ICCM2014
28-30th July, Cambridge, England

Calculation of underwater acoustic scattering problems in unbounded domain using the alpha finite element method

W. Li[1,2], *Y.F. Li[1] and Y.B. Chai[1]

[1]Department of Naval Architecture and Ocean Engineering, Huazhong University of Science and Technology, Wuhan City, P. R. China 430074

[2]Hubei Key Laboratory of Naval Architecture & Ocean Engineering Hydrodynamics (HUST)

*Corresponding Author: lyf198961@gmail.com

Abstract It is well-known that the classical finite element method fails to provide accurate results to the Helmholtz equation due to the dispersion error, which is rooted at the "overly-stiff" feature of the FEM model. By combining the "smaller wave number" model of FEM and the "larger wave number" model of NS-FEM, an alpha finite element method(α-FEM) can obtain accurate solutions. In this paper, the α-FEM has been applied to analyze 2D underwater exterior scattering problems in the unbounded domain. The non-reflecting boundary condition is imposed as an artificial boundary to model exterior acoustic problems. Several two-dimensional underwater exterior scattering problems with known exact solutions have been chosen as numerical examples. Results demonstrate the excellent properties of α-FEM.

Keywords: Alpha finite element method(α-FEM), Acoustic Scattering, Unbounded Domain, Non-reflecting boundary

Introduction

For several decades, many numerical methods have been introduced to compute the approximate solutions of acoustic problems [Suleau et al.(2000); Harari and Magoules(2004); Babuska et al(1999)]. The standard finite element method (FEM) is one of the most widely-used numerical methods in solving these acoustic problems governed by the Helmholtz equation. However it is known that the FEM fails to provides reliable predictions in high frequency range. Many studies have been done to improve resolve this defect. But such efforts have difficulties because of the well-known "pollution error".

Various numerical methods have been proposed, They are the stabilized FEM [Harari and Huhes(1992; Thompson and Pinsky(1995)], higher order methods [Petersen et al.(2006)], meshless method [Bouillard and Suleau(1998)] and so on. They all get better solutions. However, "softened" stiffness for the discrete model is more effective [Liu et al.(2009)]. The wave number in the FEM model is smaller than the actual one, leading to the so-called numerical dispersive error. The FEM model based on the standard Galerkin weak form behaviors stiffer than the continuous system. In order to "soften" the numerical system, Liu [Liu(2008; 2009)] has proposed generalized gradient smoothing technique and applied if in the meshfree setting to formulate the node-based smoothing point interpolation method (NS-PIM) and node-based finite element method (NS-FEM). But NS-PIM and NS-FEM model both behave "overly-soft". An alpha finite element method (α-FEM) was then proposed [Liu et al.(2008)] by combining the "overly-stiffness" of the FEM and the "overly-soft" of the NS-FEM through a parameter α, resulting in a numerical model with very close-to-exact stiffness.

In this work, by introducing the DtN artificial boundary condition [Givoli(1988); Givoli and Keller(1989)], the accuracy and convergence of the α-FEM is studied. Initially the scattering problem is described. Next, the weak form of α-FEM and DtN boundary condition for the two dimensions case is derived. Finally, a comparison between the α-FEM solution, the FEM solution and the numerical solution shows the performance of the α-FEM for a rigid sphere as an example.

Mathematical model of acoustic problem

Consider an acoustic problem domain Ω with boundary Γ. The acoustic wave equation can be written as following form:

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial^2 t} = 0 \tag{1}$$

where p denotes the acoustic field pressure and c be the speed of sound traveling in fluid, ∇^2 and t denote the Laplace operator and time. In the frequency domain, the acoustic can be expressed as:

$$p = p_0 e^{j\omega t} \tag{2}$$

where p_0 and ω denote the amplitude of the acoustic wave and the angular frequency, $\mathrm{j} = \sqrt{-1}$. Then p satisfies the well-known Helmholtz equation:

$$\nabla^2 p + k^2 p = 0 \tag{3}$$

where k is the wave number expressed as:

$$k = \frac{\omega}{c} \tag{4}$$

besides, the particle velocity υ in an ideal fluid is proportional to the gradient of the pressure:

$$\nabla p + j\rho\omega v = 0 \tag{5}$$

For unbounded domain problems, an artificial boundary condition is introduced. In this paper we introduce a so-called “Dirichlet-to-Neumann” boundary condition [Keller and Givili(1989)] on the outer boundary of the domain discretized with finite elements:

$$\nabla \mathrm{p} \bullet \mathrm{n} = -\mathrm{M} \bullet \mathrm{p} \tag{6}$$

where M is the DtN operator. In two dimensional problems, the DtN operator is expressed as;

$$M_n = -\frac{k}{\pi} \sum_{n=0}^{\infty} \frac{H_n^{(2)'}(kR)}{H_n^{(2)}(kR)} cosn(\theta - \vartheta) \tag{7}$$

where R is the radius of the outer boundary, $H_n^{(2)}$ is the Hankel function of the second kind, θ and ϑ are azimuth angle.

Formulation of the $\alpha - \mathrm{FEM}$

In the standard FEM, the discretized system equation can be written in the following matrix form:

$$\left[\mathbf{K}^{\mathrm{FEM}} - k^2\mathbf{M} + j\rho\omega\mathbf{C}\right]\{\mathbf{P}\} = -j\rho\omega\{\mathbf{F}\} \tag{8}$$

where

The acoustic stiffness matrix: $\mathbf{K}^{\mathrm{FEM}} = \int_\Omega (\nabla\mathbf{N})^{\mathrm{T}}(\Delta\mathbf{N})\mathrm{d}\Omega$ (9)

The acoustic mass matrix: $\mathbf{M} = \int_\Omega \mathbf{N}^{\mathrm{T}}\mathbf{N}\mathrm{d}\Omega$ (10)

The acoustic damping matrix: $\mathbf{C} = \int_\Gamma \mathbf{N}^{\mathrm{T}}\mathbf{N}A_n\mathrm{d}\Gamma$ (11)

The vector of nodal acoustic forces: $\mathbf{F} = \int_\Gamma \mathbf{N}^T v_n \mathrm{d}\Gamma$ (12)

The nodal acoustic pressure: $\mathbf{P}^{\mathrm{T}} = [p_1, p_2, \ldots, p_n]$ (13)

In the NS-FEM, the problem is first divided into N_e elements with of N_n nodes as same as those used in the FEM. Then the problem is further divided in N_n node-based smoothing domains on top of the generated cells. For 2D problems, the smoothing domain Ω_k is created by connecting sequentially the mid-edge-point to the centroids of the surrounding triangles of node k. The boundary of the smoothing domain Ω_k is labeled as Γ_k and the union of all Ω_k forms exactly the global domain Ω.

In the NS-FEM, the field variable is constructed using the linear FEM shape functions in the same way as those in the FEM. The difference is that the gradient component ∇N is replaced by the smoothing item $\overline{\nabla N}$ obtained using the node-based gradient smoothing operation [Liu et al.(2005); Zhang et al(2007)].The acoustic stiffness matrix in the NS-FEM is expressed as:

$$\mathbf{K}^{\text{NS-FEM}} = \int_{\Omega} (\overline{\nabla \mathbf{N}})^T (\overline{\nabla \mathbf{N}}) \mathrm{d}\Omega \tag{14}$$

The above integration is evaluated base on the summation of all the node-based smoothing domains as:

$$\mathbf{K}^{\text{NS-FEM}} = \sum_{k=1}^{N_n} \mathbf{K}^{(k)} \tag{15}$$

where the $\mathbf{K}^{(k)}$ is the local smoothed stiffness matrix associated with node k, and can be calculated:

$$\mathbf{K}^{(k)} = \int_{\Omega_k} \overline{\mathbf{B}}^{\mathrm{T}} \overline{\mathbf{B}} d\Omega = \overline{\mathbf{B}}^{\mathrm{T}} \overline{\mathbf{B}} A_k \tag{16}$$

where A_k is the area of the smoothing domain for node k in 2D problems, and

$$\overline{\mathbf{B}}_i(\mathbf{x}_k) = \left[\bar{b}_{i1}\, \bar{b}_{i2} \right] \tag{17}$$

$$\bar{b}_{ip} = \frac{1}{A_k} \int_{\Gamma_k} N_i(\mathbf{x}) n_p(\mathbf{x}) d\Gamma \tag{18}$$

where is the FEM shape function for node i.

In the $\alpha-$FEM, each triangular element is divided into four parts with a scaled factor α: three quadrilaterals associated with three vertexes with equal area of $\frac{1}{3}\alpha A_e$ and the remaining Y-shaped part in the middle of the element with a area of $(1-\alpha)A_e$, where the A_e is the area of the triangular element. The NS-FEM and the FEM formulations are constructed respectively in the three quadrilaterals and the Y-shaped area for each element. Then the $\alpha-$FEM will be the assembly from the entries of both the NS-FEM and FEM with the following form:

$$\mathbf{K}_{IJ}^{\alpha-\mathrm{FEM}} = \alpha^2 \sum_{m=1}^{N^e} \mathbf{K}_{IJ(m)}^{\mathrm{FEM}} + \left(1-\alpha^2\right) \sum_{n=1}^{N^n} \mathbf{K}_{IJ(n)}^{\mathrm{NS\text{-}FEM}} \tag{19}$$

In unbounded domain, according to Givoli and Kaller [Givoli(1988); Givoli and Keller(1989)], the stiffness matrix **K** consists of two matrices:

$$\mathbf{K} = \mathbf{K}^{\mathrm{FEM}} + \mathbf{K}^{b} \tag{20}$$

where $\mathbf{K}^{\mathrm{b}}$ is the DtN artificial boundary matrix, it contains the operator M and the shape functions used in the FEM:

$$\mathbf{K}_{ij}^{b} = \int_{\Gamma} \mathbf{N}_i M \mathbf{N}_j d\Gamma \tag{21}$$

Finally, the discretized system equations can be obtained and written in the following form:

$$\left[\mathbf{K}^{\alpha-\mathrm{FEM}} - k^2\mathbf{M} + j\rho\omega\mathbf{C} + \mathbf{K}^{b}\right]\{\mathbf{P}\} = -j\rho\omega\{\mathbf{F}\} \tag{22}$$

Numerical example

In this paper, to illustrate the performance and ability of $\alpha-$FEM for acoustic problems, the scattering problem on the exterior domain of a rigid sphere is dealt with. The radius of the sphere is 0.2, the radius of the artificial boundary is 1.

Consider a wave propagates in the exterior domain with two boundary condition described as follow:

on the artificial boundary: $\nabla \mathbf{p} \bullet \mathrm{n} = -\mathrm{M} \bullet \mathbf{p}$ (23)

on the boundary of the rigid sphere: $v_n = \text{-}v$ (24)

The problem has an analytical solution as follow:

$$p = -p_0 e^{j\omega t} \sum_{n=0}^{\infty} (-j)^n \varepsilon_n \frac{\dfrac{dJ_n(ka)}{dka}}{\dfrac{dH_n^{(2)}(ka)}{dka}} H_n^{(2)}(kr)\cos(n\theta), \quad \varepsilon_n = \begin{cases} 1, n=0 \\ 2, n=1,2,3 \end{cases} \tag{25}$$

Three different α values $\alpha = 0.7, \alpha = 0.8, \alpha = 0.9$ have been employed to compare the influence of α with element size of 0.02. The numerical results of acoustic pressure using $\alpha-\mathrm{FEM}$ and exact solution are plotted in Fig1. It can be seen from plot that when $\alpha=0.8$, the numerical solution is in agreement with exact solution. So $\alpha=0.8$ is used in the following computation.

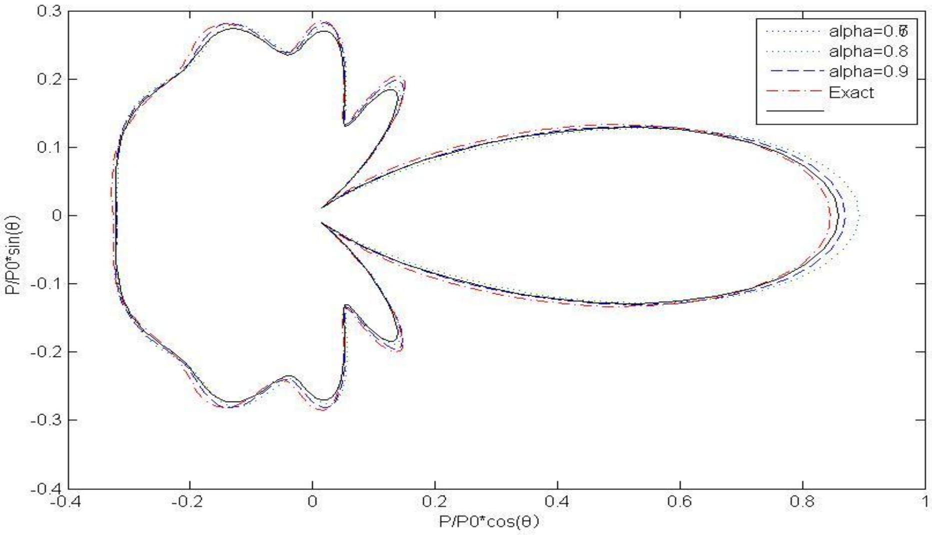

Figure 1. Analytical solution and α-FEM solution with different α values $\boldsymbol{\alpha=0.7, \alpha=0.8, \alpha=0.9}$

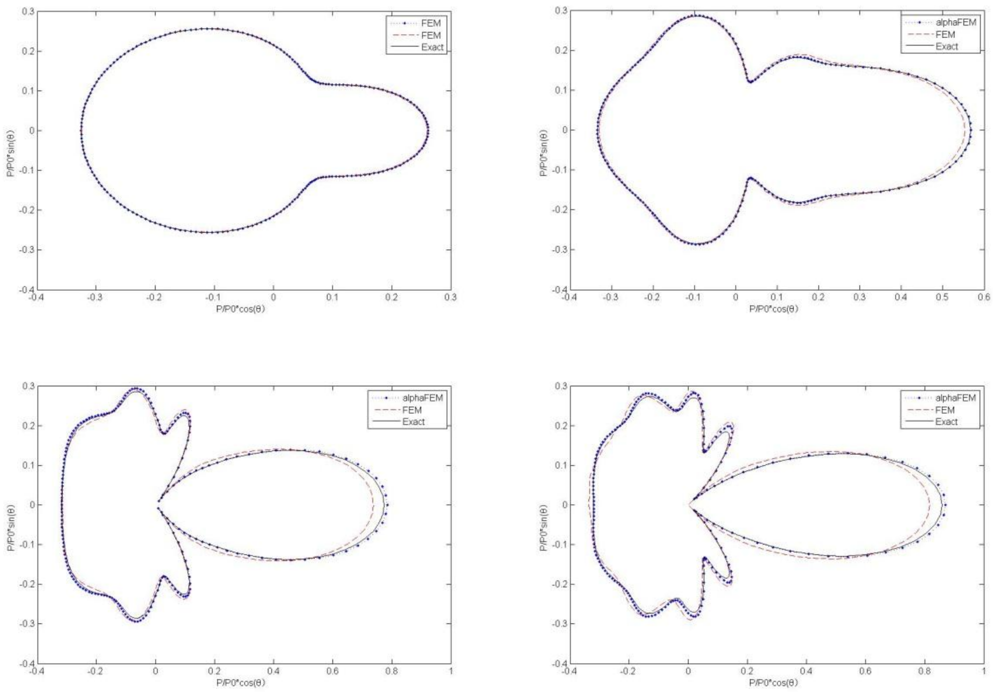

Figure. 2 Analytical solution, FEM solution and α-FEM solution on the artificial boudary for k=5, k=15, k=25, k=30

Four different wave number values k=5, k=10, k=15, k=20are employed to study the accuracy of $\alpha - \mathrm{FEM}$ on the artificial boundary in Fig2. These plots show that for lower wave number, $\alpha - \mathrm{FEM}$ and FEM all can provide close-to-exact solution. But $\alpha - \mathrm{FEM}$ is more close to exact solution than FEM solution when comparing them in the forward scattering. For higher wave number, the advantage of $\alpha - \mathrm{FEM}$ is more obvious, $\alpha - \mathrm{FEM}$ solution is still close to the exact solution on the artificial boundary, but FEM solution depart more from the exact solution.

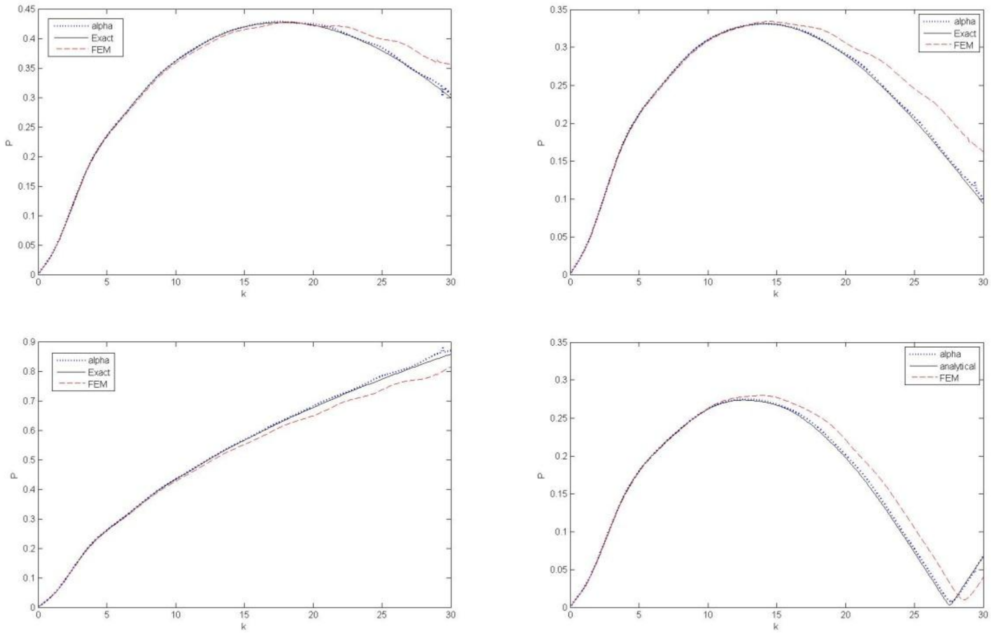

Figure 3. Calculation of $\alpha - \mathrm{FEM}$ solution, FEM solution and analytical solutionas the wave number increasing at four nodes $\theta = 0, \theta = 0.5386,\ \theta = 0.7630,\ \theta = 1.0322$

In Fig3, we compare $\alpha - \mathrm{FEM}$ and FEM solution with exact solution as the wave number increasing at four nodes $\theta = 0, \theta = 0.5386,\ \theta = 0.7630,\ \theta = 1.0322$. From these plots we can find that as wave number increasing, $\alpha - \mathrm{FEM}$ and FEM both lose their accuracy, but the error of $\alpha - \mathrm{FEM}$ is much smaller than the erroe of FEM.

Conclusions and discussions

In this work, the alpha finite element method (α-FEM) for solving scattering problems of the Helmholtz equation in two dimensions has been presented. By combining the “overly-stiff” FEM model with the “overly-soft” NS-FEM model, the α-FEM is obtained by a scaled factor $\alpha \in [0,1]$. Calculations of the scattering of a rigid sphere show the following conclusions:

1. The scaled factor α has a giant effect on the accuracy of the α-FEM.
2. The results indicate that the DtN boundary condition is a good alternative to other methods in solving scattering problems in infinite domains.

3. The α –FEM and Fem use the same mesh, which means the α-FEM model can be get from the FEM model with little change.
4. By using the gradient smoothing technology and the optimal alpha, the α-FEM appropriately softened the stiffness matrix and reduces the dispersion error. Numerical example demonstrate that the accuracy and convergence of the α-FEM is better than the FEM.

References

Suleau. S, Deraemaeker. A, Bouillard. Ph.(2000) Dispersion an pollution of meshless solution for the Helmholtz equation, Comput. Meth. Appl. Mech. Engrg. 190, 639-657.

Harari. I, Magoules. F.(2004)Numerical investigations of stabilized finite element computation for acoustics, Wave Motion 39, 339-329.

Deraemaeker. A, Babuska. I, Bouillard. Ph.(1999) Dispersion an pollution of the FEM solution for the Helmholtz equation in one, two and three dimension, Int. J. Numer. Meth. Engrg. 46, 471-499.

Harari. I, Hughes. TJR(1992)Galerkin/last squares finite element methods for the reduced wave equation with non-reflecting boundary conditions in unbounded domains. Comp Methods Appl Mech Eng 98, 411-454.

Thompson. LL, Pinksy. PM(1995)A Galerkin least squares finite element method for the two-dimensional Helmholtz equation. Int. J. Numer. Meth. Engrg. 38, 371-397.

Petersen. S, Dreyer. D, Estorff. Ov(2006)Assessment of finite and spectral element shape function or efficient iterative simulations of interior acoustics. Comp Methods Appl Mech Eng 195, 1171-1188.

Bouillard. Ph, Suleau. S(1998)Element-free Galerkin solutions for Helmholtz problems: formulation and numerical assessment of the pollution effect. Comp Methods Appl Mech Eng 162, 317-335.

He. Z. C, Liu G. R, Zhong. Z. H, Wu. S. C, Zhang. G. Y, Cheng. A. G(2009)An edge-based smoothed finite element method for analyzing three-dimensional acoustic problems. Comp Methods Appl Mech Eng 199, 20-33.

Liu. G. R.(2008)A generalized gradient smoothing technique and the smoothed bilinear form for Galerkin formulation of wide class of computational methods, Inyrtnational Journal of Computational Methods 5, 199-236.

Liu. G. R. (2008)Meshfree Methods:Moving Beyond the Finite Element Method, second ed, CRC Press, Boca Raton, USA.

Liu. G. R. Nguyen. T. T, Lan. K. Y(2009)A novel alpha finite element method for exact solution to mechanic problems using triangular and trtrahedral element. Comp Methods Appl Mech Eng 197, 3883-3897.

Givoli. D.(1988)A finite element method for large domain problems, Thesis, Stanford University.

Keller. J, Givoli. D.(1989)Exact non-reflecting boundary conditions, Journal of Computational Physics 82, 172-192.

Liu. G. R, Zhang. G. Y, Dai. K. Y, Wang. Y. Y, Zhong. Z. H, Li. G. Y, Han. X,(2005)A linear conforming point interpolation method for two-dimensional solid mechanics problems, Int J Comput Methods 2(4),645-665.

Liu. G. R, Zhang. G. Y, Dai. K. Y, Wang. Y. Y, Zhong. Z. H, Li. G. Y, Han. X.(2007) A linear conforming point interpolation method for three-dimensional elasticity problem, Int J Numer Methods Eng 72(13), 1524-1543.