APCOM & ISCM
11-14th December, 2013, Singapore

Design Optimization of Structures using A Nodal Density-Based SIMP Method

***Y. Wang, Z. Luo, and N. Zhang**

School of Electrical, Mechanical and Mechatronic System, University of Technology Sydney
Sydney, NSW 2007, Australia

*Corresponding author: Yu.Wang-11@student.uts.edu.au

This paper proposes an alternative topology optimization method for the optimal design of continuum structures, which involves a multilevel nodal density-based approximant based on the concept of conventional SIMP (solid isotropic material with penalization) model. First, to construct a material density field with global smoothness over the design domain, a family of Shepard interpolation scheme is applied as a non-local nodal density interpolation. The new nodal density field possesses non-negative and range-bounded properties to ensure a physically meaningful approximation of topology optimization design. Second, the density variables at the nodes of finite elements are used to interpolate elemental densities, as well as corresponding element material properties. In this way, the nodal density field by using the non-local Shepard function method is transformed to a practical elemental density field via a local interpolation with the elemental shape function. The low-order finite elements are utilized to evaluate the displacement and strain fields, due to their numerical efficiency and implementation easiness. So, the proposed topology optimization method is expected to be efficient in finite element implementation, and effective in the elimination of numerical instabilities, e.g. checkerboards and mesh-dependency. A benchmark numerical example in topology optimization is employed to demonstrate the effectiveness of the proposed method.

Keywords: Topology optimization method, SIMP, nodal density-based approximant, Shepard function

Introduction

In the area of structural optimization, topology optimization has experienced considerable development over the past two decades with a wide range of engineering applications (Bendsøe and Sigmund 1999). Topology optimization is essentially a systematic design methodology, which involves a numerical process to iteratively re-distribute a given amount of material inside the design domain subject to loads and boundary conditions, until a prescribed design objective is optimized under specified design constraints. Topology optimization has been recognised as the most promising but the most challenging approach in the conceptual stage of structural optimization. Many different methods have been developed for topology optimization of structures, including the homogenization method (Bendsøe and Kikuchi 1988), SIMP method (Zhou and Rozvany 1991; Mlejnek 1992; Bendsøe and Sigmund 1999), and level set-based methods (Sethian and Wiegman 2000; Wang et al. 2003; Allaire et al. 2004; Luo et al. 2008).

Topology optimization of continuum structures essentially belongs to a set of integer programming problems with a large number of discrete(0, 1) design variables. More efficient gradient-based optimization algorithms cannot be directly applied to solve such large-scale optimization problems due to the well-known combinational problem. To this end, the homogenization and SIMP methods have been widely employed to relax the discrete topology optimization problem, allowing the design variables taking intermediate densities from 0 and 1. In doing so, the original optimization

problem is changed to a regularized optimization problem with range-bounded continuous design variables. In particular, the SIMP, as an extension of the homogenization method, has won great popularity in topology optimization of solid mechanics problem, due to its conceptual simplicity and implementation easiness. In SIMP-based topology optimization methods, a 'power-law' criterion (Bendsøe and Sigmund 1999) is usually applied to penalize the intermediate densities of elements, to ensure the solution of the regularized 0-1 design close to the original binary (0,1) design as much as possible. In addition, numerical schemes such as filtering schemes (Sigmund 2001; Bourdin 2001; Luo et al. 2005) are required to be incorporated to eliminate numerical instabilities, e.g. the checkerboards and mesh-dependence (Sigmund 2001), in order to make a physically meaningful solution for topology optimization of continuum structures.

It can be found that most of the current SIMP approaches are based on element-wise design variables (Bendsøe and Sigmund 2003), which means that both the topological geometry of material distribution and the physical fields would be evaluated via elemental density variables which are piecewise constant. In topology optimization of continuum structures, the element-wise variables may be one of the reasons for the occurrence of numerical instabilities (Sigmund 2001), including checkerboards, local minima, and mesh-dependency. Moreover, the element-based topology optimization method may lead to zigzag non-smooth boundary. As a result, to overcome the shortcomings of conventional element-wise SIMP methods, several alternative methods have been proposed. More recently, there have been several approaches based on point-wise design variables (Rahmatalla and Swan 2004; Matsui and Terada 2004; Guest et al. 2004; Paulino and Le 2009; Kang and Wang 2011; Wang et al. 2012). According to these approaches, nodal densities of the finite elements are normally considered as the design variables, and subsequent element material properties are obtained in terms of nodal densities via interpolation schemes. For instance, Rahmatalla and Swan (2004) proposed several options to implement the point-wise interpolation for material density fields, although "layering" or "islanding" type numerical instabilities occurred in the design. Matsui and Terada (2004) studied a so-called CAMD (continuous approximation of material distribution) method based on the homogenization method, in which element material densities were interpolated via the nodal density values (design variables). Guest and et al. (2004) introduced nodal design variables and projection schemes into topology optimization to achieve minimum length-scale control and checkerboard-free characteristics. Nodal material densities are regarded as the design variables to calculate the element material densities and element stiffness matrices. Paulino and Le (2009) proposed a kind of hybrid low-order finite elements, in which the nodes for design variable vector are inconsistent with the nodes for displacement vector. Kang and Wang (2011) proposed a nodal density based topology optimization method, in which a non-local Shepard interpolation scheme and higher-order elements are applied to eliminate the numerical instabilities such as checkerboards.

This paper will propose a multilevel nodal density-based approximation scheme for topology optimization of continuum structures, based on the concept of SIMP method. In this study, regular Q4 (quadrilateral four-node) finite elements are applied to evaluate the displacement field vector, and the nodal densities of each Q4 element are considered as design variables. A family of Shepard functions is employed to implement a non-local density approximant with enhanced smoothness over the entire design domain. At the same time, nodal design variables are used to evaluate practical material properties of the finite elements.

Non-local Nodal Density Approximant

The regular Q4 (4-node quadrilateral) element is considered for all implementations in this paper. A family of Shepard interpolation scheme is employed in this study in a form where performed as a non-local nodal density interpolation to construct a material density field with global smoothness

over the entire design space. Via this interpolation scheme, the density field includes both the contribution of the design variables of nodes with one element and the nodes related to the neighbouring nodes within an influential support. The Shepard interpolation method is introduced, firstly. Let $\varphi_i (i = 1,2,\ldots \mathrm{n_H})$ denote a set of $\mathrm{n_H}$ non-negative data values at the associated sampling points $x_i = (X_i, Y_i)$ within the support radius r of an arbitrary point. (X_i, Y_i) defines the ith point location in the given Cartesian coordinate system. The approximation of the Shepard function method is stated as

$$\overline{\varphi}(x) = \sum_{i=1}^{\mathrm{n_H}} \Theta(x) \varphi_i \tag{1}$$

Where $\mathrm{n_H}$ is the number of the nodes that is within the support radius r of the ith point. The Shepard function $\Theta_i(x)$ is expressed as a normalized formulation

$$\Theta_i(x) = \frac{\omega_i(x - x_i)}{\sum_{j=1}^{\mathrm{n_H}} \omega_j(x - x_j)} \tag{2}$$

$\omega_i(x - x_i)$ is the weight function, in the study which is a radially linear 'hat' function defined by [Bourdin (2001)], where $x - x_i$ is the radial distance from point x to x_i. Given $D_i(x) = x - x_i = \sqrt{(X - X_i)^2 + (Y - Y_i)^2}$, the weight function can be expressed as

$$\omega_i(x - x_i) = \frac{3}{\pi r^2} \max\left(0, 1 - \frac{D_i(x)}{r}\right) \tag{3}$$

The weight function is zero outside the domain of influential support, and decays linearly with the distance from the interest point x. It means that only nearby points are considered in computing any approximated value. In this way, the cost of computation is greatly saved by eliminating calculations with distant data points. The Shepard function $\Theta_i(x)$ possesses the properties:

(1) $\sum_{i=1}^{\mathrm{n_H}} \Theta_i(x) = 1$

(2) $\Theta_i(x) > 0$

It is apparent that the Shepard function has a mechanism similar to the smoothing effect of the density filtering schemes (Bourdin 2001; Luo et al. 2005). Meanwhile, the approximated values via the Shepard function are bounded between lower and upper values of the sampling points, which is essential property for a physically meaningful density field approximant in topology optimization.

Local Nodal Density Interpolation Scheme

Here, a local nodal density-based interpolation will be presented to convert the nodal design variables into the elemental densities. In this study, the standard Lagrangian shape function in the finite element method is used to interpolate elemental material properties. The local nodal density-based interpolant is stated as

$$\rho_e = \sum_{n=1}^{n_e} N_n \overline{\rho}_n = \sum_{n=1}^{n_e} \left(N_n \sum_{i=1}^{\mathrm{n_H}} \Theta_i(x_n) \rho_i \right) \tag{4}$$

where ρ_e is the elemental density, n_e is the number of the nodes of each element(4 in Q4 element), and N_n is the standard Lagrangian shape function. For simplicity, 2×2 Gaussian points are utilized to compute the practical material properties and determine the displacement field.

Furthermore, elemental material properties, such as Young's modules and elasticity constant, can then be expressed according to the proposed the multi-level approximation scheme, respectively, as

$$\mathrm{E}_e = \sum_{n=1}^{n_e} N_n \bar{E}_n = \sum_{n=1}^{n_e} \left[N_n \left(\sum_{i=1}^{\mathrm{n_H}} \Theta_i \left(x_n \right) \rho_i \right)^p E_0 \right] \tag{5}$$

$$\mathbf{D}_e = \sum_{n=1}^{n_e} N_n \bar{\boldsymbol{D}}_n = \sum_{n=1}^{n_e} \left[N_n \left(\sum_{i=1}^{\mathrm{n_H}} \Theta_i \left(x_n \right) \rho_i \right)^p \boldsymbol{D}_0 \right] \tag{6}$$

From the above discussion, it can be found that the proposed multi-level interpolation scheme can be easily implemented and is numerically effective, due to the application of the standard low-order rather than the higher-order finite elements. The obtained nodal variables via the interpolant are bounded between [0, 1], which is crucial for generating a physically meaningful density field.

Topology optimization problem

According to the well-established theory proposed by Bendsoe and Sigmund (2003), the following structural mean compliance design is used for design sensitivity analysis. In the study, the topology optimization problem is stated as

$$\begin{cases} \text{Minimize: } \mathrm{J} = \mathbf{U}^{\mathrm{T}} \mathbf{K} \mathbf{U} = \sum_{e=1}^{\mathrm{N_e}} \left(\rho_e \right)^{\mathrm{P}} \mathbf{u}_e^{\mathrm{T}} \mathbf{k}_e \mathbf{u}_e \\ \text{Subject to:} \begin{cases} \mathrm{V} - \mathrm{f_v} \mathrm{V}_0 \le 1 \\ \rho_{\min} \le \rho_e \le 1 \\ \mathbf{KU} = \mathbf{F} \end{cases} \end{cases} \tag{7}$$

Where the objective function J is to be minimized, **U** is the displacement vector and **K** is the global stiffness matrix, **F** is the external vector. $\mathrm{N_e}$ is the number of total elements, $\mathbf{u}_e$ is the elemental displacement vector, and $\mathbf{k}_e$ is the elemental stiffness matrix. p is the penalty factor ($\mathrm{p} = 3$ in this study). V is the actual material volume and $\mathrm{f_v}$ is the specified volume fraction ratio, and V_0 is the volume of the design domain. $\rho_{\min} = 0.0001$ is the lower bound of elemental densities to avoid singularity in numerical implementation. The derivative of the objective function with respect to the nodal design variables is expressed as

$$\frac{\partial \mathrm{J}(\rho_i)}{\partial \rho_i} = -\mathbf{U}^{\mathrm{T}} \frac{\partial \mathbf{K}}{\partial \rho_i} \mathbf{U} = -\sum_{\mathrm{e} \in n_i} \mathbf{u}_{\mathrm{e}}^{\mathrm{T}} \frac{\partial \mathbf{k}_{\mathrm{e}}}{\partial \rho_i} \mathbf{u}_{\mathrm{e}} \tag{8}$$

Where n_i is an index set containing indices of all the elements connected to the ith point (Kang 2011).

Numerical ExampleS and Discussions

To have an equitable assessment of the performance of the new interpolation scheme, the well-established minimum compliance problem is chosen (Bendsøe and Sigmund 2003). Fig.1 is the design domain of the cantilever beam with an aspect ratio of 2:1 corresponding to length over height. The left side of the domain is fixed as the Dirichlet boundary while the right side is treated as a non-homogenous Neumann boundary with a concentrated force F=1 vertically applied at the centre point. The objective function is to minimize the mean compliance, and mesh level (100 ×50) is adopted. As shown in Fig.1, the design domain is discretized with 100×50 Q4 elements and design variables are located at the corners of each element.

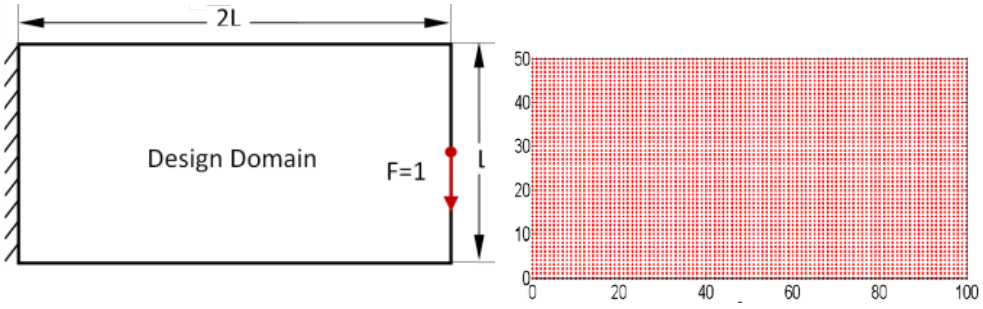

Figure 1. Design domain of the cantilever beam (Left)
Figure 2. FE nodes in the design domain(Right)

The topology optimization is converged after 319 iterations, and the overall structural mean compliance is minimized from 319.136 to 66.519. Fig.3 shows the discrete plots of the nodal densities at different design stages, in which the first figure is the initial design, the last figure is the optimal design, and the rest are the intermediate designs. The corresponding contours of the design variables are displayed in Fig.4 that shows the design gradually moves towards the lower limit 0.0001 (weak material) and upper limit 1 (solid material) during the optimization. So it can be seen that the topology optimization in this study can actually be regarded as a numerically iterative process to re-distribute a number of material density points in the design space until the convergencecriterion is satisfied.

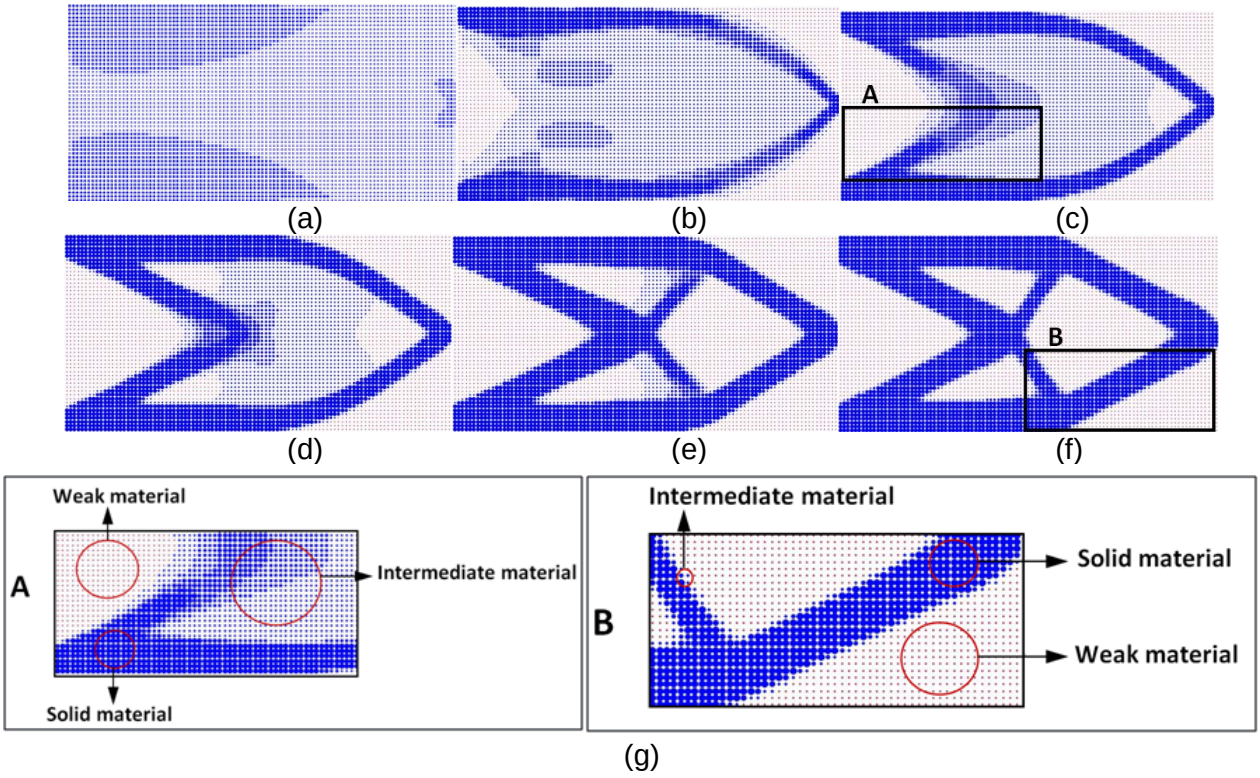

Figure 3. Topology plots: (a) initial design, (b)-(e) intermediate designs, and (f) final solution, the size of the node denoting the magnitude of nodal density values. (g) Local zoom-out plots

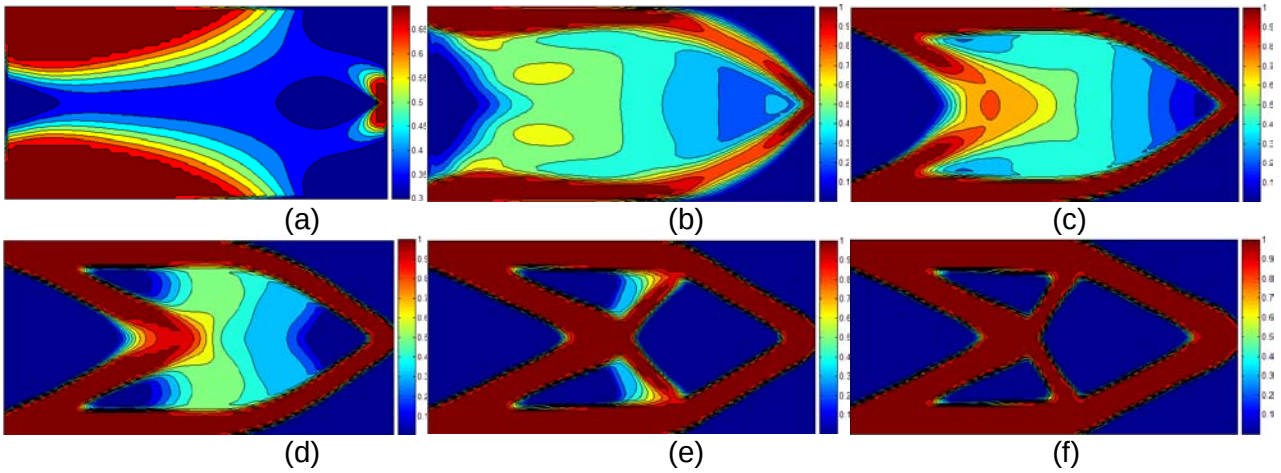

Figure 4. Contour plots of nodal design variables
(a) initial design, (b)-(e) intermediate designs, and (f) final solution

Fig.5 displays the topology plots of the element stiffness at different design stages of the optimization. The optimization using the proposed nodal density-based method can result in checkerboard-free design, and the so-called "layering" or "islanding" numerical phenomenon (Rahmatalla and Swan 2004) can also be eliminated by using the present Shepard function approximant. Fig. 6 shows curves of the objective function and the volume constraint over the iterations. It is noted that the first 75 iterations are mainly employed to implement topological optimization, and the rest iterations are used to adjust local structural shapes until a uniform distribution of the strain energy in the structure is achieved. Since the proposed method has been proved to be mesh-independent, it is possible to use a coarser finite element mesh to improve computational efficiency. According to the curve of constraint, the proposed method is well mass conservative.

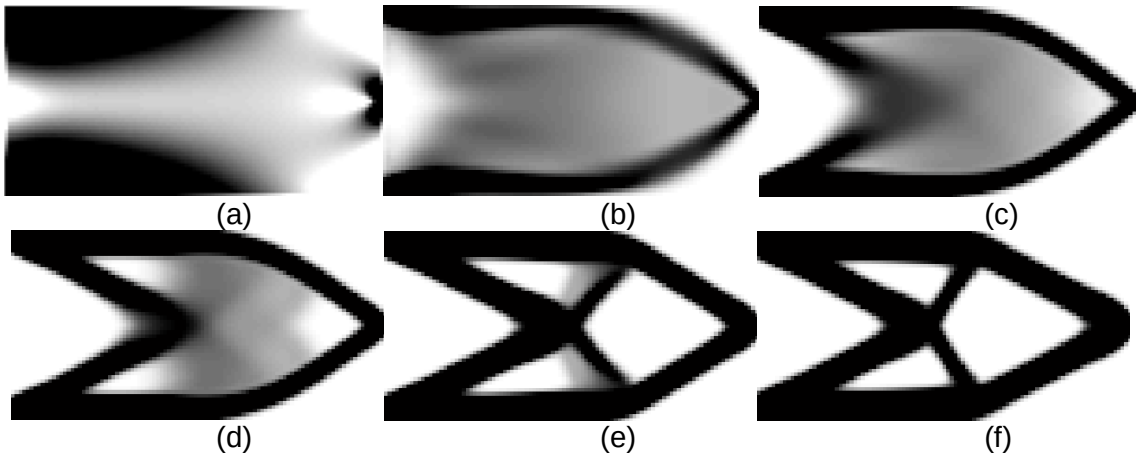

Figure 5. Topology plots of nodal design variables:
(a) initial design, (b)-(e) intermediate designs, and (f) final solution

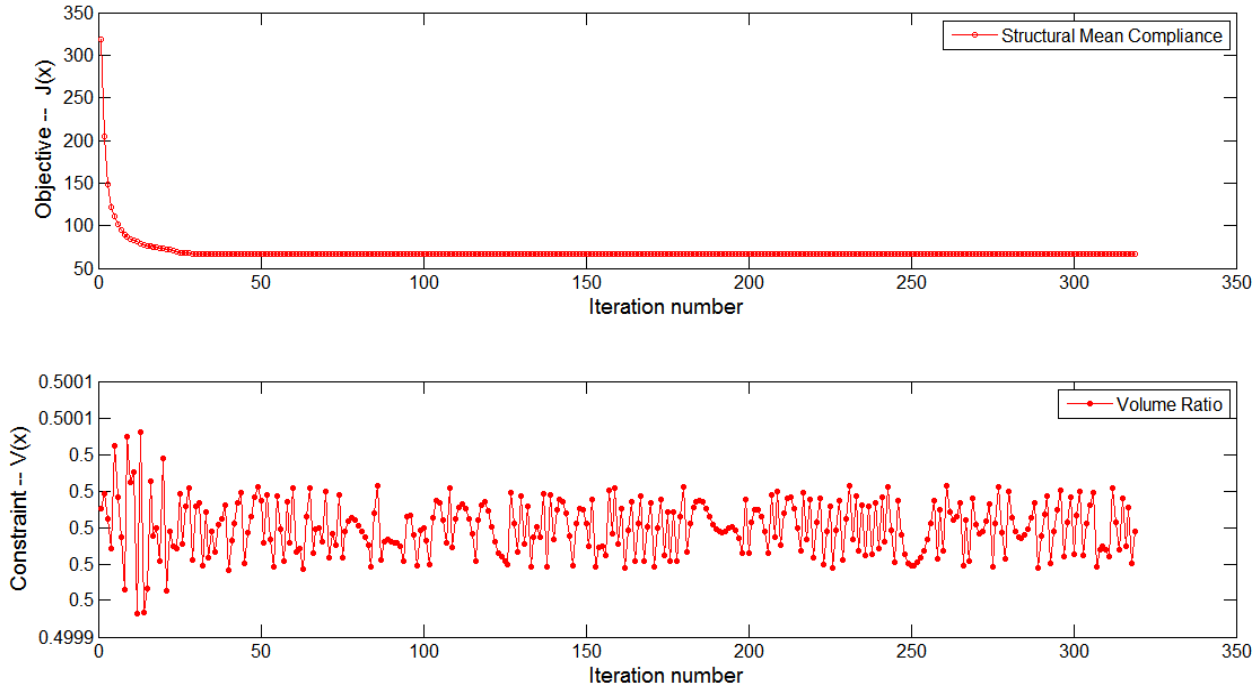

Figure 6. Iteration histories of objective function and volume constraint

Conclusions

In this paper, an alternative SIMP scheme is proposed for topological optimization of structures based on a multi-level Shepard function approximant. In this method, the nodal variables are considered as the design variables, to implement structural topology changes. A nodal density field with enhanced smoothness is constructed by using the original set of design variables via a non-local Shepard function method. The new set of nodal variables is applied to evaluate the practical material properties of finite elements, via a local interpolation scheme of the standard Lagrangian shape function. Therefore, instead of using the time-consuming higher-order elements, the lower-order finite elements can be easily employed to improve computational efficiency. The proposed topology optimization methodology is able to eliminate the typical numerical instabilities in the topology optimization of continuum structures. It is straightforward to extend the proposed multi-level topology optimization method to more advanced mechanics problems.

Acknowledgments

The research is partially supported by Chancellor's Research Fellowship (University of Technology, Sydney), and Australian Research Council, Discovery projects: DP1094451 and DP0988429.

References

Allaire, G., Jouve, F., Toader, A.M. (2004), Structural optimization using sensitivity analysis and a level-set method, *Journal of Computational Physics*, **194**, pp. 363-393.

Bendsøe, M.P., Kikuchi, N. (1988), Generating optimal topologies in structural design using a homogenization method, *Computer Methods in Applied Mechanics and Engineering*, **71**, pp. 197-224.

Bendsøe, M.P., Sigmund, O. (1999), Material interpolation schemes in topology optimization, *Archive of Applied Mechanics*, **69**, pp. 635-654.

Bendsøe, M.P., Sigmund, O. (2003), Topology optimization: Theory, Methods, and Applications, Springer, Berlin Heidelberg.

Bourdin, B. (2001), Filters in topology optimization, *International Journal for Numerical Methods in Engineering*, **50**, pp. 2143-2158.

Guest, J.K., Prevost, J.H., Belytschko, T. (2004), Achieving minimum length scale in topology optimization using nodal design variables and projection functions, *International Journal for Numerical Methods in Engineering*, **61**, pp. 238-254.

Kang, Z., Wang, Y. (2011), Structural topology optimization based on non-local Shepard interpolation of density field, *Computer Methods in Applied Mechanics and Engineering*, **200**, pp. 3515-3525.

Luo, Z., Chen, L.P., Yang, J.Z., Zhang, Y.Q., Abdel-Malek, K. (2005), Compliant mechanism design using multi-objective topology optimization scheme of continuum structures, *Structural and Multidisciplinary Optimization*, **30**, pp. 142–154.

Luo, Z., Wang, M.Y., Wang, S.Y., Wei, P. (2008), A level set-based parameterization method for shape and topology optimization, *International Journal for Numerical Methods in Engineering*, **76**, pp. 1-26.

Matsui, K., Terada, K. (2004), Continuous approximation of material distribution for topology optimization, *International Journal for Numerical Methods in Engineering*, **59**, pp. 1925-1944.

Mlejnek, H.P. (1992), Some aspects of the genesis of structures, *Structural and Multidisciplinary Optimization*, **5**, pp. 64-69.

Paulino, G.H., Le, C.H. (2009), Modified Q4/Q4 element for topology optimization, *Structural and Multidisciplinary Optimization*, **37**, pp. 255-264.

Rahmatalla, S.F., Swan, C.C. (2004), A Q4/Q4 continuum structural topology optimization implementation, *Structural and Multidisciplinary Optimization*, **27**, pp. 130-135.

Sethian, J.A., Wiegmann, A. (2000), Structural boundary design via level set and immersed interface methods, *Journal of Computational Physics*, **163**, pp. 489-528.

Sigmund, O. (2001), A 99 topology optimization code written in Matlab, *Structural and Multidisciplinary Optimization*, **21**, pp. 120-127.

Wang, Y., Luo, Z., Zhang, N. (2012), Topological optimization of structures using a multilevel nodal density-based approximant, *CMES: Computer Modeling in Engineering & Sciences*, **84**(3): pp. 229-252.

Wang, M.Y., Wang, X.M., Guo, D.M. (2003), A level set method for structural topology optimization, *Computer Methods in Applied Mechanics and Engineering*, **192**, pp. 227–246.

Zhou, M., Rozvany, G.I.N. (1991), The COC algorithm, Part II: topological, geometry and generalized shape optimization, *Computer Methods in Applied Mechanics and Engineering*, **89**, pp. 197-224.