Relations between the Poynting and axial force-twist effects

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Abstract

The relations between the Poynting effect, in which a cylinder elongates or contracts axially under torsion, and the axial force-twist effect, in which the twist of a torsionally loaded cylinder is affected by the axial loading, are investigated using second-order elasticity for an elastic homogeneous cylinder. The explicit expressions for the two effects and their relations are presented. The relations show that under tension: (a) negative Poynting effect implies negative axial force-twist effect, (b) positive axial force-twist effect implies positive Poynting effect, whereas (c) the converse statements are not true. Further results show that (a) the Poisson ratio captures the difference between the two effects, and (b) reduced elastic coefficients, which uniquely characterize the effects, lead to universal relations between the effects and the applied loading. Both effects also exhibit a strong inverse power law dependence on the radius.

Keywords: Axial force-twist effect, Poynting effect, torsion-axial loading, second-order elasticity

Introduction

Soft materials may exhibit complex nonlinear behavior such as the Poynting effect, in which a cylinder elongates or contracts axially under torsion. Poynting (1909) experimentally found that some metals exhibited the positive effect, i.e., they elongated axially under torsion. Recently, Janmey et al. (2007) found that networks of semiflexible biopolymers such as actin, collagen, fibrin and neurofilaments, exhibited the negative Poynting effect.

Wang and Wu (2014) showed that in contrast to the Poynting effect, an axial force-twist effect may also exist. It refers to their theoretical result that the twist of a cylinder under combined torsion and axial loading can be affected by the axial loading. The axial force-twist effect can also be positive or negative. The former means that both the twists produced by the axial loading and torsion are in the same direction, while the latter means that the twists produced by them are in the opposite directions. Though Wang and Wu (2014) presented the solutions for the Poynting and axial force-twist effect, the relations between them were not investigated.

This paper focuses on these relations, from which some fundamental conclusions can be drawn. The dependence of the two effects on the linear and nonlinear elastic constants is also studied. The organization of the paper is as follows. The derivation of the relations is first presented, followed by numerical results, a further discussion, and a set of conclusions.

Relations between the Poynting effect and the axial force-twist effect

Figure 1 shows a cylinder of length L and radius R under combined axial loading P and torsion T. Here P represents either a tensile or compressive stress. The materials are nonlinear elastic, isotropic and homogeneous. The initial coordinates of a particle of the cylinder are chosen as (r, θ, z). The strain energy density of Murnaghan (1951) is adopted, i.e.:

$$W = \frac{\lambda + 2\mu}{2} J_1^2 - 2\mu J_2 + \frac{l + 2m}{3} J_1^3 - 2mJ_1J_2 + nJ_3, \tag{1}$$

where λ and μ are the second-order and l, m, n the third-order elastic constants, respectively, and J_1, J_2, and J_3 are the strain invariants of the Lagrangian strain $\mathbf{E}$. The detailed solutions of the stress and displacement fields are given in Wang and Wu (2014). For the purpose of deriving the relations between the effects, the results on the axial and circumferential displacements from the earlier paper are given below.

The axial displacement u_z under pure torsion loading can be written as:

$$u_z = Dz, \tag{2}$$

where D is the Poynting effect coefficient given by:

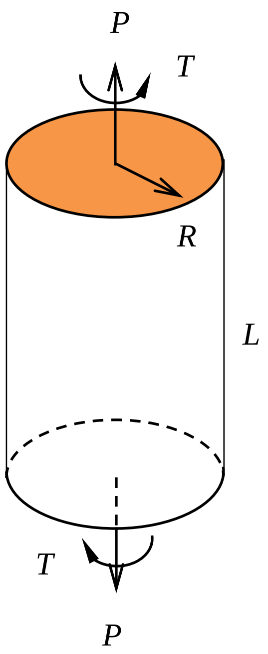

Figure 1. A homogeneous elastic cylinder with radius *R* and length *L* under combined torsion *T* and axial loading *P*.

$$D = -C_D \frac{T^2}{4\pi^2 R^6}, \quad C_D = \frac{n\lambda + 4\mu m + 4\lambda\mu + 8\mu^2}{\mu^3(3\lambda + 2\mu)}. \tag{3}$$

Note that a change in the direction of T does not change the sign of D. The parameter C_D is a reduced coefficient of the four elastic constants. It uniquely characterizes the quadratic relation between the Poynting effect and T. If a modified Poynting effect coefficient $\bar{D} = D / C_D$ is defined, then a universal relation between $\bar{D}$ and T can be obtained:

$$\bar{D} = -\frac{T^2}{4\pi^2 R^6}. \tag{4}$$

Furthermore, the circumferential displacement under combined axial loading P and torsion T is:

$$u_\theta = u_\theta^L + u_\theta^{NL}, \tag{5}$$

where u_θ^L represents the linear twist due to torsion T:

$$u_\theta^L = \frac{2T}{\pi R^4 \mu} rz, \tag{6}$$

and u_θ^{NL} represents the nonlinear twist associated with the axial force-twist effect:

$$u_\theta^{NL} = -\frac{PT(n\lambda + 4\mu m + 6\lambda\mu + 8\mu^2)}{2\pi R^4 \mu^3 (3\lambda + 2\mu)} rz. \tag{7}$$

The axial force-twist effect coefficient can be defined as:

$$H = \frac{u_\theta^{NL}}{u_\theta^L} = -C_H \frac{P}{4}, \quad C_H = \frac{n\lambda + 4\mu m + 6\lambda\mu + 8\mu^2}{\mu^2(3\lambda + 2\mu)}. \tag{8}$$

C_H is a reduced coefficient which characterizes the relation between H and P. It is similar in form to C_D. By defining the modified axial force-twist effect coefficient $\bar{H} = H / C_H$, a universal linear relation between $\bar{H}$ and P can be obtained:

$$\bar{H} = -\frac{P}{4}. \tag{9}$$

It can be seen from Eq. (7) that the axial force-twist effect only exists under combined axial loading and torsion, i.e., $P \neq 0$ and $T \neq 0$. Eq. (8) implies that if *H* is positive, u_θ^{NL} has the same direction as u_θ^{L} and the axial force twist effect is positive; otherwise, it's negative. Because *H* depends on *P* and not *T*, two further observations can be made from Eq. (8):

(1) Change of the direction of *T* does not change the sign of *H*.
(2) Change of the sign of *P* changes the sign of *H*.

Eqs. (3) and (8) show that materials with different elastic constants can have the same Poynting effect or the axial force-twist effect, provided the reduced coefficients of these materials are the same. Another observation of Eq. (3) is that for a particular μ, if *m* and *n* are chosen in a way that makes $(n+4\mu)/3=(4m+8\mu)/2$, or $8\mu+6m-n=0$, then λ has no influence on the Poynting effect. A similar conclusion can be made for *H* on the basis of Eq. (8). If *m* and *n* are chosen such that $(n+6\mu)/3=(4m+8\mu)/2$, or $6\mu+6m-n=0$, then λ has no influence on the axial force-twist effect.

The relation between *H* and *D* in dimensionless form can be obtained easily from Eqs. (3) and (8):

$$\frac{H}{P/4\mu} = \frac{D}{T^2/4\pi^2R^6\mu^2} - \frac{2\lambda}{3\lambda+2\mu}. \tag{10}$$

Since $\nu = \lambda/(2\lambda+2\mu)$, the above equation can be rewritten as:

$$\frac{H}{P/4\mu} = \frac{D}{T^2/4\pi^2R^6\mu^2} - \frac{2\nu}{1+\nu}. \tag{11}$$

The term on the left-hand side represents the axial force-twist effect coefficient normalized by the axial loading, while the first term on the right-hand side represents the Poynting effect normalized by the torsion. An explicit relationship between the axial force-twist effect and the Poynting effect is thus established.

Since ν is positive generally, several conclusions can be drawn from Eq. (11), assuming that the axial loading *P* is tensile:

(a) If $D < 0$, then necessarily $H < 0$,
(b) If $H > 0$, then necessarily $D > 0$.
(c) If H and D have different signs, then necessarily $H < 0$ and $D > 0$.

It should be emphasized that the converses of (a) and (b) are not true, i.e., $H < 0$ does not necessarily imply $D < 0$, and $D > 0$ does not necessarily imply $H > 0$. A further observation is that the sign of H will change if the sign of P changes. Thus for the case of compressive axial loading, the above three conclusions should be changed to:

(d) If $D < 0$, then necessarily $H > 0$,
(e) If $H < 0$, then necessarily $D > 0$.
(f) If H and D have the same sign, then necessarily $H > 0$ and $D > 0$.

The Poisson ratio plays a key role since the difference between the normalized H and the normalized D is the term $2\nu/(1+\nu)$. This difference reaches its maximum when $\nu = 0.5$, i.e., the material is incompressible.

Finally, the size dependence of the Poynting effect can be judged from Eq. (3) to be inversely proportional to the sixth power of the cylinder radius. For the axial force-twist effect, Eq. (7) shows that the maximum circumferential displacement ($r = R$) is inversely proportional to the third power of the cylinder radius. Hence, the Poynting effect is relatively more important than the axial force-twist effect for small cylinders, and the reverse holds for large cylinders.

Numerical results

This section focuses on the influence of the elastic constants on the Poynting effect and the axial force-twist effect. The elastic constants of the soft materials were adapted from Wang and Wu (2013, 2014) for poly(acrylic acid) (PAA) gels and capillary muscles, respectively, and Catheline et al. (2003) for an agar-gelatin. The geometry of the cylinder is fixed as R = 0.002 m and L = 0.01 m. The applied axial loading and torsion may vary for different figures.

Fig. 2 plots the $H = 0$ and $D = 0$ contours in the μ–ν space, for $m = -2420$ kPa and $n = -2350$ kPa. The axial loading P is chosen as positive. It can be seen that the μ–ν space is partitioned into three regions: Region I with $H > 0$ and $D > 0$, Region II with $H < 0$ and $D < 0$ and Region III with $H < 0$ and $D > 0$.

Several interesting phenomena can be observed, in agreement with the conclusions (a) to (c) stated above. First, negative Poynting effect ($D < 0$) implies negative axial force-twist effect ($H < 0$) as shown in Region II. However, the converse is not true, i.e., negative axial force-twist effect ($H < 0$) does not imply negative Poynting effect ($D < 0$) necessarily, as shown in the small Region III. Secondly, positive axial force-twist effect implies positive Poynting effect (i.e., $H > 0$ means $D > 0$, as shown

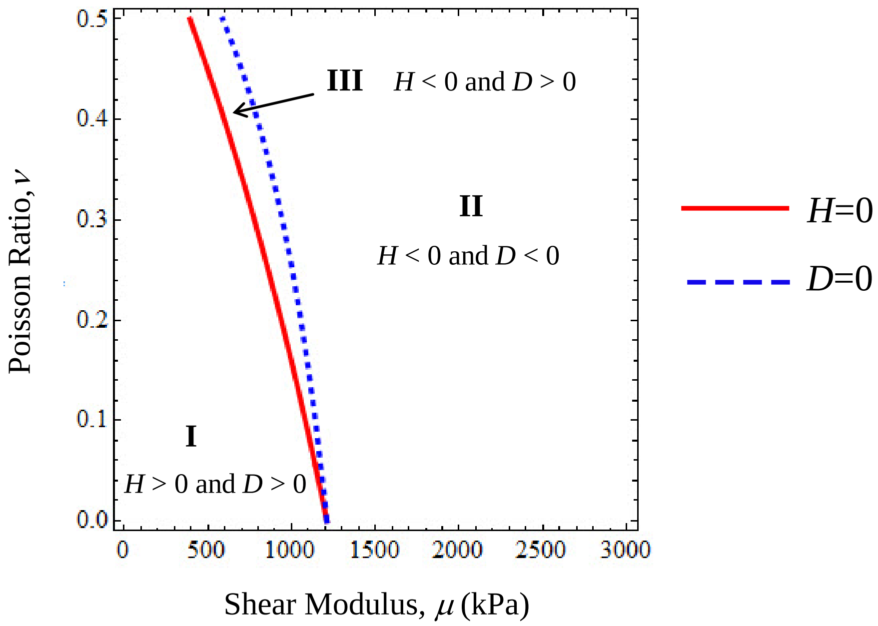

Figure 2. Contours of the Poynting effect coefficient D = 0 (dashed line) and axial force-twist effect coefficient H = 0 (solid line) in μ–ν space for a homogeneous elastic cylinder. The contours partition the space into three regions.

in Region I). However, the converse is not true. Positive Poynting effect does not imply positive axial force-twist effect (i.e., $D > 0$ does not necessarily imply $H > 0$ as shown in region III). Moreover, when the two effects differ in sign, the Poynting effect must be positive and the axial force-twist effect must be negative (as shown in Region III). Region III, where the two effects have different signs, is generally small, suggesting that only careful choices in the material parameters can lead to different signs for the two effects.

Fig. 3 plots H and D against the Poisson ratio ν. The material parameters are based on those of polymers with μ = 10.3 kPa, m = –24.2 kPa and n = –23.5 kPa. The loadings are P = 10 kPa and T = 300 kPa·m^3. It can be seen that when ν increases, both H and D decrease from positive to negative monotonically. Thus, the Poisson ratio can be an important parameter in controlling the two effects. Secondly, the magnitudes of H and D are of the order of 10^{-1}, suggesting that the nonlinear effects can be significant. Note that ν_1 and ν_2 are the particular Poisson ratios which make H = 0 and D = 0, respectively. This figure further shows that (a) if $H > 0$, then $D > 0$, as shown when $\nu < \nu_1$, (b) if $D < 0$, then $H < 0$, as shown when $\nu > \nu_2$, and (c) if H and D have different signs, then $H < 0$ and $D > 0$, as shown when $\nu_1 < \nu < \nu_2$.

Fig. 4 shows how the linear elastic constants μ and ν affect the Poynting effect and the axial force-twist effect. The parameters are m = –360 kPa, n = 20 kPa, P = 10 kPa and T = 1000 kPa·m^3. It can be seen that there exists a μ_1 for which H is independent of ν. Similarly, there exists a μ_2 for which D is independent of ν. As mentioned above,

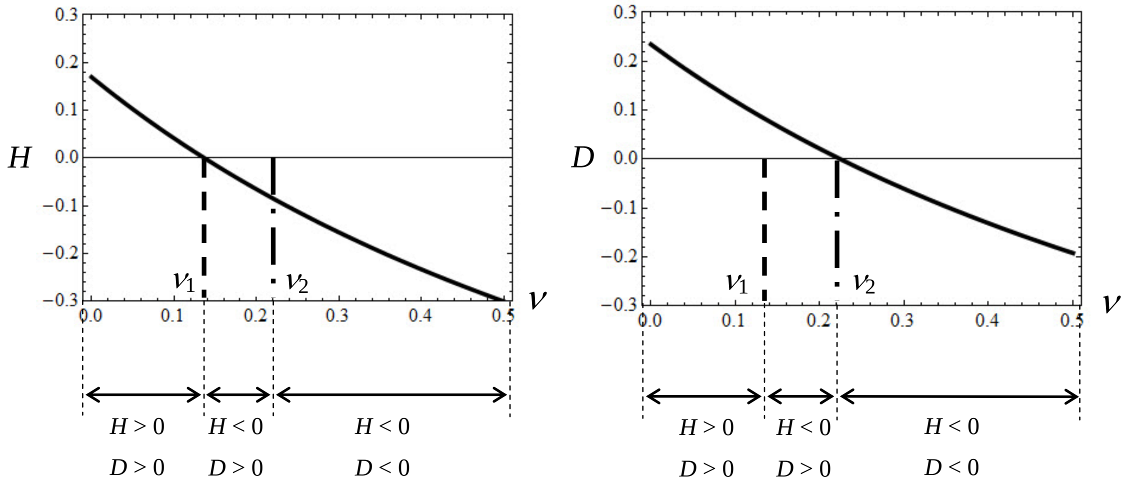

Figure 3. Dependence of *H* and *D* on the Poisson ratio ν, with μ = 10.3 kPa, *m* = –24.2 kPa, and *n* = –23.5 kPa.

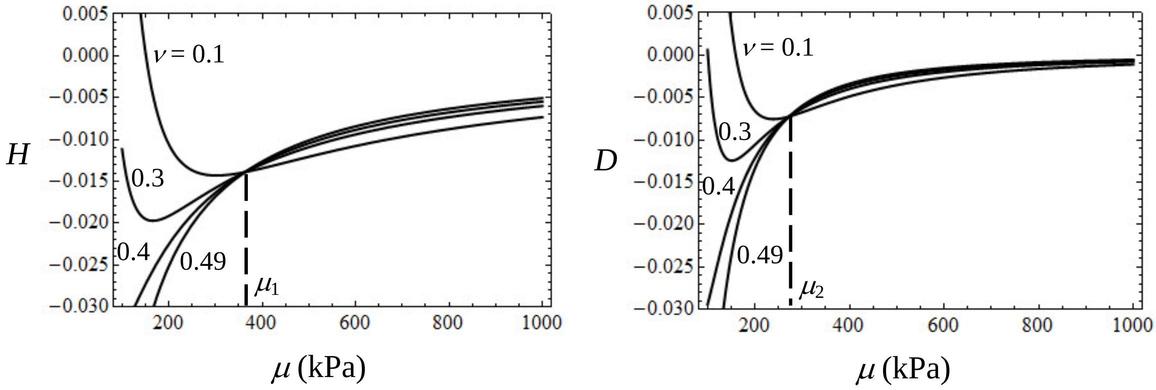

Figure 4. Dependence of *H* and *D* on the shear modulus μ for different Poisson ratios ν =0.1, 0.3, 0.4 and 0.49, with *m* = –360 kPa and *n* = 20 kPa.

μ_1 and μ_2 can be determined from the equations $(n+6\mu_1)/3=(4m+8\mu_1)/2$ and $(n+4\mu_2)/3=(4m+8\mu_2)/2$, respectively, yielding μ_1 = 363.3 kPa and μ_2 = 272.5 kPa. A further observation is that the negative *H* and *D* values appear to have upper bounds, while the positive values are unbounded. More generally, however, *D* or *H* may either have a positive or negative bound, depending on the values of *m* and *n*.

Fig. 5 shows how the nonlinear elastic constant *m* can significantly influence both effects. Here *H* and *D* are plotted against μ for $m=\pm2\times10^6$, $\pm10^6$ and 0 kPa. The other elastic parameters are λ = 60 kPa and *n* = –23.5 kPa. The loadings are *P* = 0.01 kPa and *T* = 10 kPa·m^3. For this set of parameters, increasing μ will decrease the magnitudes of the coefficients. Secondly, both effects are positive for negative *m* and negative for positive m. Changing the sign of m will change the sign of both H and D. Thirdly, decreasing the magnitude of m will also decrease the magnitudes of H and D. The magnitudes can reach the order of 10^{-2} to 10^{-1} when μ is small; thus the nonlinear behavior can be significant when the material is very soft with a small μ.

Fig. 6 plots H and D versus μ for the same sets of m, with λ = 35700 kPa and n = –23500 kPa. The loadings are P = 0.01 kPa and T = 10 kPa·m^3. The nonlinear effects are different from those shown in Fig. 5. For m positive, both H and D decrease to a negative maximum and subsequently decrease slowly to zero with increasing μ. However, for m negative, they decrease monotonically to zero with μ.

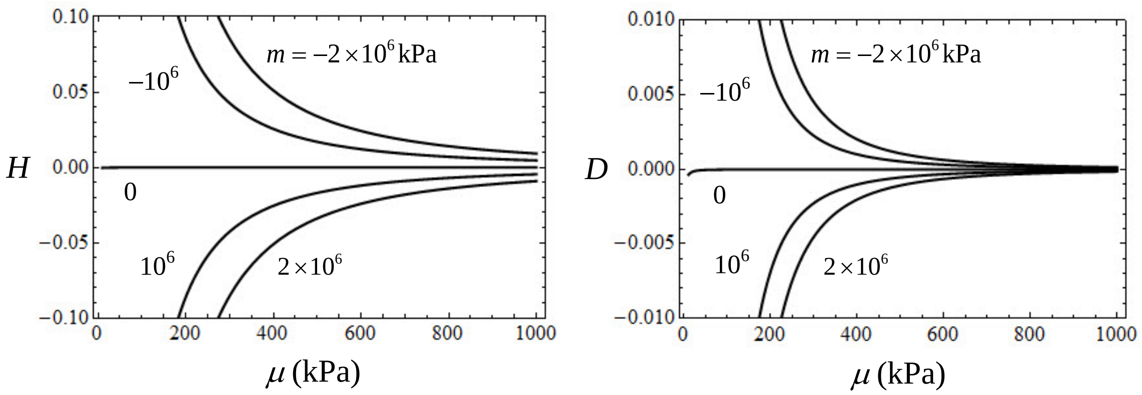

Figure 5. Dependence of *H* and *D* on the shear modulus μ for *m* = –2×10^6, –10^6, 0, 10^6, 2×10^6 kPa, with λ = 60 kPa and *n* = –23.5 kPa. The loadings *P* = 0.01 kPa and *T* = 10 kPa·m^3.

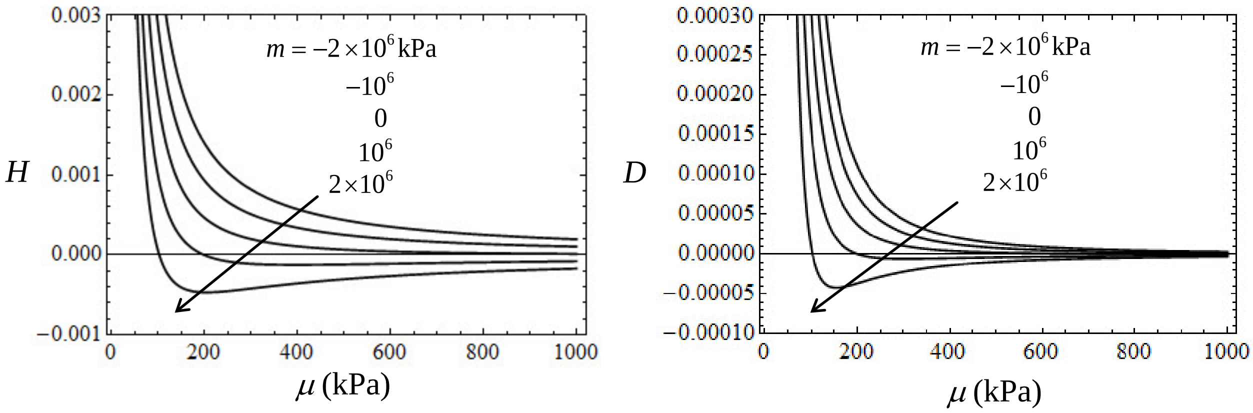

Figure 6. Dependence of *H* and *D* on the shear modulus μ for *m* = –2×10^6, –10^6, 0, 10^6, 2×10^6 kPa, with λ = 35700 kPa and *n* = –23500 kPa. The loadings *P* = 0.01 kPa and *T* = 10 kPa·m^3.

Discussion

Many biological materials, from soft to hard, are subjected to complex loading in their physiological environment. A few examples are described here. Arterial walls associated with human brain aneurysms were subjected to combined extension, torsion and inflation in finite element studies, in order to mimic the real physiological conditions (Tóth et al., 2005). The behavior of lumbar spinal units under torsion, compression and flexion/extension were also experimentally studied (Haberl et al., 2004). It is also well-known that articular cartilage is subjected to combined compression and shear during normal activities (Mansour, 2003). Fatigue tests were conducted on cylindrical bovine cortical bone specimens under axial, torsional and combined axial-torsional loadings (Vashishth et al., 2001). Finite extension and torsion were applied on capillary muscles in order to characterize their behavior under physiological conditions (Criscione et al., 1999).

Because of the prevalence of combined loadings, the Poynting effect and the axial force-twist effect may be highly relevant. In particular, large stresses may be generated by both effects if the specimen is confined in one way or another, i.e., the additional axial and rotational displacements are restrained. These large stresses can, for instance, alter the overall force balance and the cytoskeleton structure of cells, or the movement of a human red blood cell through narrow capillaries. The diameter of a human red blood cell is 7.0-8.5 μm, while that of narrow capillaries is smaller than 3 μm (Bao and Suresh, 2003).

The effects can also be utilized in the design of devices such as actuators and sensors. One can imagine a bio-inspired polymer actuator based on the axial force-twist effect, i.e., a torsionally loaded cylinder may generate an additional output twist, if subjected to an input axial force. By carefully selecting the elastic parameters of the materials and the structural dimensions, the amount of twist can be increased significantly and the desired output can be achieved.

Conclusions

Explicitly expressions for the Poynting effect, the axial force-twist effect and their relation are presented in this paper. The dependence of the relation on elastic constants is investigated.

The results show that under a tensile stress P, (a) negative Poynting effect implies negative axial force-twist effect, (b) positive axial force-twist effect implies positive Poynting effect, and (c) if the two effects differ in sign, the Poynting effect must be positive and the axial force-twist effect negative. The loadings P and T are such that (d) changing the direction of T will not change the sign of both effects, and (e) changing the direction of P will change the direction of the axial force-twist effect. Moreover, the Poynting and axial force-twist effects exhibit a very significant size dependence, respectively of the inverse sixth and third power of the cylinder radius.

Reduced elastic coefficients characterize universal relations between the effects and the applied loadings. The elastic constants μ, ν and m have significant influence on the magnitude and direction of the Poynting and axial force-twist effects. For certain combinations of elastic constants, changing the sign of m can directly change the sign of the two effects. The two effects may have a positive or negative bound, depending on the elastic constants. From the perspective of material design, the elastic constants are thus of vital importance.

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