Complex modal analysis using undamped modes

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Abstract

Based on undamped modes, a novel method is presented to efficiently calculate complex eigenpairs by combining the Neumann series and the reduced basis technique. To avoid the modal truncation problem, the reduced basis is calculated by using a Neumann series expansion and only requires the undamped eigenpair of interest. The sufficient condition for the convergent Neumann series is derived and the computational complexity of the proposed method is discussed. Useful characteristics on the accuracy and the advantages of the proposed method over the exact state-space method, as well as over the common approximate procedure of ignoring the modal coupling, are shown and discussed in terms of some case studies. It is shown that the complex eigenpairs can be calculated by simply postprocessing of undamped eigenpairs.

Keywords: Complex modal analysis; Eigensolution; Undamped modes; Non-classically damped systems; Modal coupling; Frequency response function

Introduction

The equation of motion of an N DOF linear viscously damped system can be given by

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{C}\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=\mathbf{f}(t) \tag{1}$$

where **M**, **C** and **K** are real mass, damping and stiffness matrices, respectively; **q**(t) and **f**(t) are displacement vector and force vector, respectively. In this paper, assume that **M** is a positive definite symmetric matrix, **K** and **C** are non-negative definite symmetric matrices. The inclusion of the influence of damping in structural and mechanical systems is extremely important if a model is to be applied in predicting vibration levels, transient responses, transmissibility and design problems dominated by energy dissipation. The eigenvalue problem of the viscously damped system can be written in matrix form as

$$\left(\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K}\right)\boldsymbol{\varphi}_j=\mathbf{0}\quad\forall\, j=1,\ 2,\ \ldots,2N \tag{2}$$

Here λ_j and $\boldsymbol{\varphi}_j$ denote the jth eigenvalue and eigenvector. Although several efficient approximation techniques were suggested, the most common is so-called the proportional approximation method (PAM), which is simply to ignore the mode coupling by using undamped modal shapes. The undamped mode shapes (normal modes) can be obtained by solving the undamped eigenproblem $\mathbf{K}\mathbf{u}_j=\omega_j^2\mathbf{M}\mathbf{u}_j$ where ω_j is the jth undamped frequency and in order of ascent; $\mathbf{u}_j$ denotes the mode shape corresponding to the jth frequency ω_j The PAM may be the most efficient approximate method, but the results of the PAM are not always with acceptable accuracy. It was shown by many studies (Hasselman, 1976; Warburton and Soni, 1977; Gawronski and Sawicki, 1997; Udwadia, 2009) that the light damping, the diagonal dominance of the transformed damping matrix

and the good separation property of normal modes are not any more the sufficient conditions for the accuracy of the proportional approximation method (these conditions were once believed to produce small errors for the PAM). When the non-proportional part of the damping is local, some method developed by (Özgüven, 1987; Denoël and Degée, 2009) can be used to efficiently calculate the frequency responses of a non-classically damped system in terms of the undamped modes. The complex modal analysis may be also a good choice to accurately calculate the frequency response (Adhikari, 2013). If only the lower modes are available, the frequency responses can be efficiently calculated by the method developed by (Li et al., 2014b)(Li et al., 2014a). In addition, complex modes can be used to transform any viscously damped system with N DOF into N independent second-order equations [see, e.g., (Kawano et al., 2013; Morzfeld et al., 2011; Ma et al., 2010) for details]. Note that in the dynamic response analysis, the primary computational effort is spent on the solution of the complex modes of the eigenproblem (2).

Real eigensolution techniques can be easily extended to handle the damped eigenproblem in terms of 2N-space (state-space) formulation, where N is the system dimension [see e.g., (Veletsos and Ventura, 1986) for details]. Although these state-space methods are exact in nature, they usually need heavy computational cost in practice due to the double size of system matrices. The state-space methods are not only computationally expensive, but also lack the physical insight provided by the superposition of the complex modes in the original physical space. To avoid the disadvantages mentioned previously, some efficiently computational methods in the original space were developed to compute the complex modes [see, e.g., (Kwak, 1993; Adhikari, 2011; Fischer, 2000; Holz et al., 2004; Rajakumar, 1993; Lee et al., 1998)].

In this paper, based on undamped modes, an efficient method is presented to calculate the complex eigenpairs by combining the Neumann series and the reduced basis technique. To avoid the modal truncation problem, the reduced basis is calculated by using a Neumann series expansion and only requires the undamped eigenpair of interest (i.e., it is not necessary to calculate all the undamped eigenpairs to hold the accuracy of engineering required). The sufficient condition for the Neumann series is derived and the computational complexity of the proposed method is discussed.

Solution of complex eigenvalue problem using classical normal modes

Premultiplying the damped eigenproblem by using $(\lambda_j^2\mathbf{M})^{-1}$ yields

$$\left(\mathbf{I}_N + \kappa_j\mathbf{M}^{-1}\mathbf{C} + \kappa_j^2\mathbf{M}^{-1}\mathbf{K}\right)\boldsymbol{\varphi}_j = \mathbf{0} \text{ with } \kappa_j = 1/\lambda_j \tag{3}$$

Here $\mathbf{I}_N$ denotes the identity matrix of size N. The previous equation can be rewritten as

$$\left(\mathbf{I}_N + \kappa_j\mathbf{M}^{-1}\mathbf{C}\right)\boldsymbol{\varphi}_j = -\kappa_j^2\mathbf{M}^{-1}\mathbf{K}\boldsymbol{\varphi}_j \tag{4}$$

By using the Neumann series expansion, one obtains

$$\boldsymbol{\varphi}_j = -\kappa_j^2 \sum_{k=0}^{\infty} \left(-\kappa_j \mathbf{M}^{-1}\mathbf{C}\right)^k \mathbf{M}^{-1}\mathbf{K}\boldsymbol{\varphi}_j \tag{5}$$

On the condition that the mode shapes are not changed significantly, it may be convenient to approximate the eigenvectors using undamped mode shapes

$$\boldsymbol{\varphi}_j \approx -\sum_{k=0}^{\infty} \left(-\kappa_j\right)^{k+2} \mathbf{r}_k^{(j)} \tag{6}$$

where

$$\mathbf{r}_0^{(j)} = \mathbf{M}^{-1}\mathbf{K}\mathbf{u}_j \text{ and } \mathbf{r}_k^{(j)} = \mathbf{M}^{-1}\mathbf{C}\mathbf{r}_{k-1}^{(j)} \qquad \forall\ k \geq 1 \tag{7}$$

In view of $\mathbf{M}^{-1}\mathbf{K}\mathbf{u}_j = \omega_j^2 \mathbf{u}_j$, Eq. (7) can be further simplified as

$$\mathbf{r}_0^{(j)} = \omega_j^2 \mathbf{u}_j \text{ and } \mathbf{r}_k^{(j)} = \mathbf{M}^{-1}\mathbf{C}\mathbf{r}_{k-1}^{(j)} \qquad \forall\ k \geq 1 \tag{8}$$

The basis vectors can be efficiently calculated using the matrix decomposition of the mass matrix **M**, which only needs to be obtained once for different eigenpairs. Once the matrix decomposition is available, the calculation of the series vectors by Eq. (8) involves only forward and backward substitutions, which are trivial. However, the coefficient of each basis vector is unknown.

In the reduced basis technique, the approximate result is searched within a subspace spanned by using reduced basis vectors. We define a reduced subspace $\mathbf{R}_j$ for each complex eigenpair in terms of the first *r* terms of the Neumann series expansion. That is

$$\mathbf{R}_j = \text{span}\left\{\mathbf{r}_0^{(j)}, \quad \mathbf{r}_1^{(j)}, \quad \ldots, \quad \mathbf{r}_{r-1}^{(j)}\right\} \in \mathbb{R}^{N\times r} \tag{9}$$

One obtains the approximate complex eigenvectors as

$$\boldsymbol{\varphi}_j \approx \sum_{k=0}^{r-1} \alpha_k^{(j)} \mathbf{r}_k^{(j)} = \mathbf{R}_j \boldsymbol{\alpha}_j \text{ where } \boldsymbol{\alpha}_j = \left\{\alpha_1^{(j)}, \quad \alpha_2^{(j)}, \quad \ldots, \quad \alpha_{r-1}^{(j)}\right\}^T \in \mathbb{R}^{r\times 1} \tag{10}$$

Since the approximate eigenvectors should satisfy the eigenproblem (2), we determine these unknown coefficients $\boldsymbol{\alpha}_j$ by substituting the approximate eigenvectors back into Eq. (2) and pre-multiplying it using $\mathbf{R}_j^T$, that is,

$$\left(\lambda_j^2 \mathbf{M}_R + \lambda_j \mathbf{C}_R + \mathbf{K}_R\right)\boldsymbol{\alpha}_j = \mathbf{0} \tag{11}$$

with

$$\mathbf{M}_R = \mathbf{R}_j^T \mathbf{M}\mathbf{R}_j \in \mathbb{R}^{r\times r},\ \mathbf{C}_R = \mathbf{R}_j^T \mathbf{C}\mathbf{R}_j \in \mathbb{R}^{r\times r} \text{ and } \mathbf{K}_R = \mathbf{R}_j^T \mathbf{K}\mathbf{R}_j \in \mathbb{R}^{r\times r} \tag{12}$$

The robustness of the rectangular basis matrix $\mathbf{R}_j$ can be improved by using a Gram–Schmidt orthonormalization procedure. The ($r\times r$) reduced system matrices are dense matrices, but they are symmetric and much smaller in size than the ($N\times N$) original system matrices. On the condition that the mode shapes of the systems do not have significantly changed, the approximate modified eigenvalue λ_j can be chosen by finding the eigenvalue

of the reduced eigenproblem (11) such that the absolute largest value of $\omega_j-\lambda_j$ is minimized. Next, based on the frequency shifting technique, we give a method to obtain the eigenvalue λ_j. The reduced eigenproblem by Eq. (11) can be reformed as

$$\left(\tilde{\lambda}_j^2\mathbf{M}_R+\tilde{\lambda}_j\bar{\mathbf{C}}_R+\bar{\mathbf{K}}_R\right)\boldsymbol{\alpha}_j=\mathbf{0} \text{ with } \tilde{\lambda}_j=\lambda_j-\omega_j \tag{13}$$

where the equivalent damping and stiffness matrices are

$$\bar{\mathbf{C}}_R=\left(\mathbf{C}_R+2\omega_j\mathbf{M}\right) \text{ and } \bar{\mathbf{K}}_R=\left(\mathbf{K}_R+\omega_j\mathbf{C}_R+\omega_j^2\mathbf{M}_R\right) \tag{14}$$

Once the first-order mode of the reduced eigenproblem (13) is solved. The approximate modified eigenvalue λ_j can be calculated. Once the vector $\boldsymbol{\alpha}_j$ is calculated, the modified eigenvectors can be obtained using Eq. (10). It is interesting to note that the first reduced basis vector is the undamped mode shape in nature. If the complex eigenpair can be calculated to satisfy suitable accuracy requirements by using one reduced basis vector, it means the system is close to the classically damped system and the PAM may produce small errors. If the complex eigenpairs need more than one reduced basis vector, under such circumstance, the PAM will produce unexpected errors and the proposed method can be used for suitable accuracy requirements.

In view of Eq. (2), the number r in Eq. (10) can be then determined if

$$e(\mathrm{i}\omega)=\left\|\left(\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K}\right)\boldsymbol{\varphi}_j\right\|_2<\varepsilon \tag{15}$$

where the parameter ε is a given accuracy for the absolute error. Alternatively, the number r in Eq. (10) may be determined by

$$\theta<\varepsilon_\theta \quad (\text{here } \cos\theta=\frac{\left|\mathbf{r}_r^H\mathbf{r}_{r-1}\right|}{\sqrt{\left(\mathbf{r}_{r-1}^H\mathbf{r}_{r-1}\right)\left(\mathbf{r}_r^H\mathbf{r}_r\right)}}) \tag{16}$$

Here the parameter ε_θ is a given accuracy. It means that increasing the reduced basis vectors can be stoped if the reduced basis vectors become linearly-dependent.

Computational considerations

For the convergent Neumann series expansion, one obtains the necessary and sufficient condition $\rho(\kappa_j\mathbf{M}^{-1}\mathbf{C})<1$. Here $\rho(\bullet)$ denotes the spectral radius of matrix $(\bullet)$. It means that all the eigenvalues of matrix $\kappa_j\mathbf{M}^{-1}\mathbf{C}$ have absolute values less than one. The maximal eigenvalues of matrix $\mathbf{M}^{-1}\mathbf{C}$ can be found by solving the minimal eigenvalues of $\mathbf{M}\mathbf{y}_j=s_j\mathbf{C}\mathbf{y}_j$. Once the minimal eigenvalue $s_{\min}$ is solved, the convergence condition can be given by

$$\left|s_{\min}\kappa_j\right|<1 \quad \text{or} \quad \left|s_{\min}\right|<\left|\lambda_j\right| \tag{17}$$

which can be approximated determined using the known undamped frequencies

$$\left|s_{\min}\right|<\left|\omega_j\right| \tag{18}$$

When the size N is large, the operation count to solve the undamped eigenproblem is $O(N^3)$ (Adhikari, 2013). The operation count for the $\mathbf{LDL}^T$ decomposition of the mass matrix $\mathbf{M}$ is $0.5Nb^2$ (Bathe, 1996), where b is the semi-bandwidth of the mass matrix. The process of forward and backward substitutions for these basis vectors given by Eq. (7) is $2LrNb$. Since the number of the basis vectors $r \ll N$ and the number of the calculated complex modes $L \ll N$, the other operation count is trivial. Therefore, the operation count of the proposed method is $O(N^3+0.5Nb^2+2LrNb)$. Since the half-bandwidth b is roughly proportional to $N^{0.5}$ (Bathe, 1996), the flop can be simplified as $O(N^3+0.5N^2+2LrN^{1.5})$ for the consistent mass matrix and $O(N^3+2LrN^{1.5})$ for the lumped mass matrix. For non-classically damped systems, the state-space method (the size of the state-space matrix is $2N$) to calculate the complex eigenproblem is $O(8N^3)$ (Adhikari, 2013). Therefore, the proposed method shows a clear advantage over the state-space method in engineering applications as $O(8N^3) > O(N^3+0.5N^2+2LrN^{1.5})$ in the case the consistent mass matrix and $O(8N^3) > O(N^3+2LrN^{1.5})$ in the case of the lumped mass matrix.

Examples and discussions

Example 1: Three-DOF viscously damped system. A three DOF viscously damped system is considered here. The mass, damping and stiffness matrices are

$$\mathbf{M} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \ \mathbf{C} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.175 & -0.175 \\ 0 & -0.175 & 0.175 \end{pmatrix}, \ \mathbf{K} = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix} \quad (19)$$

To illustrate the accuracy of the proposed method, two cases for distinct damping matrices are considered.

Case 1: consider the damping matrix given by Adhikari (Adhikari, 2011). In this case, the system does not satisfy the convergence condition for the Neumann series expansion and the assumption condition that the mode shapes are not changed significantly. **Table 1** lists the undamped frequencies and the complex eigenvalues using the exact state-space method and the proposed method for case 1. The damping coefficient of non-classically damped systems can be defined as (McLean, 2010)

$$g_j = -2\alpha_j \big/ \left|\omega_{dj}\right| \quad (\text{here } \lambda_j = \alpha_j + \mathrm{i}\omega_{dj})$$

which is popular in the aeroelastic filed. The damping coefficient g_j is approximately twice times the value of the conventional modal damping ratio. The MAC between the undamped mode shapes and exact complex mode shapes is 0.9953 for the first mode, 0.7734 for the second mode and 0.6587 for the third mode (the MAC value close to unity denotes the similarity whereas it close to zero denotes no similarity). As can be seen, the MAC of the last two modes is not much similarity in comparison with that of the first mode and the last two undamped frequencies are close. Therefore the system shows a significant modal coupling. The high modal coupling means that the system does not satisfy the assumption condition that the mode shapes are not changed significantly. The minimal eigenvalue of $\mathbf{M}\mathbf{y}_j = s_j\mathbf{C}\mathbf{y}_j$ is 0.8571 ($s_{\min}$=0.8571). It means that the convergence

condition for the Neumann series expansion is not satisfied. Although the convergence problem and high modal coupling exist in the special case, the proposed method with two basis vectors shows a good accuracy (the maximal error is 0.7469% and the minimum MAC is 0.9822). Therefore, although the convergence condition given by (17) is the necessary and sufficient condition for the Neumann series expansion, it may be only a sufficient condition for the proposed method. In addition, the condition, which assumes the mode shapes are not changed significantly, maybe not a necessary condition for the proposed method. Now, we use these calculated modes to calculate the frequency response functions (FRFs). For the sake of comparison, the FRFs are also calculated using the direct frequency response method (DFRM) and the PAM. The DFRM, which requires a matrix decomposition of the dynamic stiffness matrix at each excitation frequency, is considered as an exact result. Two typical FRFs are shown in **Figure 1**. The damping coefficient is 0.0328 for the first mode, 0.0771 for the second mode and 0.7546 for the third mode. Only two resonance peaks are visible since the damping coefficient of the third mode is high. It is shown that an unacceptable error is produced in the FRFs obtained by the PAM. However, the proposed method improves the results.

Table 1. Eigenvalues using state-space method and the proposed method for case 1.

Mode	Complex eigenvalue (exact)	Undamped frequency	One basis vector (r=1)			Two basis vectors (r=2)		
			Eigenvalues	Error (%)	MAC	Eigenvalues	Error (%)	MAC
1	-0.0103 + 0.6298i	0.6249	-0.0125 + 0.6248i	0.8734	0.9953	-0.0107 + 0.6295i	0.0715	0.9995
2	-0.0478 + 1.2407i	1.1547	-0.1458 + 1.1455i	11.0078	0.7734	-0.0406 + 1.2350i	0.7469	0.9822
3	-0.5252 + 1.2890i	1.5087	-0.4250 + 1.4476i	13.4770	0.6587	-0.5250 + 1.2825i	0.4708	0.9917

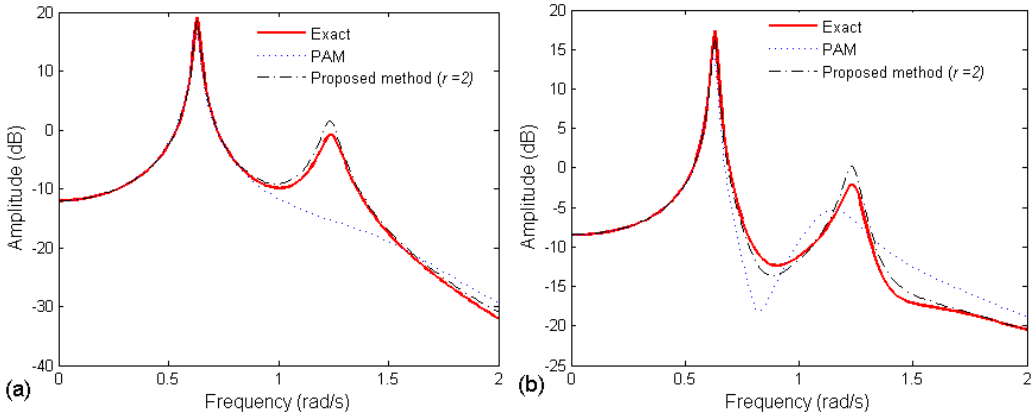

Figure 1. FRFs for case 1. (a) The FRF excited at the first DOF and measured at the second DOF. (b) The FRF excited at the three DOF and measured at the three DOF.

Case 2: the damping matrix is considered as Eq. (19). In this case, the system satisfies the assumption condition that the mode shapes are not changed significantly but does not satisfy the convergence condition for the Neumann series expansion. **Table 2** lists the eigenvalues for case 2. The MAC of the undamped mode shapes and exact complex mode shapes is 0.9999 for the first mode, 0.9962 for the second mode and 0.9935 for the third mode. These MAC values close to unity show the system satisfy the assumption condition that the mode shapes are not changed significantly. The minimal eigenvalue of $\mathbf{My}_j=s_j\mathbf{Cy}_j$ is 8.5714 ($s_{\min}$= 8.5714), which means that the system does not satisfy the

convergence condition for the Neumann series expansion. It is shown that, although the system does not satisfy the convergence condition in the special case, the proposed method with one basis vectors shows a good accuracy (the maximum error is 0.1378 and the minimum MAC is 0.9935) and the results of the proposed method with two basis vectors almost coincides with the exact results. As can be seen, the assumption condition can significantly affect the accuracy of the proposed method. In this case, two typical FRFs are shown in **Figure 2**. The damping coefficient is 0.0040 for the first mode, 0.0251 for the second mode and 0.0565 for the third mode. It is shown that, although the damping coefficient is relatively light, an unacceptable error is also produced in the FRFs obtained by the PAM. However, the proposed method improves the results and its result shows a good agreement with that of the DFRM.

Table 2. Eigenvalues using state-space method and the proposed method for case 2.

Mode	Complex eigenvalue (exact)	One basis vector (r=1)			Two basis vectors (r=2)		
		Eigenvalues	Error (%)	MAC	Eigenvalues	Error (%)	MAC
1	-0.0012 + 0.6250i	-0.0013 + 0.6249i	0.0095	0.9999	-0.0012 + 0.6250i	0.0008	1.0000
2	-0.0145 + 1.1561i	-0.0146 + 1.1546i	0.1281	0.9962	-0.0145 + 1.1563i	0.0161	0.9995
3	-0.0426 + 1.5060i	-0.0425 + 1.5081i	0.1378	0.9935	-0.0425 + 1.5061i	0.0049	0.9999

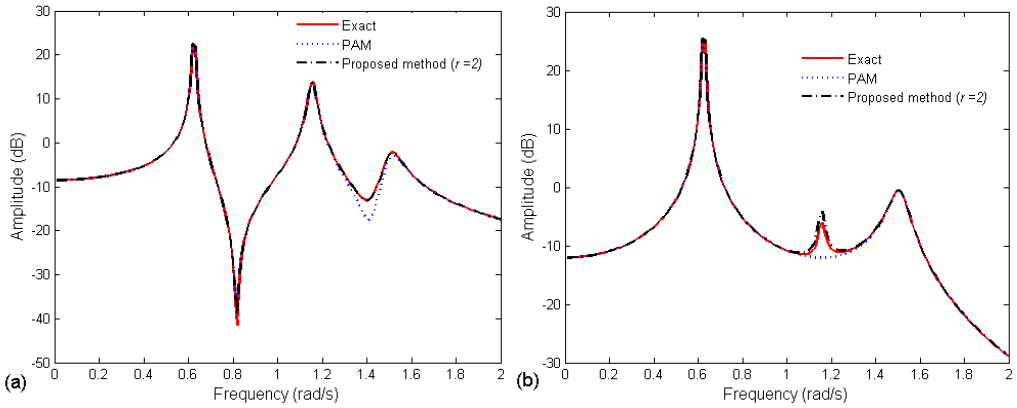

Figure 2. FRFs for case 2. (a) The FRF excited at the first DOF and measured at the first DOF. (b) The FRF excited at the first DOF and measured at the second DOF.

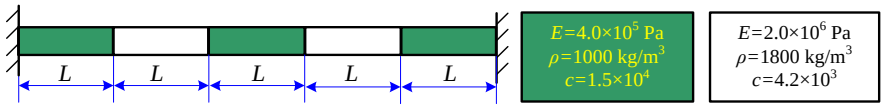

Figure 3. A damped truss structure.

Example 2: Damped truss structure. A simple but representative truss structure, shown in **Fig. 3**, is considered. For an element e, the elementary matrices are defined as follows:

$$\mathbf{M}_e = \frac{A\rho l_e}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \mathbf{K}_e = \frac{EA}{l_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \text{ and } \mathbf{C}_e = \frac{cA}{l_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

where A is the cross-section area, ρ is the density, E is the elastic modulus, c is the damping coefficient and l_e is the length of the truss element (l_e=0.5 m). here L=10 m and

A=0.0001 m^2. The DOF of the truss structure is 99. **Figure 4** shows the first DOF of the FRF excited at the first DOF and measured at the first DOF. The complex modal superposition method should be used to accurately calculate the FRF since the results of the PAM do not give an acceptable accuracy. **Table 3** lists the undamped frequencies and the complex eigenvalues using the exact state-space method and the proposed method. Two typical mode shapes calculated by using the exact state-space method and the proposed method with three basis vectors are shown in **Figure 5**. It is shown that the proposed method shows a good agreement with the exact results. The computational time of obtaining the first five complex eigenpairs is 3.2396e-2 seconds for the proposed method and 4.8613e-2 seconds for the state-space method. It means that the proposed method is faster than the state space method. If the DOF becomes larger, the time of the state-space method will increase rapidly and the proposed method will show a clear advantage over the state-space method as it discussed previously.

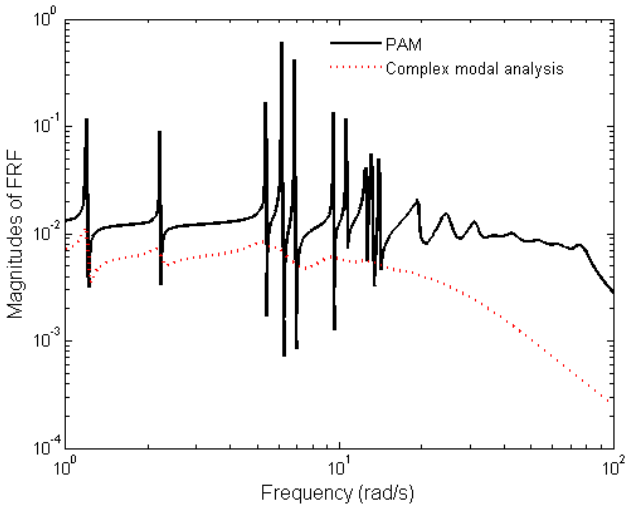

Figure 4. The FRF calculated by the PAM and the complex modal analysis.

Table 3. Eigenvalues using state-space method and the proposed method.

Mode	Complex eigenvalue (exact)	Undamped frequency	One basis vector (r=3)		Two basis vectors (r=4)	
			Eigenvalues	Error (%)	Eigenvalues	Error (%)
1	-2.4839e-2 +1.1883i	1.1885	-2.4836e-2 +1.1882i	5.2116e-3	-2.4836e-2 +1.1882i	5.1716e-3
2	-8.6932e-2 +2.2064i	2.2078	-8.6913e-2 +2.2061i	1.2450e-2	-8.6913e-2 +2.2061i	1.2273e-2
3	-3.5758e-1 +5.3648i	5.3479	-3.5588e-1 +5.3385i	4.8983e-1	-3.5592e-1 +5.3388i	4.8479e-1
4	-5.7556e-1 +6.1127i	6.1088	-5.7054e-1 +6.0848i	4.6164e-1	-5.7060e-1 +6.0851i	4.5686e-1
5	-7.9690e-1 +6.7949i	6.8224	-7.9366e-1 +6.7809i	2.1052e-1	-7.9375e-1 +6.7812i	2.0528e-1

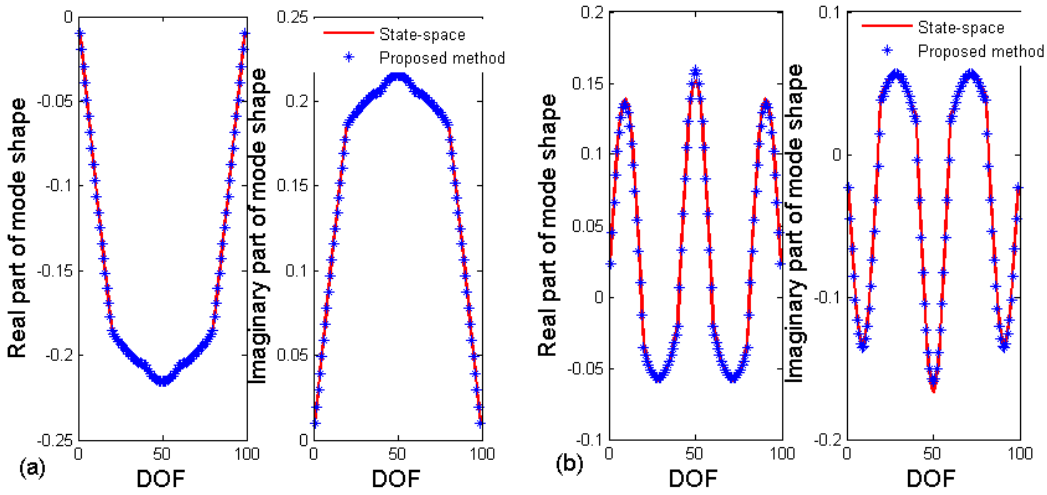

Figure 5. Complex mode shape. (a) First-order mode. (b) Fifth-order mode.

Conclusions

Based on undamped modes, an efficient method is presented to calculate complex eigenpairs by combining the Neumann series and the reduced basis technique. The method only requires the undamped eigenpair of interest. Some interest characteristic is shown and discussed in terms of numerical examples. It is shown that the complex eigenpairs can be calculated by simply postprocessing of undamped eigenpairs. Note that the first reduced basis vector is the undamped mode shape in nature. If complex eigenpair can be calculated by using one reduced basis vector to satisfy suitable accuracy, it means the system is close to the classically damped system and the proportional approximation method (PAM) may produce small errors. Therefore, it can be concluded that the proposed method can be reduced to the PAM when only the first reduced basis vector is used. If complex eigenpairs need more than one reduced basis vector to satisfy suitable accuracy requirements, under such circumstance, the PAM will produce unexpected errors and the proposed method can be used for suitable accuracy. Therefore, the proposed method is also developed as a criterion for choosing the PAM.

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