results, the convergence properties are improved by the Tikhonov regularization method. Future tasks for inverse scattering analysis are to improve the accuracy of the estimation of the spatial spreads of the fluctuation.

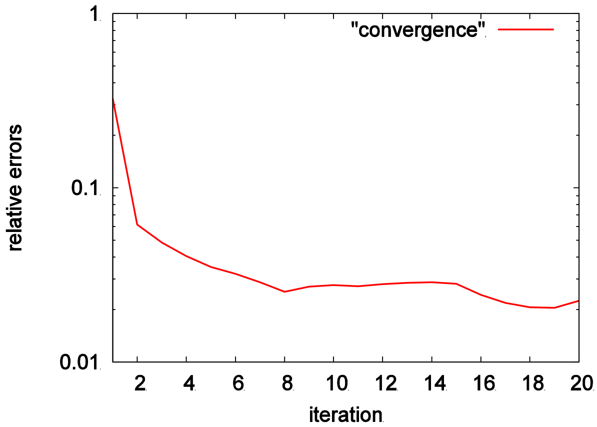

Figure 5 Convergence ($\alpha = 0.0$)

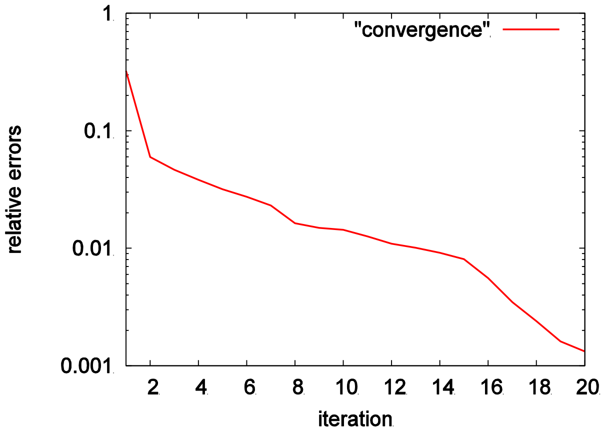

Figure 6 Convergence ($\alpha = 1.0 \times 10^{-8}$)

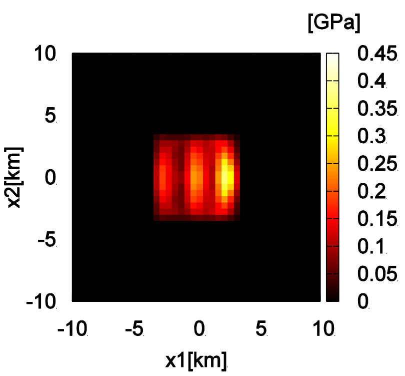

Figure 7 Reconstruction of the fluctuation $\tilde{\mu}$ (α=0.0)

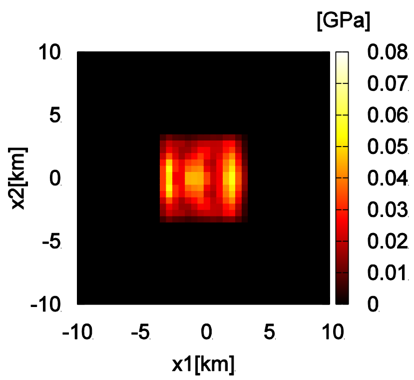

Figure 8 Reconstruction of the fluctuation $\tilde{\mu}$ ($\alpha = 1.0 \times 10^{-8}$)

Conclusion

Inverse scattering analysis was carried out based on the fast volume integral equation method in this article. The Tikhonov regularization method was employed to examine the convergence properties of the solution of the inverse scattering analysis. According to the numerical examples, the Tikhonov regularization method was effective to improve the convergence properties. For the future, however, accuracy of the reconstruction of the results has to be improved.

References

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