ODE-Solver-Oriented Computational Method for the Structural Dynamic Analysis of Super Tall Buildings

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Abstract

The paper is to introduce a computational methodology that is based on ordinary differential equations (ODE) solver for the structural systems adopted by super tall buildings in their preliminary design stage so as to facilitate the designers to adjust the dynamic properties of the adopted structural systems. The construction of the study is composed by following aspects. The first aspect is the modelling of a structural system. As a typical example, a mega frame-core-tube structural system adopted by some famous super tall buildings such as Taipei 101 building, Shanghai World financial center, is employed to demonstrate the modelling of a computational model. The second aspect is the establishment of motion equations constituted by a group of ordinary differential equations for the analyses of free vibration and resonant response. The solutions of the motion equations (that constitutes the third aspect) resorted to ODE-solver technique. Finally, some valuable conclusions are summarized.

Keywords: ODE-solver-oriented computational methodology, tall building structures, structural dynamic analysis, computational model of a mega frame-core-tube structural system, free vibration and resonant response, ODE solver

Introduction

Nowadays, we are experiencing an unprecedented level of activity in the design and construction of super tall buildings because of the limitation of land resources and advanced construction technology, ad hoc in China [X. zhao et al. (2011)]. The world architecture history has been rewritten by the multiformity of structural systems, the complexity of component arrangements and the variation of architectural styles of current super tall buildings. However, the analytical level for the investigation of dynamic properties of various structural systems adopted by super tall buildings lags behind their construction level. Both computational models and numerical methods for the dynamic analyses proposed hitherto by existing literatures are quite limited in their ability to model and to determine the three-dimensional motion of the structural systems.

For instance, Reza Kamgar, Mohammad Mehdi and Saadatpour [Reza et al. (2011)] developed a simple mathematical model based on Euler-Bernoulli beam theory to determine the first natural frequency of tall buildings including a framed tube, a shear core, a belt truss and an outrigger system with multiple jumped discontinuities in the cross section of the framed tube and shear core. Hong Fan, Q.L. Li, Alex Y. Tuan and Lihua Xu [Hong Fan et al. (2009)] investigated the seismic analysis of the structural system of Taipei 101, a mega-frame system with a central braced core connected to perimeter columns on each building face, by employing a 5-storey frame computational model composed by 3-D beams, 3-D columns and floor slabs. Wen-Hae Lee [Lee (2007)] simplified a tube-in-tube tall-building system as an Euler-

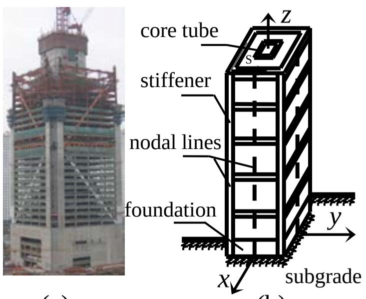
Bernoulli beam with variable cross-sections and then formulated an approximate solution procedure for the free vibration analysis.

In order to render the computational model of a super tall building system closer to practical engineering as well as the corresponding numerical method more efficient, the purpose of this paper is to present an ODE-solver-oriented computational methodology for the structural systems adopted by super tall buildings in their preliminary design stage so as to facilitate the designers to adjust the dynamic properties of the adopted structural systems. The construction of the study is composed by following aspects. The first aspect is the modelling of a structural system adopted by a super tall building. As a typical example, a mega frame-core-tube structural system as showed in Figure 1(a) adopted by some famous super tall buildings such as Taipei 101 building, Shanghai World financial center, is employed to demonstrate the modelling of a computational model. The second aspect is the establishment of motion equations constituted by a group of ordinary differential equations (ODE) for the analyses of free vibration and resonant response. The establishment utilized semi-discretization, displacement quantification and motion-field quantification techniques. The solutions of the motion equations (that constitutes the third aspect) resorted to an ODE solver technique (Yuan Si [Yuan (1991, 1993)]). Finally, some valuable conclusions are summarized.

1. Modelling of a super-tall building system

Figure 1(a) shows a mega frame-core-tube system adopted by some famous super tall buildings such as Taipei 101 building, Shanghai World financial center, etc. On structural aspects, the space mega frame is composed by two grades of members. The first grade is mega columns and beams, and the second grade is interiorly supplementary frames in the mega frame. The mega columns are generally made by tubes or other mega-substructures, which are jointed by the giant beams in every several floors. Since the geometric dimension (cross sectional area and inertial moment, etc.) of the members of the mega frame is very large, comparing with that of the supplementary ones, the characteristic makes this kind of structure has great load bearing capacity, strong sidesway stiffness. By analyzing the structural performance of the mega frame-core-tub system shown in Figure 1(a), we can conduct following two basic assumptions:

(1) Rigid floor slab assumption, that is, each floor is infinite rigid in its own plane;
(2) Strain state assumption, that is, the axial strain of a mega beam is negligible comparing with that of a mega column.



(a) (b)

Figure 1: A mega-frame-core-tube system and its computational model

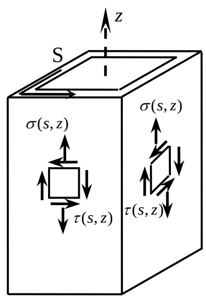


Figure 2: Stress state of the tube

Based on the two assumptions, we might simplify a mega frame-core-tube system shown in Figure 1(a) as a generalized equivalent continuous stiffened thin-walled tube-in-tubes as shown in Figure 1(b), and conclude that the wall of the thin-walled tubes is subjected to a plane stress state of longitudinal normal stress and horizontal as well as vertical shear stress, which are the functions with respect to the curved coordinate S, along the direction of centerline of the thin-walled tubes, and the longitudinal coordinate (vertical axis Z) of the tubes, as showed in Figure 2.

The subgrade or the foundation soil of the structural system is idealized as a semi-infinite elastic body, whose equivalent stiffness equations in the normal and tangential direction at the bottom and walls of a foundation pit have been formulated by employing the displacement equations of Mindlin [Mindlin (1936)] in a semi-infinite elastic body subject to a concentrated force acting in the interior of the semi-infinite elastic body. Using the equivalent stiffness equations [Gong (2007)], the interactions between the foundation and the subgrade (foundation soil) can be readily taken into considerations quantitatively.

Consequently, the computational model of a mega frame-core-tube system of a super tall building will be a generalized equivalent continuous tubular shell constituted by stiffened thin-walled tubes-in-tubes supported on a semi-infinite elastic body as shown in Figure 1(b).

2. Formulation of motion equations

2.1 *Semi-discretization technique and displacement quantification*

As shown in Figure 1(b), if we use one-variable functions $v_{0x}(z)$, $v_{0y}(z)$ and $\theta(z)$ defined on the vertical axis Z of the tubular shell, which are piecewise functions in most cases, the transverse displacements of the cross section of the tubular shell in the X and Y directions, and the rotation around the longitudinal axis Z will be represented as

$$\{v_0(z)\} = \left\{ \begin{Bmatrix} v_{0x}(z) \\ v_{0y}(z) \\ \theta(z) \end{Bmatrix}_j \right\}. \tag{1}$$

Similarly, if we make a semi-discretization along the cross-section central line S by the vertical lines named nodal lines and employ the one-variable functions $w_j^{in}(z)$, and $w_j^{ex}(z)$, respectively defined on the inner and outer nodal lines, and interpolation functions $\varphi_j(s_{in})$, and $\varphi_j(s_{ex})$ between the inner and outer nodal lines, the axial displacement or longitudinal warping of the tubular shell will be expressed as

$$\{u(s,z)\} = \begin{Bmatrix} [\varphi(s_{in})]\{w_{in}(z)\} \\ [\varphi(s_{ex})]\{w_{ex}(z)\} \end{Bmatrix}_j . \tag{2}$$

where $j = 1, 2, \cdots n$ is the segment number of the nodal lines in the longitudinal direction Z (1 may represent foundation and 2 to 6 may stand for the first to fifth floors and so on, for example), and the segment number depending upon the property variation of the building system up the height is the intersection number between nodal lines and the curvilinear coordinate (central line of the cross section) S; $\{v_0(z)\}$ and $\{u(s,z)\}$ are function sets, constituted by all of the basic unknown functions; $[\varphi(s)]$ a row vector, and $\{w(z)\}$ a column vector, respectively.

2.2 *Motion-field quantification*

For free vibration analysis, the longitudinal and transverse dynamic displacements of the structural system (the tubular shell) can be respectively written in Galerkin's form as

$$\{u(s,z,t)\} = \left\{ \begin{Bmatrix} [\varphi(s_{in})]\{w_{in}(z)\} \\ [\varphi(s_{ex})]\{w_{ex}(z)\} \end{Bmatrix}_j e^{i\omega t} \right\}, \tag{3}$$

$$\{v_0(z,t)\} = \left\{ \{v_0(z)\}_j \, e^{i\omega t} \right\}. \tag{4}$$

For forced vibration steady-state response analysis, if giving an arbitrary vertical ground-motion of $\{u_g(t)\}$, an arbitrary horizontal ground-motion of $\{T_g(t)\}$ in the X and Y directions, and the rotation around the longitudinal axis Z, for instance, the motion field of the computational model can be readily quantified as follow

$$\left.\begin{aligned}\left\{u^t(s,z,t)\right\}&=\left\{u_g(t)\right\}+\left\{u(s,z)\right\}r(t)\\ \left\{v_o^t(z,t)\right\}&=\left\{T_g(t)\right\}+\left[f(t)\right]\left\{v_0(z)\right\}\end{aligned}\right\}, \tag{5}$$

where $r(t)$ and $f(t)$ are the time functions concluded by means of the time-change law of $u_g(t)$ and $T_g(t)$, respectively.

2.3 *Motion equations or governing equations*

By employing above motion field, the total kinetic energy as well as the potential energy of the structural system including the strain energy stored in the subgrade can be readily estimated. Then, by using a Hamiltonian principle, the governing equations of the structural system can be derived conveniently, which are the ordinary differential equations (ODE) and corresponding boundary conditions. For instance, the motion equations for free vibration will lead to

$$\left.\begin{aligned}&\left\{F_s^u\right\}_1^{in}+\left\{F_i^u\right\}_1^{in}=\{0\},\quad \left\{F_s^u\right\}_1^{ex}+\left\{F_i^u\right\}_1^{ex}=\{0\}\\ &\left\{F_s^v\right\}_1^{in}+\left\{F_s^v\right\}_1^{ex}+\left\{F_i^v\right\}_1^{in}+\left\{F_i^v\right\}_1^{ex}-\left\{F_r^t\right\}_1^{ex}=\{0\}\end{aligned}\right\}, \tag{6}$$

$$\left.\begin{aligned}&\left\{F_s^u\right\}_j^{in}+\left\{F_i^u\right\}_j^{in}=\{0\},\quad \left\{F_s^u\right\}_j^{ex}+\left\{F_i^u\right\}_j^{ex}=\{0\}\\ &\left\{F_s^v\right\}_j^{in}+\left\{F_s^v\right\}_j^{ex}+\left\{F_i^v\right\}_j^{in}+\left\{F_i^v\right\}_j^{ex}=\{0\}\end{aligned}\right\}, \tag{7}$$

in which,

$$\left\{F_s^u\right\}=E[A]\{w''(z)\}-G[B]\{w(z)\}-G[C]\{v_0'(z)\},\quad \left\{F_i^u\right\}=m\omega^2[A]\{w(z)\},$$

$$\left.\begin{aligned}&\left\{F_s^v\right\}=G[D_1]\{v_0''(z)\}+G[C]^T\{w'(z)\},\left\{F_i^v\right\}=m\omega^2[D_2]\{v_0(z)\},\\ &\left\{F_r^t\right\}=C_r k_{nH}[E]\{v_0(z)\}\end{aligned}\right\}.$$

Equations (6) and (7) are the motion equations for the foundation and other segments of the computational model respectively, and their corresponding boundary conditions at the bottom of the foundation will be

$$\left.\begin{aligned}&\left\{E[A]\{w'(0)\}=k_{zD}[A]\{w(0)\}\right\}^{in},\quad \left\{E[A]\{w'(0)\}=k_{zD}[A]\{w(0)\}\right\}^{ex}\\ &\left[\left(G[D_1]\right)^{in}+\left(G[D_1]\right)^{ex}\right]\{v_0'(0)\}+\left\{\left(G[C]^T\{w(0)\}\right)^{in}\right\}\\ &\quad+\left\{\left(G[C]^T\{w(0)\}\right)^{ex}\right\}+k_{tD}[S]\{v_0(0)\}=\{0\}\end{aligned}\right\}. \tag{8}$$

The boundary conditions at the top of the computational model become as

$$\left.\begin{aligned}&\left\{E[A]\{w'(H)\}\right\}^{in}=\{0\},\quad \left\{E[A]\{w'(H)\}\right\}^{in}=\{0\}\\ &\left[(G[D_1])^{in}+(G[D_1])^{ex}\right]\{v_0'(H)\}+\left\{(G[C]^T\{w(H)\})^{in}+(G[C]^T\{w(H)\})^{ex}\right\}=\{0\}\end{aligned}\right\}. \tag{9}$$

Also the displacement consistence and generalized internal force equilibrium conditions at each connection of the computational model must be

$$\left.\begin{aligned}&\{w^{in}(H_k)\}_k=\{w^{in}(H_k)\}_{k+1},\ \{w^{ex}(H_k)\}_k=\{w^{ex}(H_k)\}_{k+1}\\ &\{v_0(H_k)\}_k=\{v_0(H_k)\}_{k+1}\end{aligned}\right\}, \tag{10}$$

$$\left.\begin{array}{l}\left\{E[A]\{w'(H_k)\}\right\}_k^{in}=\left\{E[A]\{w'(H_k)\}\right\}_{k+1}^{in},\\ \left\{E[A]\{w'(H_k)\}\right\}_k^{ex}=\left\{E[A]\{w'(H_k)\}\right\}_{k+1}^{ex},\\ \left\{[(G[D_1])^{in}+(G[D_1])^{ex}]\{v_0'(H_k)\}\right\}_k+\left\{G[C]^T\{w(H_k)\}\right\}_k^{in}\\ \quad+\left\{G[C]^T\{w(H_k)\}\right\}_k^{ex}=\left\{[(G[D_1])^{in}+(G[D_1])^{ex}]\{v_0'(H_k)\}\right\}_{k+1}\\ \quad+\left\{G[C]^T\{w(H_k)\}\right\}_{k+1}^{in}+\left\{G[C]^T\{w(H_k)\}\right\}_{k+1}^{ex}\end{array}\right\}. \tag{11}$$

The meaning of the matrices such as [A], [B], etc. can be referred to [Gong (2010)]. It is observed, in mathematic view, that the problem about the free vibration of a super tall building system is an eigenvalue problem, and its governing ordinary differential equations (ODE) can be theoretically solved by an ODE solver such as COLSYS [Ascher (1981)], a general purpose program developed to solve various ODE problems. However, the normal ordinary differential equation solver can only solve the standard ODE problem. Consequently, a computational software package known as EIGENCOL [Yuan (1991, 1993)] has been developed to solve the eigenvalues and corresponding modes efficiently [Yaoqing Gong (2010)].

3. ODE-Solver Method

As mentioned previously, the free vibration of a super tall building system is an eigenvalue problem of a group of ordinary differential equations, which can be theoretically solved by an ODE solver. However, a normal ordinary differential equation solver can only solve a standard ODE problem. In order to find the eigenvalues, a computational software package known as EIGENCOL [Yuan (1991, 1993)] has been developed to solve the eigenvalues and corresponding modes efficiently. According to the technique proposed in the literatures, before the ordinary differential equations with eigenvalues are solved, they should be transformed into the standard ODE forms accepted by COLSYS [Ascher (1981)]. The procedure includes following steps.

3.1 *Coordinate transformation*

The solving interval of standard ordinary differential equations must be $[0,1]$. Thusly, the coordinate transformation must be performed for a practical problem with the solving interval of $[0, H_j]$, for example. At this point, the transformation technique will be

$$\xi = x / H_i, \quad \frac{d(\ )}{dx} = \frac{d(\ )}{d\xi}\cdot\frac{d\xi}{dx} = \frac{1}{H_i}\frac{d(\ )}{d\xi}.$$

3.2 *Trivial ODE conversion technique*

Because eigenvalues are undetermined constants and also a part of the solution of a group of ODEs, the determination of the unknown constants become a key point for the solution of the group of ODEs. Therefore, a trivial ODE is necessary to convert the ODEs with eigenvalues into a new set of standard ODEs in which an eigenvalue, say ω^2, has been made as an unknown function. In view of the derivative of a constant is zero, the trivial ODE can be thusly established as

$$\lambda' = d(\omega^2) / d\xi = 0\,. \tag{12}$$

The addition of equation (12) will lead to one more corresponding boundary condition. Finding the condition introduces another technique, equivalent ODE technique.

3.3 *Equivalent ODE technique*

If we define a normalized function with respect to the forgoing mentioned basic unknown functions as

$$R(\xi)=\frac{\int_0^{\xi}\left(\{w(\zeta)\}_{in}^T\{w(\zeta)\}_{in}+\{w(\zeta)\}_{ex}^T\{w(\zeta)\}_{ex}+\{v_0(\zeta)\}^T\{v_0(\zeta)\}\right)d\varsigma}{H^2}, \tag{13}$$

where H is the total height of the structural system. Equation (13) can be recognized as a generalized inner production of the basic unknown functions, and obviously

$$\frac{dR}{d\xi}=R'(\xi)=\frac{\left(\{w(\xi)\}_{in}^T\{w(\xi)\}_{in}+\{w(\xi)\}_{ex}^T\{w(\xi)\}_{ex}+\{v_0(\xi)\}^T\{v_0(\xi)\}\right)}{H^2}. \tag{14}$$

Also if we set

$$R(1)=\frac{\int_0^{1}\left(\{w(\zeta)\}_{in}^T\{w(\zeta)\}_{in}+\{w(\zeta)\}_{ex}^T\{w(\zeta)\}_{ex}+\{v_0(\zeta)\}^T\{v_0(\zeta)\}\right)d\varsigma}{H^2}=1, \tag{15}$$

the equation will become a standard normalized condition, and we can find two useful boundary conditions as follows

$$\left.\begin{matrix}R(0)=0\\R(1)=1\end{matrix}\right\}. \tag{16}$$

By employing above trivial ODE conversion and equivalent ODE techniques, one can transform ordinary differential equations with eigenvalues into a new group of standard ODEs. For instance, equations (6), (7), (12) and (14) constitute a group of standard ODEs, and equations (8), (9), (10), (11) and (16) become their corresponding boundary conditions. The group of ordinary differential equations can be readily solved by a normal ODE solver such as COLSYS [Ascher (1981)].

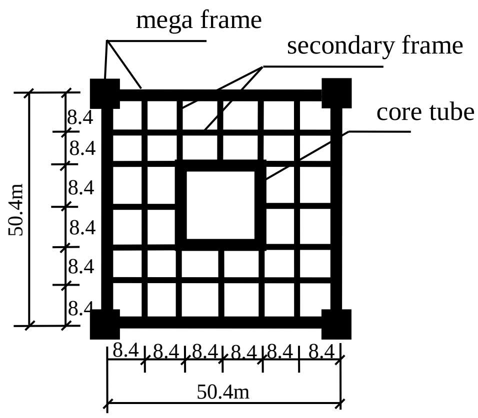


Figure 3. cross-section of a mega frame-core-tube system

4. Example and computational result analysis

The purpose of the section is to demonstrate the numerical determination of resonant response for a super tall building system due to a given complex ground motion.

4.1 *Example*

Figure 3 shows the cross section of a mega frame-core-tube system adopted by a reinforced concrete super tall building under its structural construction as shown in Figure 1 (a). The height of the main superstructure is 261.9 meters, and the height of its foundation structure is 21 meters. The cross-section area of all the mega columns and beams is $2.4\times2.4\text{m}^2$; the cross-section size for all the columns and beams of the secondary frame in the mega frame is $0.7\times0.7\text{m}^2$; the distance between two columns is 8.4m. A box-pile foundation is implemented, and the foundation soil is clayey silt. The equivalent stiffness of soil at the bottom and side faces of the foundation pit is respectively utilized as follows.
$K_{zD}=r_d\,32.4\times10^5$ KPa/m, $K_{tD}=r_d\,25.1\times10^5$ KPa/m, $K_{tH}=r_d\,25.1\times10^5$ KPa/m, $K_{nH}=r_d\,32.8\times10^5$ KPa/m. r_d is a coefficient that depends on the realistic site conditions (1.0 is used in this example); C_r is the contact coefficient between the foundation and the subgrade (takes 0.5 in this example); the materials used in the structural system are respectively: the thickness of the wall for the foundation tube is

0.6m, the concrete level is $C50$, and the concrete level for the mega frame is $C50$ too; the concrete level executed for the secondary frame is $C35$, the thickness of the wall for the inner tube is 0.40m, 0.45m for the outer tube, and the concrete level for the tubes is $C40$.

4.2 *Computational result analysis*

In the following tables and figures P_x, P_y, P_z and P_θ represent a group of natural periods of the structural system in the X, Y, Z, and θ (around axis Z) directions, respectively. It is implied that if the ground motion periods coincide with the group of natural periods of the structural system, the resonance of the structural system will occur. That is, if $2\pi/\omega_x=P_x$, $2\pi/\omega_y=P_y$, $2\pi/\omega_z=P_z$ and $2\pi/\omega_\theta=P_\theta$, resonance will occur; $\overline{v}_{0x}$, $\overline{v}_{0y}$, θ, $\overline{w}_{in}$, and $\overline{w}_{ex}$ stand for the resonant displacement amplitudes at the top of the structural system in the transvers directions as well as the resonant warping at the top of inner and outer tubes of the computational model.

Table 1. Resonant response of coupling vibration

P_x	P_y	P_z	P_θ	$\overline{v}_{0x}$	$\overline{v}_{0y}$	$\overline{w}_{in}$	$\overline{w}_{ex}$
3.9	4.9	4.2	21	1866.75	52.70	342.66	372.85
4.5	4.9	4.2	1.3	38.66	34.46	13.99	15.07
2.9	3.1	3.1	3.1	1533.69	11.15	319.34	344.75

Table 2. Influence of dynamic-property adjustment on resonant response

P_x	$\overline{v}_{0y}$	$\overline{w}_{in}$	$\overline{w}_{ex}$
7.9	0.45	0.05	0.11
3.5	1.62	0.46	0.28
2.2	33.86	5.05	6.52
1.7	151.96	28.95	28.38
1.5	245.71	45.16	44.71

Table 3. Influence of structural stiffness on resonant response

structural stiffness	$\overline{v}_{0x}$	$\overline{v}_{0y}$	$\overline{\theta}$	$\overline{w}_{in}$	$\overline{w}_{ex}$
3.15	3.51	3.25	-0.82	0.45	2.53
3.5	1.69	-0.16	-0.82	0.26	1.22
4.15	-3.34	-26.24	-0.82	-1.87	-5.25
4.55	9.97	463.79	-0.82	38.67	107.55

Table 4. Influence of foundation stiffness on resonant response

foundation stiffness	$\overline{v}_{0x}$	$\overline{v}_{0y}$	$\overline{\theta}$	$\overline{w}_{in}$	$\overline{w}_{ex}$
2.2	3.58	4.60	-0.82	0.58	2.59
3.2	3.55	4.53	-0.82	0.57	2.56
4.2	3.53	4.49	-0.82	0.57	2.55

5. Conclusions

The structural resonance will occur when the ground motion period in one direction is very close to the natural period of the structural system in the identical direction. The characteristic of the computational result is that the dynamic response value is very large (should be infinite theoretically), as shown in Figure 4, a step change is happening.

A designer must pay attention to the coupling natural periods of a structural system as long as the movement of its foundation soil during an earthquake is very hard to predict or evaluate quantitatively. The adjustment of a structural system, including the change of its material, arrangement of its components, etc., will lead to the change of its dynamic property, especially its coupling natural period that possess many combinations, as listed in table 1. As shown in Figure 5, when the ground motion period changes to a certain degree, the structural system might experience a different coupling resonant state. Also as listed in table 2, they implicitly teach us that the dynamic property improvement of a structural system just in a single direction could render the structural system to stay in a potential coupling resonant state.

The resonant periods or natural periods of a structural system strongly depend on its global rigidity, as shown in table 3. The reduction of global rigidity of a structural system will make its natural period become longer, and vice versa.

The influence of the foundation stiffness of a structural system on its resonant periods or natural periods is not obvious if the superstructure remain unchanged, as shown in table 4. These computational results tell us that it is not a wise way to improve the dynamic property of a structural system by means of increasing the size of the foundation in its aseismic design.

The methodology presented in the paper is helpful for the determination of coupling frequencies or periods of a complex structural system, which are very hard to find in the published literatures hitherto and to determine by utilizing other numerical methods.

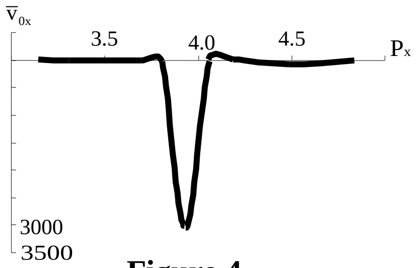


Figure 4.
variation law between $\overline{v}_{0x}$ and P_x

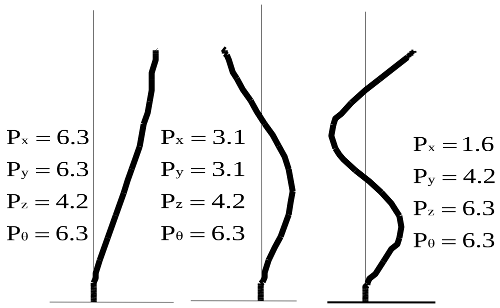


Figure 5.
the first three coupling resonent modes of v_{0x}

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