Symbolic Calculation of Free Convection for Porous Material of Quadratic Heat Generation in a Circular Cavity

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Abstract We consider the two-dimensional problem of steady natural convection in a circular cavity with quadratic volumetric generation filled with porous material. We use Darcy's law for this cavity filled with porous material. The solution is governed by dimensionless parameter Darcy-Rayleigh number. The solution is expanded for low Darcy-Rayleigh number as was done by [1] and extended to 18 terms by computer. Analysis of these expansions allows the exact computation for arbitrarily accuracy up to 50000 figures. Although the range of the radius of convergence is small but Pade approximation leads our results to be good even for higher value of the similarity parameter.

Introduction

We consider the two-dimensional problem of steady natural convection in a circular cavity with linear volumetric heat flux filled with porous material. We use Darcy's law for this cavity filled with porous material. The solution is governed by dimensionless parameter Darcy-Rayleigh number. The solution is expanded for low Darcy-Rayleigh number as was done by [1], [2],[3] and extended to 18 terms by computer. Analysis of these expansions allows the exact computation for arbitrarily accuracy up to 50000 figures. Although the range of the radius of convergence is small but Pade approximation leads our result to be good even for higher value of the similarity parameter. This investigation is in a porous circular cavity driven by heating in the horizontal direction is our interest. We use Darcy's law for this cavity filled with porous material. The solution is governed by dimensionless parameter Darcy-Rayleigh number. The solution is expanded up to 18 terms by computer in powers of Darcy-Rayleigh number. Analysis of these expansions allows the exact computation for arbitrarily accuracy up to 50000 figures. Although the range of the radius of convergence is small but Pade approximation leads our result to be good even for higher value of the similarity parameter. The analysis yields a solution for all values of Rayleigh number from zero to finite value in a continuous fashion. The natural convection in a cavities filled with porous medium has received much attention because of the theoretical interest of [4], [5], [6], [7] for isothermal surfaces and [8] for isothermal inner and sinusoidal outer boundaries. As far as the numerical works for the case of isothermal surfaces a parameter study of diameter-ratio effects on the heat transfer coefficient was performed by [4] and angle of heating by [8] and other related problem of Natural Convection Non-Darcy effect by [9] and finally experimental work of [4]. The question of Hydrodynamic instability induces steady or oscillatory flows have been subject of many studies for example [4] and [10]. I hope the present exact solution of steady flow will help to answer such question more clearly. We recently have done the same present approach of symbolic calculation for laminar flow through heated horizontal pipe [3] and similar work done for concentrically spheres [11]. The similar approach was done for rectangular cavity by [12] and they found boundary limit solution from their regular perturbation for small Darcy-Rayleigh number.

Statement of Problem

The governing equations for porous materials with Darcy’s law can be written in dimensionless form as:

$$\nabla \dot{V} = 0 \qquad (1)$$

$$\dot{V} = -\frac{K}{\mu}(\nabla \dot{P} - \dot{\rho} g J) \qquad (2)$$

$$q''' + \dot{\rho} c(V.\dot{V})\dot{T} = k\nabla^2 \dot{T} \qquad (3)$$

$$\dot{\rho} = \rho_r (1 - \beta(T - T_r))$$

Where $\dot{V}$ is the velocity vector, $\dot{\rho}$ density, temperature, μ viscosity, $\dot{P}$ pressure J= $(\cos\theta, -\sin\theta)$ is a unit vector in the direction of gravity and $\lambda = \frac{g\beta\Delta T_q (KK) R}{\nu\alpha}$ is defined the internal Rayleigh number. The equations (5), (6) have been non-dimensionalzed by scaling length, velocity and temperature using the inner radius of cavity as the length scale, $\frac{k}{R(\rho c)_r}$, ΔT_q . ΔT_q Is calculated from uniform heat generation q''', $\Delta T_q \sim \frac{q''' R^2}{\alpha}$ is somehow certain gradient of temperature across the cavity, β the coefficient of volumetric expansion of the fluid) and g the acceleration due to gravity. Introducing the stream function in order to satisfy Eq. (1), eliminating the pressure from Eq. (2) and writing the resulting equation in cylindrical polar coordinates leads to the dimensionless equations as used in [13].

We consider the two-dimensional problem of steady natural convection Mansour [1]. We introduced cylindrical coordinates (r, θ) θ measured from upward vertical.

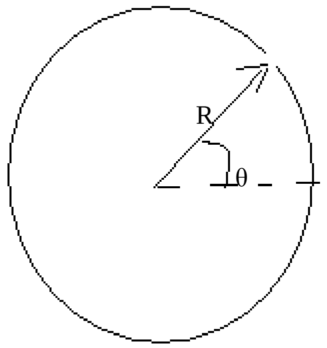

Fig.1 Geometry of the Problem

We mainly follow Mansour's notation [1]. Then the velocity component U, V, W in the fluid are functions of r and θ only. The continuity equation can be satisfied by introducing a Stokes stream function ψ for the cross flow The governing equations for porous materials with Darcy'law can be written in dimensionless form as:

$$\nabla^2 \psi = -\lambda \left(\sin\theta \frac{\partial T}{\partial r} + \frac{\cos\theta}{r} \frac{\partial T}{\partial \theta} \right) \qquad (4)$$

$$\nabla^2 T = \frac{1}{r}\left(-\frac{\partial \psi}{\partial \theta}\frac{\partial T}{\partial r} + \frac{\partial \psi}{\partial r}\frac{\partial T}{\partial \theta} \right) + 9r \qquad (5)$$

$$\nabla^2 = \frac{\partial^2}{\partial \mathbf{r}^2} + \frac{1}{\mathbf{r}}\left(\frac{\partial}{\partial \mathbf{r}} \right) + \frac{1}{\mathbf{r}^2}\left(\frac{\partial^2}{\partial \boldsymbol{\theta}^2} \right) \qquad (6)$$

The corresponding boundary conditions expressing the impermeability of the wall, the no-slip conditions, and imposed temperature distribution are respectively:

$\psi = 0, \qquad at\ r = 1$

$\partial\psi/\partial r = 0 \qquad at\ r = 1$

$T = 1 \qquad at\ r = 1,$

$\lambda = \frac{g\beta\Delta T_q (KK) R}{\nu\alpha}$ is the Rayleigh number. Coordinates are non-dimensional zed by using the radius of cavity as the length scale. Here ΔT_0 is imposed temperature difference (ν is kinematics viscosity, β the coefficient of volumetric expansion of the fluid) and g the acceleration due to gravity.

Series Derivation and Computer Extension

In this work we delegated the mounting algebra to the computer and, for our boundary condition (9) one can systematically improve on this approximation).

$$\begin{cases} T = T_0 + (\frac{\lambda}{32})\,T_1 + (\frac{\lambda}{32})^2 T_2 + (\frac{\lambda}{32})^3 T_3 + \ldots. \\ \psi = (\frac{\lambda}{32})\,\psi_1 + (\frac{\lambda}{32})^2 \psi_2 + (\frac{\lambda}{32})^3 \psi_3 + \ldots\ldots \quad (7) \end{cases}$$

We substitute the expansions (7) into our simplified equations (4) and (5) then equating like powers of K Gives for T_0 the equation:

$$\frac{\partial^2 T_0}{\partial r^2} + \frac{1}{r}\frac{\partial T_0}{\partial r} = 16r^2$$

Subject to boundary condition, It is easy to show that $T_0 = r^4$ and $\psi_0 = 0$. Therefore, the basic solution is the state of simple conduction. Substituting and equating like powers of λ yields this sequence of successive linear equations, together with boundary conditions

$$\psi_1 = \left(\frac{16}{3}\right) * r * (-1 + r^4) * Cos(\theta)$$

$$\psi_2 = (-\left(\frac{16}{45}\right)) * r^2 (4 - 5 * r^4 + r^8) Sin(2\theta)$$

$$\psi_3 = (\frac{1}{42525}) * (32 * (3 * r * (1517 - 1995 * r^2 + 665 * r^6 - 217 * r^{10} + 30 * r^{14})\ Cos(\theta) - 35 * r^3 * (-19 + 27 * r^4 - 9 * r^8 + r^{12}) * Cos(3\theta)))$$

$$\psi_4 = -((\frac{1}{7016625}) * (16 * (2 * r^2 * (-706043 + 800560 * r^2 + 250305 * r^4 - 457380 * r^6 + 139920 * r^{10} - 30030 * r^{14} + 2668 * r^{18})\ Sin(2\theta) - 5 * r^4 * (14495 - 23100 * r^4 + 10593 * r^8 - 2156 * r^{12} + 168 * r^{16}) Sin(4\theta))))$$

$$\psi_5 = (\frac{1}{948191619375}) * (32 * (6 * r * (-26649918834 + 46048261255 * r^2 - 19049830800 * r^4 - 9882692820 * r^6 + 11124883770 * r^8 + 1553442891 * r^{10} - 3856272420 * r^{12} + 831228255 * r^{16} -$$

$$127869742 * r^{20} + 8768445 * r^{24})Cos(\theta) -$$
$$r^3 * (57276117182 - 57801113370 * r^2 -$$
$$37637043444 * r^4 + 47328421140 * r^6 +$$
$$4509995490 * r^8 - 17437955535 * r^{10} +$$
$$4400505252 * r^{14} - 684018335 * r^{18} +$$
$$45091620 * r^{22})Cos(3\theta) +$$
$$99 * r^5 * (-15241026 + 27220700 * r^4 - 16041025 * r\text{^}8 +$$
$$4711980 * r^{12} - 692125 * r^{16} + 41496 * r^{20}) *$$
$$Cos(5\theta)))$$

$$T_1 = (-(\frac{4}{45})) * r * (7 - 10 * r^4 + 3 * r^8)Sin(\theta)$$

$$T_2 = (\frac{1}{14175}) * (4 * (4 * (368 - 735 * r^2 + 595 * r^6 - 273 * r^{10} +$$
$$45 * r^{14}) - 105 * r^2 * (-3 + r^4)^2 * (-1 + r^4) *$$
$$Cos(2\theta)))$$

$$T_3 = -((\frac{1}{7016625}) * (8 * (33 * r * (-31227 + 44730 * r^2 +$$
$$15170 * r^4 - 43400 * r^6 + 19334 * r^{10} -$$
$$5145 * r^{14} + 538 * r^{18})Sin(\theta) -$$
$$10 * r^3 * (9175 - 17325 * r^4 + 10593 * r^8 -$$
$$2695 * r^{12} + 252 * r^{16}) * Sin(3\theta))))$$

$$T_4 = -((\frac{1}{316063873125}) *$$
$$(4 * (10 * r^2 * (7377934765 - 8889961080 * r^2 -$$
$$6272840574 * r^4 + 10723280568 * r^6 +$$
$$1052332281 * r^8 - 5412390984 * r^{10} +$$
$$1711184904 * r^{14} - 313196884 * r^{18} +$$
$$23657004 * r^{22}) * Cos(2\theta) -$$
$$3003 * (22 * (-1 + r^2)^3 * (826973 - 29841 * r^2 -$$
$$696192 * r^4 + 59260 * r^6 + 380115 * r^8 +$$
$$29721 * r^{10} - 105442 * r^{12} - 25374 * r^{14} +$$
$$10995 * r^{16} + 3665 * r^{18}) +$$
$$5 * r\text{^}4 * (-196994 + 418780 * r^4 - 317295 * r^8 +$$
$$113916 * r^{12} - 19775 * r^{16} + 1368 * r^{20}) *$$
$$Cos(4\theta)))))$$

(It is possible to introduce the quantity average of temperature defined as:

$$T_{ave} = \frac{1}{\pi}\int_{r=0}^{1}\int_{0}^{2\pi} T r\, d\theta$$

$$T_{ave} = 4 + \frac{40 * (\frac{\lambda}{32})^2}{27} - \frac{178784224 * (\frac{\lambda}{32})^4}{91216125} + \frac{33127587128816 * (\frac{\lambda}{32})^6}{9050920003125} - \frac{4152464766691835964920 32 * (\frac{\lambda}{32})^8}{524214273367444816406 25} + \frac{13856116561981926942506797974752 * (\frac{\lambda}{32})^{10}}{7438168062308513802831152343 75} - \frac{2589929811683749738873764604340745236900296 * (\frac{\lambda}{32})^{12}}{5603018798818335312236335167319614257812 5} + \cdots$$

Of course for lack of space we omit showing the calculation further than this order if any reader interested to have more calculation please contacts the author.

Analysis of Series and Discussion

Pade approximants has been used in original forms to enable us to increase the range of applicability of the series as has been used in the works of Mansour [2] and Mansour [3]. This method does not necessarily require any information about the radius of convergence. The Pade approximants provide an approximation that is invariant under an Euler transformation of the independent variables. The theory of Pade approximants has been used extensively in Mansour [1]. Briefly stated, the Pade approximant is the ratio P (λ)/Q (λ) of polynomials P and Q of degree m and n, respectively, that, when expanded, agrees with the given series through terms of degree m+n, and normalized by P (Q) =1. Such rational fractions are known to have remarkable properties of analytic continuation. The coefficients of the power series must be known to degree m+n. By equating like power of g(λ) and P(λ)/Q(λ), the linear system of m+n+1 equation must be solved to obtain the coefficients in the functional form P(λ)/Q(λ) Pade approximation of orders [1/2], [2/3] and [3/4] for T_{ave} series are respectively:

pade[1/2]: pade[2/3]: pade[3/4]:

$$\frac{4}{1-\frac{10*(\frac{\lambda}{32})^2}{27}}$$

$$\frac{4+\frac{114417112*(\frac{\lambda}{32})^2}{16891875}}{1+\frac{22348028*(\frac{\lambda}{32})^2}{16891875}}$$

$$\frac{4+\frac{(12026891273204)}{(1419129742275)}(\frac{\lambda}{32})^2}{1+\left(\frac{2481119210051*(\frac{\lambda}{32})^2}{1419129742275}-\frac{(431576903292518 6)(\frac{\lambda}{32})^4}{(2739629967461887 5)}\right)}$$

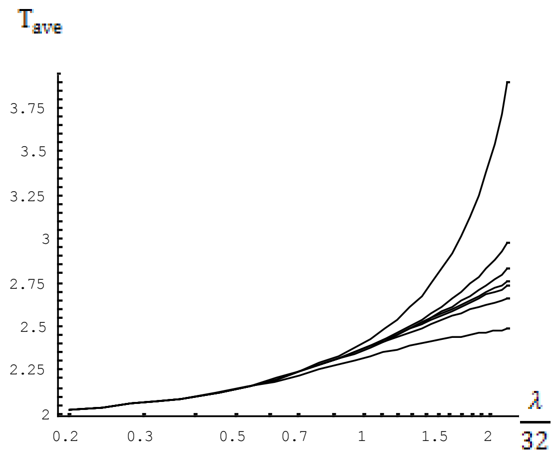

Fig.2. Plots of [8/9], [7/8],[6/7],[5/6],[4/5], [3/4]and [2/3] of the Pade approximants for T_{ave} versus $\frac{\lambda}{32}$

When we form the ratios [8/9], [7/8], [6/7],[5/6], [4/5], [3/4]and [2/3] of the Pade approximants, It can be shown, they agreed up to the value$\frac{\lambda}{32}$=.7. This conclusion is confirmed as is plotted in Figure 2.

Summary

This is problem of the two-dimensional problem of steady natural convection in a horizontal cylinder filled with porous medium due to quadratic volumetric generation. The solution is expanded for low Darcy-Rayleigh number and the series extended by means of symbolic calculation up to 18 terms. Analysis of these expansions allows the exact computation for arbitrarily accuracy up to 50000 figures. Although the range of the radius of convergence is small but Pade approximation leads our result to be good even for much higher value of the similarity parameter we have found extending terms exactly by means of symbolic calculation up to 18th order. Then we tried to make analytic continuation by using Pade approximation. In other words, we have solved the nonlinear partial differential equation exactly by means of computer and that is a real success. Finally we mention recent work of [14] with viscous fluid for micro gap which shows our analytical approach more attractive and successful.

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