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Displacement Function Method of Space Problem for Transversely Isotropic Foundation Based on Damage Theory

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Abstract

This paper investigates the solution of space problem for transversely isotropic foundation based on damage theory. Firstly, the modified Galerkin's displacement function is introduced into the basic equations of the transversely isotropic elastomer space problem. Secondly, employing Hankel integration transform and Bessel function theory, we obtain a general three-dimensional solution in the image field by the displacement function method. Finally, by means of the Hankel integration inversion shift theory, the fundamental expressions of strain and stress in the transversely isotropic foundation are presented for different cases that the characteristic roots were equal or not. The solution could be used to solve some specific axisymmetric and asymmetrical problems in semi-infinite space under different boundary conditions.

Keywords: transversely isotropic foundation, damage, displacement function method, Hankel integral transform， stress.

1 Introduction

The research achievements of many researchers at home and abroad and a large number of engineering practice show that it is feasible and more representative that the cohesive soil foundation is simplified transversely isotropical elastic half-space model in many circs. Since the transversely isotropic foundation model was put forward, many domestic and foreign researchers have carried out in-depth research on the model. In 1940, a general solution of axisymmetric problem about the transversely isotropical body of revolution was firstly solved by Soviet scholar Lekhnitskii. And then, Elliott derived a particular solution about a three-dimensional problem that can be degenerated to Leknitskii solution in the axisymmetric condition by means of two harmonic functions in 1948. Based on the equilibrium equation of displacement expression. Eubanks and Sternberg obtained the Leknitskii solution systematically in 1954, and proved the completeness of Elliott solution. The detailed analytic solution expression of the surface and interior of the transversely isotropic half-space under point loads or circular loads was given by C.M.Gerrard(1980). And by referencing the three displacement functions, Pan and Chou(1979) made use of the Green's function method to obtain the whole expressions of fundamental solution of transversely isotropic semi-infinite body.

Interiorly, it is Hu haichang who studied that problem first. In 1953, he obtained the general solution of space problem of transversely isotropic elastic body by means of two potential functions, specifically discusses the problems of transversely isotropic semi-infinite elastic body, and got the fundamental solution when $s_1 \neq s_2$. Ding Haojiang, through the Hu Haichang's solution, used integral method to obtain the fundamental solution of axisymmetric problems of transversely isotropic materials, which can be directly degenerated to the fundamental solution of axisymmetric problems of isotropic materials. Dun zhiling and his graduates, by modifying Love displacement function in isotropic elastic mechanics, made use of displacement method and Hankel integral-transform theory to get the general solution of transversely isotropic foundation in image field. And by using the inversion of Hankel integral, the analytical solutions of displacement and stress of semi-infinite foundation under several common loads were obtained and the results of numerical calculation were given. Based on the basic equations of transversely isotropic elastic body and Hankel transform, the generation solutions of displacement and stress of space problem of layered transversely isotropy under non-axisymmetric loads were solved by using transfer matrix method(Chen guangjing and Zhao xihong, 1998). The Biot’s wave equations of transversely isotropic saturated poroelastic media excited by non-axisymmetrical harmonic source were solved by means of Fourier expansion and Hankel transform. Then the components of total stress in porous media are expressed with the solutions of Biot’s wave equations.(Zhang Yinke and Huang Yi, 2001).

Based on the theory of damage mechanics, the elastic damage of transversely isotropic foundation was considered and a general three-dimensional solution for transversely isotropic foundation in the image field was strictly obtained by the displacement function method. The Galerkin’s displacement function for isotropic elasticity was modified, and Hankel integration transform and Bessel function theory were employed in the solution. With the Hankel integration inversion shift theory, the analytical expressions of stress and displacement in the transversely isotropic foundation were presented.

2 Damage theory

Shen Zhujiang put forward that the damage characteristics of the material are described by the cementitious bar element. The linear elastic damage model is adopted to simulate the transverse isotropic soil. This model is a single-spring model, which is composed of cementitious bar and spring and includes a lot of cementitious node. The soil of the mechanical process is analyzed as a synthetic material including undisturbed and damage soil. With the increase of deformation, cementitious nodes will be gradually destroyed and mechanical properties of soil gradually deteriorate.

The effective stress in damage mechanics is called the equivalent stress in this article to distinguish the concept of effective stress in soil mechanics. According to Lemaitre equivalent strain principle, the stress-strain relationship of one-dimensional damaged soil is as follows: $\varepsilon = \sigma / \tilde{E} = \tilde{\sigma} / E = \sigma / E(1-D)$ （1）

Where $\tilde{\sigma}$ is equivalent stress, E is the elastic modulus of non-destructive materials, $\widetilde{E}$ is effective elastic modulus, and D is damage variable.

For the three-dimensional anisotropic damage, equivalent stress can be expressed as: $\widetilde{\sigma} = M(D) : \sigma$ （2）

In the formula, $M(D)$ is the damage effective tensor. considering the matrix is symmetric and can be reduced to an matrix of isotropic damage problem, so damage tensor $M(D)$ is elected: （3）

$$[M(D)] = \begin{bmatrix} \frac{1}{1-D_1} & & & & & \\ & \frac{1}{1-D_1} & & & & \\ & & \frac{1}{1-D_2} & & & \\ & & & \frac{1}{\sqrt{(1-D_1)(1-D_2)}} & & \\ & & & & \frac{1}{\sqrt{(1-D_1)(1-D_2)}} & \\ & & & & & \frac{1}{(1-D_1)} \end{bmatrix}$$

where D_1,D_2 respectively respect the damage variable of the horizontal plane and the vertical plane. The damage variables can be determined according to the literature[15].

3 The general solution of space problem for transversely isotropic foundation based on damage theory

3.1 The equivalent stress of transversely isotropic space problem in cylindrical coordinate

In cylindrical coordinate (r,θ,z), it is assumed that the z-axis is perpendicular to the isotropic plane of physical properties, so elasticity is isotropic in the plane (r,θ). Let u、v、w be the displacement of points along the three directions r， θ， z .The geometric equation of transversely isotropic space problem in cylindrical coordinate is as follows:

$$\{\varepsilon\}=\{\varepsilon_r,\varepsilon_\theta,\varepsilon_z,\gamma_{\theta z},\gamma_{rz},\gamma_{r\theta}\}^T=\left\{\frac{\partial u}{\partial r},\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta},\frac{\partial w}{\partial z},\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial \theta},\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r},\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\right\}^T \tag{4}$$

Therefore the relationship of stress components and displacement components of undamaged foundation can be expressed by the displacement in cylindrical coordinates:

$$\{\tilde\sigma\}=\{\tilde\sigma_r,\tilde\sigma_\theta,\tilde\sigma_z,\tilde\sigma_z,\tilde\tau_{rz},\tilde\tau_{r\theta}\}^T=[C]\{\varepsilon\}=[C]\left\{\frac{\partial u}{\partial r},\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta},\frac{\partial w}{\partial z},\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial \theta},\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r},\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\right\}^T \tag{5}$$

Where $[C]=\begin{bmatrix} c_{11} & c_{12} & c_{13} & & & \\ c_{12} & c_{11} & c_{13} & & & \\ c_{13} & c_{13} & c_{33} & & & \\ & & & c_{44} & & \\ & & & & c_{44} & \\ & & & & & c_{66}\end{bmatrix}$, $\lambda=E_2\big/\left[(1+\mu_1)(1-\mu_1-2n\mu_2^2)\right]$, $c_{11}=\lambda n(1-n\mu_2^2)$,

$c_{12}=\lambda n(\mu_1+n\mu_2^2)$, $c_{13}=\lambda n\mu_2(1+\mu_1)$, $c_{33}=\lambda(1-\mu_1^2)$, $c_{44}=G_2$, $c_{66}=E_1\big/[2(1+\mu_1)]$, $n=E_1/E_2$, E_1 and E_2 is the modulus of elasticity in a horizontal plane and a vertical plane respectively; μ_1 and μ_2 is the Poisson ratio in a horizontal plane and a vertical plane respectively; G_2 is the shear modulus in a vertical plane.

From Eq.(2), the stress of transversely isotropic foundation with damage can be obtained by the stress of undamaged foundation in the follow form as

$$\{\sigma\}=\begin{bmatrix} \tilde c_{11} & c_{12} & c_{13} & & & \\ c_{12} & \tilde c_{11} & c_{13} & & & \\ c_{13} & c_{13} & \tilde c_{33} & & & \\ & & & \tilde c_{44} & & \\ & & & & \tilde c_{44} & \\ & & & & & \tilde c_{66}\end{bmatrix}\left\{\frac{\partial u}{\partial r},\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta},\frac{\partial w}{\partial z},\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial \theta},\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r},\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\right\}^T \tag{6}$$

Where $\tilde c_{11}=\lambda n(1-n\mu_2^2)(1-D_1)$, $\tilde c_{33}=\lambda(1-\mu_1^2)(1-D_2)$,

$\tilde c_{44}=G_2\sqrt{(1-D_1)(1-D_2)}$, $\tilde c_{66}=E_1(1-D_1)\big/[2(1+\mu_1)]$

3.2 The Displacement Function Method of transversely isotropic space problem in cylindrical coordinate

When analyzing elastic semi-infinite body and non-axisymmetric problem of the thick plate in 1960, Ruki R introduced two displacement functions φ and ψ:

$$u=-\frac{\partial^2\varphi}{\partial r\partial z}-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\quad v=-\frac{1}{r}\frac{\partial^2\varphi}{\partial\theta\partial z}+\frac{\partial\psi}{\partial r}\quad w=2(1-\mu)\nabla^2\varphi-\frac{\partial^2\varphi}{\partial z^2}\tag{7}$$

These are the general expression of the isotropic spatial problems. And they can be used to solve the isotropic spatial problem in a variety of different boundary conditions specifically.

Based on damage theory, the relation between the displacement components and the displacement function for transversely isotropic spatial problem can be obtained through complex calculations (specific process can be seen in reference [6], [16] and [17]). And it is the revised Galerkin's displacement function.

$$u=-\frac{\partial^2 F}{\partial r\partial z}-\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\quad v=-\frac{1}{r}\frac{\partial^2 F}{\partial\theta\partial z}+\frac{\partial\varphi}{\partial r}\quad w=\left(\tilde{a}\nabla^2+\tilde{b}\frac{\partial^2}{\partial z^2}\right)F\tag{8}$$

where, $\tilde{a}=\frac{\tilde{c}_{11}}{c_{13}+\tilde{c}_{44}},\tilde{b}=\frac{\tilde{c}_{44}}{c_{13}+\tilde{c}_{44}}$, $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}+\frac{\partial^2}{r^2\partial\theta^2}$ is Laplace Operator. F and φ is two displacement function for transversely isotropic body. Hu Haichang proved that the displacement function, F and φ, represents a complete solution of the space problem.

What's more, F and φ should go for the following compatible equation.

$$\left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}+\frac{\partial^2}{r^2\partial\theta^2}+\frac{\partial^2}{\tilde{s}_1{}^2\partial z^2}\right)\cdot\left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}+\frac{\partial^2}{r^2\partial\theta^2}+\frac{\partial^2}{\tilde{s}_2{}^2\partial z^2}\right)F=0,\ \left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}+\frac{\partial^2}{r^2\partial\theta^2}+\frac{\partial^2}{\tilde{s}_3{}^2\partial z^2}\right)\varphi=0\tag{9}$$

Where $\tilde{s}_1=\sqrt{(\overline{c}_{13}-c_{13})(\overline{c}_{13}+c_{13}+2\tilde{c}_{44})/4\tilde{c}_{33}c_{44}}+\sqrt{(\overline{c}_{13}+\tilde{c}_{13})(\overline{c}_{13}-c_{13}-2\tilde{c}_{44})/4\tilde{c}_{33}\tilde{c}_{44}}$

$$\tilde{s}_2=\sqrt{(\overline{c}_{13}-c_{13})(\overline{c}_{13}+c_{13}+2\tilde{c}_{44})/4\tilde{c}_{33}\tilde{c}_{44}}-\sqrt{(\overline{c}_{13}+c_{13})(\overline{c}_{13}-c_{13}-2\tilde{c}_{44})/4\tilde{c}_{33}\tilde{c}_{44}}$$

$$\tilde{s}_3^2=\frac{\tilde{c}_{66}}{\tilde{c}_{44}}\quad\overline{c}_{13}=\sqrt{\tilde{c}_{11}\tilde{c}_{33}}\tag{10}$$

In order to obtain the expression of stress and displacement components, the derivative of the Eq. (8) for r， θ and z are substituted into Eq. (6).Then we can obtain the expression of the stress components in cylindrical coordinates.

$$\sigma_r=\frac{(c_{12}-\tilde{c}_{11})}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}+(ac_{13}-\tilde{c}_{11})\frac{\partial^3 F}{\partial r^2\partial z}+\frac{(c_{12}-\tilde{c}_{11})}{r^2}\frac{\partial\varphi}{\partial\theta}+\frac{(ac_{13}-c_{12})}{r}\frac{\partial^2 F}{\partial r\partial z}+\frac{(ac_{13}-c_{12})}{r^2}\frac{\partial^3 F}{\partial\theta^2\partial z}+bc_{13}\frac{\partial^3 F}{\partial z^3}$$

$$\sigma_\theta=\frac{(\tilde{c}_{11}-c_{12})}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}+(ac_{13}-c_{12})\frac{\partial^3 F}{\partial r^2\partial z}-\frac{(\tilde{c}_{11}-c_{12})}{r^2}\frac{\partial\varphi}{\partial\theta}+\frac{(ac_{13}-\tilde{c}_{11})}{r}\frac{\partial^2 F}{\partial r\partial z}+\frac{(ac_{13}-\tilde{c}_{11})}{r^2}\frac{\partial^3 F}{\partial\theta^2\partial z}+bc_{13}\frac{\partial^3 F}{\partial z^3}$$

$$\sigma_z=(a\tilde{c}_{33}-c_{13})\frac{\partial^3 F}{\partial r^2\partial z}+\frac{1}{r}(a\tilde{c}_{33}-c_{13})\frac{\partial^2 F}{\partial r\partial z}+\frac{1}{r^2}(a\tilde{c}_{33}-c_{13})\frac{\partial^3 F}{\partial\theta^2\partial z}+b\tilde{c}_{33}\frac{\partial^3 F}{\partial z^3}$$

$$\tau_{\theta z}=\frac{\tilde{c}_{44}}{r}(\tilde{b}-1)\frac{\partial^3 F}{\partial\theta\partial z^2}+\tilde{c}_{44}\frac{\partial^2\varphi}{\partial r\partial z}+\frac{\tilde{c}_{44}}{r}\tilde{a}\left(\frac{\partial^3}{\partial\theta\partial r^2}+\frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial^3}{\partial\theta^3}\right)F$$

$$\tau_{zr}=\tilde{c}_{44}(b-1)\frac{\partial^3 F}{\partial r\partial z^2}-\frac{\tilde{c}_{44}}{r}\frac{\partial^2\varphi}{\partial\theta\partial z}+\tilde{a}\tilde{c}_{44}\left(\frac{\partial^3}{\partial r^3}+\frac{1}{r}\frac{\partial^2}{\partial r^2}-\frac{1}{r^2}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^3}{\partial\theta^2\partial r}-\frac{2}{r^3}\frac{\partial^2}{\partial\theta^2}\right)F$$

$$\tau_{r\theta}=\tilde{c}_{66}\left(-\frac{2}{r}\frac{\partial^3 F}{\partial r\partial\theta\partial z}-\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial^2 F}{\partial\theta\partial z}+\frac{\partial^2\varphi}{\partial r^2}-\frac{1}{r}\frac{\partial\varphi}{\partial r}\right)\tag{11}$$

Eq. (8) and Eq. (11) are the general solution of transversely isotropic space problem and they are represented by F and φ .As can be seen from the expression, if displacement function, F and φ, is determined appropriately, we can determine the corresponding displacement components and stress components. Therefore, Hankel integration transform is introduced and displacement functions, F and φ, are expressed in the form of series as the following.

$$F=F(r,\theta,z)=\sum_{k=0}^{\infty}F_k(r,z)\cos k\theta,\quad \varphi=\varphi(r,\theta,z)=\sum_{k=0}^{\infty}\varphi_k(r,z)\sin k\theta \qquad (12)$$

After Eq. (12) put into Eq. (9), we obtain:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}-\frac{k^2}{r^2}+\frac{\partial^2}{\tilde{s}_1^2\partial z^2}\right)\cdot\left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}-\frac{k^2}{r^2}+\frac{\partial^2}{\tilde{s}_2^2\partial z^2}\right)F_k=0,\ \left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}-\frac{k^2}{r^2}+\frac{\partial^2}{\tilde{s}_3^2\partial z^2}\right)\varphi_k=0 \qquad (13)$$

Let $\bar{F}_k(\xi,z)=\int_0^\infty rF_k(r,z)J_k(\xi r)dr,\ \bar{\varphi}_k(\xi,z)=\int_0^\infty r\varphi_k(r,z)J_k(\xi r)dr$ (14)

Then using the Hankel transform and the nature of Bessel functions

$$\int_0^\infty\left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}-\frac{n^2}{r^2}\right)f(r)J_n(\xi r)dr=-\xi^2\bar{f}(\xi) \qquad (15)$$

After k-order Hankel transform is applied to, the Eq. (13) can be transformed into as follows:

$$\left(\frac{\partial^2}{\tilde{s}_1^2\partial z^2}-\xi^2\right)\left(\frac{\partial^2}{\tilde{s}_2^2\partial z^2}-\xi^2\right)\bar{F}_k=0,\ \left(\frac{\partial^2}{\tilde{s}_3^2\partial z^2}-\xi^2\right)\bar{\varphi}_k=0 \qquad (16)$$

The inversion formula of Hankel transform is :

$$F_k(r,z)=\int_0^\infty\xi\bar{F}_k(\xi,z)J_k(\xi r)d\xi,\quad \varphi_k(r,z)=\int_0^\infty\xi\bar{\varphi}_k(\xi,z)J_k(\xi r)d\xi \qquad (17)$$

After the Eq. (17) put into Eq. (11), the displacement function can be expressed as:

$$F_k(r,\theta,z)=\sum_{k=0}^{\infty}\int_0^\infty\xi\bar{F}_k(\xi,z)J_k(\xi r)\cos k\theta d\xi,\ \varphi_k(r,\theta,z)=\sum_{k=0}^{\infty}\int_0^\infty\xi\bar{\varphi}_k(\xi,z)J_k(\xi r)\sin k\theta d\xi \qquad (18)$$

According to the nature of Bessel functions, we can obtain its derivative express

$$\frac{d}{dr}J_k(\xi r)=\xi J_{k-1}(\xi r)-\frac{k}{r}J_k(\xi r),\ \frac{d^2}{dr^2}J_k(\xi r)=-\xi^2J_k(\xi r)+\xi\left[\frac{k+1}{2r}J_{k+1}(\xi r)+\frac{k-1}{2r}J_{k-1}(\xi r)\right] \qquad (19)$$

Using Eq. (19), we can calculate the derivative of the displacement Eq. (18) for r, θ and z. And after putting them into the Eq. (8) and Eq. (11) respectively, the general solution of transversely isotropic spatial problems within quadrants is obtained (only u 、 and $\tau_{\theta z}$ was listed).

$$u=-\sum_{k=0}^{\infty}\int_0^\infty\xi^2\frac{d\bar{F}_k}{dz}J_{k-1}(\xi r)\cos k\theta d\xi+\sum_{k=0}^{\infty}\frac{k}{r}\int_0^\infty\xi\left(\frac{d\bar{F}_k}{dz}-\bar{\varphi}_k\right)J_k(\xi r)\cos k\theta d\xi$$

$$\tau_{\theta z}=\sum_{k=0}^{\infty}\int_0^\infty\left[\frac{\tilde{c}_{44}}{r}(1-\tilde{b})k\frac{d^2\bar{F}_k}{dz^2}-\frac{\tilde{c}_{44}}{r}k\frac{d\bar{\varphi}_k}{dz}+\frac{\tilde{c}_{44}}{r}\tilde{a}k\xi^2\bar{F}_k\right]\xi J_k(\xi r)\sin k\theta d\xi+\tilde{c}_{44}\sum_{k=0}^{\infty}\int_0^\infty\xi^2\frac{d\bar{\varphi}_k}{dz}J_{k-1}(\xi r)\sin k\theta d\xi \qquad (20)$$

3.3 The displacement components and the stress components of space problems for transversely isotropic foundation

The Eq. (20) shows that, if the functions $\bar{F}_k$ and $\bar{\varphi}_k$ can be obtained, we can get the analytical expressions of displacement components and stress components of the elastic space problem that is transversely isotropic. According to the material characteristic roots of transversely isotropic body, s_1 and s_2, the situation can be divided into $s_1=s_2$ and $s_1\neq s_2$ when solving $\bar{F}_k$ and $\bar{\varphi}_k$ and getting their derivatives for z. Because the anisotropy of most foundation rock materials can be reflected by $s_1\neq s_2$, the solution of $\tilde{s}_1\neq\tilde{s}_2$ is only solved as follow.

When $\tilde{s}_1 \neq \tilde{s}_2$, the solution of ordinary differential equation of Eq. (16) is

$$\overline{F}_k(\xi,z)=A_\xi e^{-\tilde{s}_1\xi z}+B_\xi e^{-\tilde{s}_2\xi z}+C_\xi e^{\tilde{s}_1\xi z}+D_\xi e^{\tilde{s}_2\xi z},\overline{\varphi}_k(\xi,z)=E_\xi e^{-\tilde{s}_3\xi z}+F_\xi e^{\tilde{s}_3\xi z} \quad (21)$$

Put the Eq. (21) and its derivative for z into Eq. (20). when $\tilde{s}_1 \neq \tilde{s}_2$, the general expression of displacement components and stress components can be obtained (only u 、 and $\tau_{\theta z}$ were listed).

$$u=-\sum_{k=0}^{\infty}\int_0^{\infty}\frac{1}{2}\left[\left(-\tilde{s}_1Ae^{-\tilde{s}_1\xi z}-\tilde{s}_2Be^{-\tilde{s}_2\xi z}+\tilde{s}_1Ce^{\tilde{s}_1\xi z}+\tilde{s}_2De^{\tilde{s}_2\xi z}\right)+\left(E_\xi e^{-\tilde{s}_3\xi z}+F_\xi e^{\tilde{s}_3\xi z}\right)\right]J_{k-1}(\xi r)\cos k\theta d\xi$$
$$+\sum_{k=0}^{\infty}\frac{k}{r}\int_0^{\infty}\frac{1}{2}\left[\left(-\tilde{s}_1Ae^{-\tilde{s}_1\xi z}-\tilde{s}_2Be^{-\tilde{s}_2\xi z}+\tilde{s}_1Ce^{\tilde{s}_1\xi z}+\tilde{s}_2De^{\tilde{s}_2\xi z}\right)-\left(E_\xi e^{-\tilde{s}_3\xi z}+F_\xi e^{\tilde{s}_3\xi z}\right)\right]J_{k+1}(\xi r)\cos k\theta d\xi$$

$$\tau_{\theta z}=\sum_{k=0}^{\infty}\int_0^{\infty}\left\{\frac{\tilde{c}_{44}}{2}\left[\xi\left(1-\tilde{b}\right)\left(\tilde{s}_1{}^2Ae^{-\tilde{s}_1\xi z}+\tilde{s}_2{}^2Be^{-\tilde{s}_2\xi z}+\tilde{s}_1{}^2Ce^{\tilde{s}_1\xi z}+\tilde{s}_2{}^2De^{\tilde{s}_2\xi z}\right)\right]\right.$$
$$\left.+\tilde{s}_3\xi\left(E_\xi e^{-\tilde{s}_3\xi z}-F_\xi e^{\tilde{s}_3\xi z}\right)+\tilde{a}\xi\left(Ae^{-\tilde{s}_1\xi z}+Be^{-\tilde{s}_2\xi z}+Ce^{\tilde{s}_1\xi z}+De^{\tilde{s}_2\xi z}\right)\right\}J_{k+1}(\xi r)\sin k\theta d\xi \quad (22)$$
$$+\sum_{k=0}^{\infty}\int_0^{\infty}\left\{\frac{\tilde{c}_{44}}{2}\left[\xi\left(1-\tilde{b}\right)\left(\tilde{s}_1{}^2Ae^{-\tilde{s}_1\xi z}+\tilde{s}_2{}^2Be^{-\tilde{s}_2\xi z}+\tilde{s}_1{}^2Ce^{\tilde{s}_1\xi z}+\tilde{s}_2{}^2De^{\tilde{s}_2\xi z}\right)\right]\right.$$
$$\left.-\tilde{s}_3\xi\left(E_\xi e^{-\tilde{s}_3\xi z}-F_\xi e^{\tilde{s}_3\xi z}\right)+\tilde{a}\xi\left(Ae^{-\tilde{s}_1\xi z}+Be^{-\tilde{s}_2\xi z}+Ce^{\tilde{s}_1\xi z}+De^{\tilde{s}_2\xi z}\right)\right\}J_{k-1}(\xi r)\sin k\theta d\xi$$

The analytic expressions of displacement components and stress components expressed in the Eq. (22) apply to space issues for transversely isotropic foundation with injury under a variety of non-axisymmetric loads. For the specific space problem of transversely isotropic foundation, when it is known that the specific boundary conditions of transversely isotropic foundation and five independent engineering elastic constants , E_1 、 E_2 、 μ_1 , μ_2 , G_2 ,and damage variable D_1 、 D_2 , displacement and stress field distribution in the corresponding boundary conditions can be obtained through using the result.

4 The example analysis

In order to verify the validity of these theoretical approaches and indicate the impact on transversely isotropic foundation when damage considered, the engineering elastic constants of transversely isotropic foundation in the reference[9] was selected: $E_1 = 35.6\text{MPa}$, $E_2 = 20.8\text{MPa}$, $G_2 = 9.09\text{MPa}$, $\mu_1 = 0.299$, $\mu_2 = 0.146$, The foundation bears the circular uniform unidirectional horizontal load. Its load collection degree, p, is 10kPa. And the load radius r is 1m. Because of space limitations and time constraints, this numerical example just makes the comparative analysis of the shear stress component in the z-axis, $\tau_{\theta z}\big|_{r=0}$, for transversely isotropic foundation without injury. The mathematical software used is Matlab.

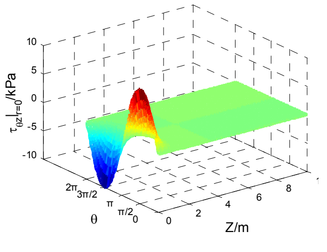

Figure1 Spatial distribution of shear stressas as *r*=0

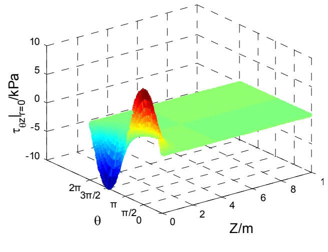

Figure2 Spatial distribution of shear stress as *r*=0 and D_1=0.2

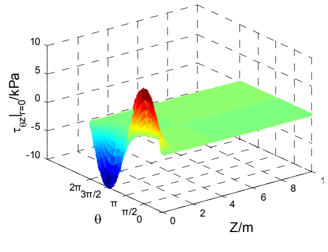

Figure3 Spatial distribution of shear stress as *r*=0 and D_1=0.2

Fig. 1, Fig. 2 and Fig. 3 show that the spatial distribution curves of the shear stress component in the z-axis, $\tau_{\theta z}\big|_{r=0}$, for transversely isotropic foundation with damage or not , are almost identical with the changing of the depth z and the polar angle θ. Then we can make an qualitative analysis of the difference of foundation shear stress $\tau_{\theta z}\big|_{r=0}$ in two different situations when choosing two kinds of the plane state.

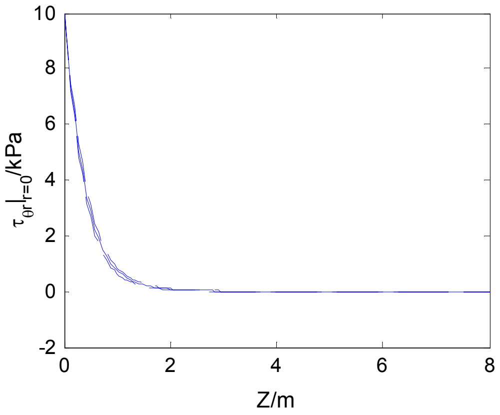

Figure 4 The comparison of shear stress as $\theta=\pi/2$ for two kind of foundation

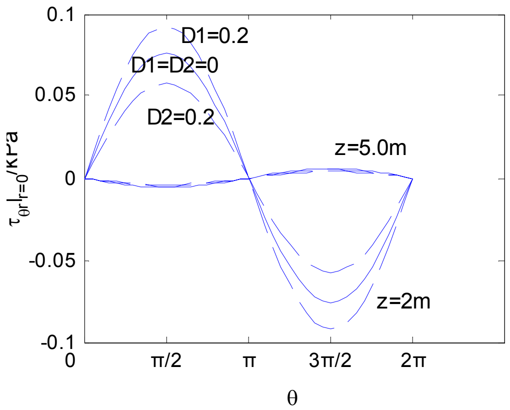

Figure 5 The comparison of shear stress as z=2m and z=5m for two kind of foundation

(Note: The dotted line represents damage foundation and solid line represents the foundation without damage in figures)

Fig. 4 and Fig. 5 show that: ①When θ fixed, with the z value increases, the foundation's shear stress $\tau_{\theta z}\big|_{r=0}$ in three cases is continuously attenuated . And when the depth z is smaller than 4m (equivalently 4 times of circular load's radius), the attenuation is in a fast rate. When z is greater than 4m, the attenuation speed becomes slower. And with the increasing of the depth, three shear stress values are close to equal ultimately; ② When z is constant, the foundation's shear stress $\tau_{\theta z}\big|_{r=0}$ in three cases, conforms with the sinusoidal line as θ changing. In view of D_1, The shear stress $\tau_{\theta z}\big|_{r=0}$ of transversely isotropic foundation is greater than that without damage, while it is smaller than that in view of D_2. And when the depth is to 4m(equivalently 4 times of circular load's radius), the shear stress is almost equal in three cases. Thus the damage has an impact on the transversely isotropic foundation's shear stress within the scope of 4 times of load's radius, but the impact can be negligible beyond the scope. This conclusion is identical to the results of practical engineering test, which can describe that the displacement function method here is reasonable and effective.

5 Conclusions

In this paper, it is a important work that the Galerkin displacement potential function has been revised. Base on this, a general solution of the transversely isotropic elastic space's problem is obtained in the image field, which is based on the elastic damage theory. Moreover, The analytic solution expression of displacement and stress components for the space problem of transversely isotropic foundation is derived. The solution doesn't only apply to solve both axisymmetric and non-axisymmetric problem of the transversely isotropic elastic space no matter whether or not the damage is considered., but also plays a important role in studying displacement and stress field of transversely isotropic foundation under specific boundary conditions.

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