Topology Optimization of Anisotropic Materials under Harmonic Response

Based on ICM Method

*H.L. Ye, Y.M. Li, N. Chen, Y. K. SUI

College of Mechanical Engineering and Applied Electronics Technology, Beijing University of Technology, China *Corresponding author: yehongl@bjut.edu.cn

Abstract

Topology optimization of anisotropic materials is one of the most challenging research topics in the field of structural optimization and full of innovation. This paper aims at to find the optimal lay-out of anisotropic materials under harmonic response loads within a specified region. The optimization model which subject to response amplitude of the harmonic excitation is established and solved by using Independent, Continuous, Mapping (ICM) method. The filter function of elemental mass matrix, elemental stiffness matrix and elemental weight would be introduced, by which the three matrixes are updated in iteration putted into the dynamic topology optimization of differential equation to analyses the design sensitivity and optimize the structure. An explicit expression of constraint(s) with respect to the topological variables is obtained based on Rayleigh's quotient and sequential approximation method with filter functions. Then, the mathematical formulation of optimal problem of anisotropic materials is established and solved by dual sequence quadratic programming (DSQP). Finally, Numerical examples are provided to demonstrate the validity and effectiveness of the ICM method.

Keywords: Topology optimization; anisotropic materials; harmonic response; ICM method; dynamic topology optimization

Introduction

The vibration Control is a major problem in industrial engineering. How to make materials property achieve optimization and lightweight as far as possible at the same time, is bottleneck of the highend equipment manufacturing and aerospace industry, which urgently need to be solved. The traditional parameter optimization for structure optimization design cannot meet the requirement of lightweight design in the engineering field. However, through the method of topology optimization, the structure of the topology configuration could be redesign, which is of a new train thought of the structural dynamic design.

A large number of examples have proved that topology optimization is a magic tool to improve the material mechanics performance. Now, many studies have been carried out on dynamic response topology optimization. SIMP method [*Allahdadian S, Boroomand B, Barekatein A R(2012).*] is applied to study the optimal topology of the support structure with minimum compliance design under harmonic force. Kang [*Kang Z, Zhang X, Jiang S, et al(2012).*] aims at damping structure composite board to research minimum amplitude of topology optimization under harmonic excitation by SIMP and GCMMA method. Rong [*Rong J H, Xie Y M, Yang X Y, et al(2000)*] applies the ESO method to study the random dynamic response of minimum weight topology optimization problems. Zhang [*Zhang Qiao, Zhang Weihong, Zhu Jihong. (2010)*] make use of the RAMP methods research the dynamic response topology optimization subject to the random dynamic response of white noise excitation. Recently there are some literatures such as [*Motamarri*]

P, *Ramani A*, *Kaushik A*(2012)]take advantage of equivalent statics to solve the problem of dynamic response, it avoids to solve the complicated problems such as dynamic equation problems.

In engineering field, engineers often want to get the optimal structure of minimizing weight with satisfying some mechanical constraint at a certain point of interest. As for the minimum weight problem subjected to dynamic response displacement, the objective function and constraints include higher nonlinear implicit function equations. It is more difficult to have it explicit, and difficult to analysis the design sensitivity. However, sensitivity analysis of dynamics is a major problem for topology optimization.

In this paper, ICM method [*Sui Yunkang*(1996), *Sui Yunkang* & *Ye Hongling*(2013)]is extended to construct the optimal model of anisotropic materials under harmonic response loads within a specified region. And the optimal model is solved by dual sequence quadratic programming. Numerical examples show that this method is effective and valid for the problem of topology optimization subjected to dynamic response displacement.

The ICM (Independent Continuous Mapping) method for anisotropic materials

ICM method, namely Independent, Continuous and Mapping method, designs a special type of topological variable independent of specific physical quantity to indicates the 'exist-null' of elements, which is proposed by Sui (1996) for skeleton and continuum structures.

The "polish function" and "filter function" are the key points in the ICM method, which are used to map discrete variables and inverse continuous variables. By introduce material's retention ratio v, the discrete variables "0-1" could indicate the optimal lay-out of or topological structure, namely

$$t_{i} = H(v) = \begin{cases} 1 & v_{i} / v_{i}^{0} \in (0, 1] \\ 0 & v_{i} / v_{i}^{0} = 0 \end{cases}$$
(1)

Where t_i is topological variable, v_i^0 is material's initial value, H(v) is mapping relation, which can be regard as a step function like Fig.1. The "polish function" means using a smoothed curve to approximate the step function like Fig.2. And the "filter function" is inverse function to the "polish function" like Fig.4. The filter function can also be considered as the inverse function of the step function like Fig.3.



The important role of filter function is use to identify the physical parameters, the element weight, element quality matrix and element stiffness matrix cloud be recognized.

$$w = \overline{w} f_w(t_i), \boldsymbol{M} = \boldsymbol{M} f_m(t_i), \boldsymbol{K} = \boldsymbol{K} f_k(t_i)$$
(2)

Where \bar{w} , \bar{M} , \bar{K} are respectively initial element weight, initial element mass matrix, and initial element stiffness matrix. *w*, *M*, *K* are weight, mass matrix and stiffness matrix of *i*th element. $f_w(t_i)$, $f_k(t_i)$, $f_m(t_i)$ are respectively filter functions of weight, mass matrix, stiffness matrix of *i*th element. Thus, the model of continuum topology optimization with dynamic response constraints can be written as follows

$$\begin{cases} find \quad t \in E^{N} \\ make \ W = \sum_{i=1}^{N} f_{w}(t_{i})w_{i}^{0} \rightarrow \min \\ s.t. \quad u_{j}(f_{k}(t_{i}), f_{m}(t_{i})) \leq \overline{u_{j}}(j = 1, \cdots, J) \\ \quad 0 \leq t_{i} \leq 1 \quad (i = 1, \cdots, N) \end{cases}$$

$$(3)$$

Where $\overline{u_j}$ is allowable amplitude constraint, J is the total number of effective displacement constraints. As the equations of continuum problem, the filter function could recognize the elastic tensor and density of structure

$$\begin{cases} C_{ijkl} = f_k C_{ijkl}^0 \\ \rho = f_m \rho^0 \end{cases}$$
(4)

Where f_k and f_m are filter function equations of different physical parameters. As we know, there is a relationship between C_{ijkl} and Young's modulus. Assume that there is a matrix A

$$\boldsymbol{C}_{ijkl} = \boldsymbol{E}\boldsymbol{A} = f_k \boldsymbol{C}_{ijkl}^0 = f_k \boldsymbol{E}^0 \boldsymbol{A}$$
(5)

For 2d orthotropic material, assuming $\sigma_{33} = \sigma_{32} = \sigma_{31} = 0$, and ignoring the *z* axis direction of the two modulus, elastic tensor can be expressed as:

$$\boldsymbol{C}_{ijkl} = f_k \boldsymbol{C}_{ijkl}^0 = \begin{bmatrix} f_k \frac{E_1^0}{1 - v_{12}v_{21}} & f_k \frac{E_2^0 v_{21}}{1 - v_{12}v_{21}} & 0 \\ f_k \frac{E_2^0 v_{21}}{1 - v_{12}v_{21}} & f_k \frac{E_2^0}{1 - v_{12}v_{21}} & 0 \\ 0 & 0 & f_k \boldsymbol{G}_1^0 \end{bmatrix}$$
(6)

So, for the anisotropic material, Eq(3) could be expressed

$$E_1 = f_k E_1^0, E_2 = f_k E_2^0, G_{12} = f_k G_1^0$$
(7)

Usually, the composite material's each independent modulus displays a marked difference. The high ratio in low topology variable (null zone) may impact on structure mechanics of exist zone performance, as the local mode in dynamic topology optimization. So ICM identification equations for 2d orthotropic material could be

$$E_{1}, E_{2}, G_{12} = \begin{cases} E_{1}^{0} f_{k}, E_{2}^{0} f_{k}, G_{12}^{0} f_{k} & 0.1 \le t \le 1.0\\ \max\left(E_{1}^{0}, E_{2}^{0}, G_{12}^{0}\right) f_{k} & t_{\min} \le t \le 0.1 \end{cases}$$
(8)

In ICM method, the polish function is applied to eliminate intermediate variables

$$E = E_0 \bullet f(t) \bullet P_\beta(t) \tag{9}$$

Where $\boldsymbol{E} = (E_1, E_2, G_{12}), \boldsymbol{E}_0 = (E_1^0, E_2^0, G_{12}^0)$. In order to prevent the numerical instability, define

$$\mu_{ij} = \frac{d(i,j)}{\sum_{j=1}^{N} d(i,j)}$$
(10)

Eq.9 could be

$$E = \sum_{i=1}^{i < j} \mu_{ij} \bullet E_i \bullet f(t) \bullet P_{\beta}(t)$$
(11)

The β of Eq.9 is a variable parameter. The topology variable t_i is approaching to 0-1 with the β increasing. Throughout the iterative process, β is changed in stages.

Dynamic response equations

Dynamic response equation has been extensively researched.

$$M\ddot{u} + C\dot{u} + Ku = P \tag{12}$$

Define the structural damping coefficient is R, the dynamic equation can be turned into

$$\left(-\omega^{2} \left[\boldsymbol{M}\right] + \left(i\omega R + 1\right) \left[\boldsymbol{K}\right]\right) \left\{\boldsymbol{u}\right\} = \left\{\boldsymbol{P}\right\}$$
(13)

In the process of optimization, the reciprocal transformation to design variables as follows

$$x_i = \frac{1}{f_k(t_i)} \tag{12}$$

Derivation with design variables for both sides of equation, then

$$\left(-\omega^{2} \left[\boldsymbol{M}\right] + (i\omega R + 1)\left[\boldsymbol{K}\right]\right) \left\{\frac{\partial \boldsymbol{u}}{\partial x}\right\} = \left\{\frac{\partial \boldsymbol{P}}{\partial x}\right\} - \left(-\omega^{2} \frac{\partial \left[\boldsymbol{M}\right]}{\partial x} + (i\omega R + 1)\frac{\partial \left[\boldsymbol{K}\right]}{\partial x}\right) \left\{\boldsymbol{u}\right\}$$
(15)

P is constant, so $\frac{\partial \mathbf{P}}{\partial x} = 0$, define

$$\begin{cases} \left(-\omega^{2}\frac{\partial[\boldsymbol{M}]}{\partial x} + (i\omega\boldsymbol{R}+1)\frac{\partial[\boldsymbol{K}]}{\partial x}\right)\{\boldsymbol{u}\} = \{\boldsymbol{Z}\} \\ \frac{\partial\boldsymbol{u}}{\partial x} = \{\boldsymbol{\varphi}\} \end{cases}$$
(16)

Through the filter functions, the derivative of stiffness matrix and mass matrix is as followed,

$$\frac{\partial [\mathbf{K}]}{\partial x} = \frac{\partial f_k(t_i)}{\partial x} [\mathbf{K}_0], \frac{\partial [\mathbf{M}]}{\partial x} = \frac{\partial f_M(t_i)}{\partial x} [\mathbf{M}_0]$$
(17)

So solving the sensitivity is transformed into the problem of solving differential equations

$$\left(-\omega^{2} \left[\boldsymbol{M}\right] + (i\omega R + 1) \left[\boldsymbol{K}\right]\right) \{\boldsymbol{\varphi}\} = \{\boldsymbol{Z}\}$$
(18)

Solve Eq(12) could get the dynamic response \boldsymbol{u} and Eq(17) could get the sensitivity $\boldsymbol{\varphi}$.

Numerical solution of topology optimization

In order to solve the optimal model of Eq(3), the objective function needs to be modified by second order Tailor expansion. The power function is adopted to recognize structure weight as filter function. The objective function of

$$w = \sum_{i=1}^{N} w^{0} f_{w}(t_{i}) = \sum_{i=1}^{N} \frac{w^{0}}{x_{i}^{\alpha}}, \alpha = \frac{a}{b}$$
(19)

For completely eliminating the intermediate variable, the optimization model is also need to be modified, so polish function is introduced to objective function

$$w = \sum_{i=1}^{N} \frac{w^{0}}{\left(P_{\beta}\left(x_{i}\right)\right)^{\alpha}}$$
(20)

Solve the first and second order partial derivative

$$\begin{cases} \frac{\partial w}{\partial x} = \sum_{i=1}^{N} \frac{-\alpha \left(e^{-\beta} + \beta e^{-\beta x}\right)}{x \left(1 + x e^{-\beta} - \beta e^{-\beta x}\right)^{(\alpha+1)}} w^{0} \\ \frac{\partial^{2} w}{\partial x^{2}} = \sum_{i=1}^{N} \frac{-\alpha \left(\alpha + 1\right) \left(e^{-\beta} + \beta e^{-\beta x}\right)^{2}}{\left(1 + x e^{-\beta} - \beta e^{-\beta x}\right)^{(\alpha+2)}} + \frac{\alpha \beta^{2} e^{-\beta x}}{\left(1 + x e^{-\beta} - \beta e^{-\beta x}\right)^{(\alpha+1)}} w^{0} \end{cases}$$
(21)

Omit the constant terms, the objective function is approximated as

$$W = \sum_{i=1}^{N} (a_i x_i^2 + b_i x_i)$$
(22)

Constraint function $u_j(f_k(t_i), f_m(t_i)) \le \overline{u_j}$ of Eq(3) can also be explicited by Taylor's approximation. However, as the local approximation, it will take more truncation errors. So we have to find another effective way to make it explicit.



Fig.1 The process card of sequential approximate



Fig.2 The process card of double mapping

Fig.1 shows the process of sequential approximate. Where v denotes the number of iterations of topology optimization process. PO is the original mathematical programming, PA is mapping mathematical programming. M denotes mapping relationship, M^{-1} denotes inverse relationship $x^{(k+1)}$ is the solution of PA. The solution of the original mathematical programming x^* is approximated to $x^{(k+1)}$, namely $x^* = Ex^{(k+1)}$, E is defined as the unit matrix. Sequential

approximation can be understood as the PO to PA when the iterative process start every times. It is turn into a parallel mapping and inversion, as $x^{(0)} \rightarrow x^{(1)} \rightarrow \cdots \rightarrow x^{(k)} \rightarrow \cdots \rightarrow x^*$, equal to $x^* = M^{-1}x^{(0)}$. According to RMI(Relation Mapping Inverse)(Sui, 1996), there is slight error in the sequence of approximate approach. The inverse relationship M^{-1} is simply putted as E, The process of sequential approximate of Fig.1 could be improved as following card.

Fig.2 shows the process of double mapping. PO can be mapped to PD by precise mapping M_e , which is the process sequential approximate. This is a series-parallel connection mapping and inversion method. PO and PA are in series by PD, which is formed a double mapping. The sequential approximate PD to PA^(k) formed a multiple parallel, then

$$\left(RMI\right)^{2} = \left(RMI_{1}^{(j)}\right) \times \left(\sum_{k} RMI_{2}^{(k)}\right)$$
(23)

Where the *j* is the outer circulation variable, *k* is the inner circulation variable. The mapping relation M_a is defined linear Taylor expansion.

Now, the response amplitude constraints $u_j(f_k(t_i), f_m(t_i)) \le \overline{u_j}$ could be

$$\boldsymbol{u}_{j} = \boldsymbol{u}_{j}^{k-1} + \sum_{i=1}^{N} \frac{\partial \boldsymbol{u}_{j}}{\partial t_{i}} \left(x_{i} - x_{i}^{(k-1)} \right) \leq \overline{\boldsymbol{u}_{j}}$$
(24)

Define

where

$$\boldsymbol{\phi}_{ji} = \overline{\boldsymbol{u}_{j}} - \boldsymbol{u}_{j}^{(k-1)} + \sum_{i=1}^{N} \left| \boldsymbol{u}_{j}^{\prime} \right| \boldsymbol{x}_{i}^{(k-1)}$$
(25)

Eq(24) can be transformed

$$\sum_{i=1}^{N} \left| \boldsymbol{u}_{j}^{\prime} \right| \boldsymbol{x}_{i} < \boldsymbol{\phi}_{ji}$$

$$(26)$$

From above calculation, the optimization mode is

$$\begin{cases} find \quad \mathbf{x} \in E^{N} \\ make \quad W = \sum_{i=1}^{N} (a_{i}x_{i}^{2} + b_{i}x_{i}) \rightarrow \min \\ s.t. \quad \sum_{i=1}^{N} \left| \overline{u'}_{j}x_{i} \right| \leq \phi_{ji} (j = 1, \cdots, J) \\ \underline{x_{i}} \leq x_{i} \leq \overline{x_{i}} (i = 1, \cdots, N) \\ \overline{\omega}_{a} \leq \omega \leq \omega_{b} \end{cases}$$
(13)

Considering that the number of design variables is frequently quite bigger than that of constraints in topology optimization of continuum structure, the programming discussed-above could be converted into dual programming according to dual programming theory in order to solve the optimal model simply.

Now, we employ the Dual Quadratic Programs to solve the optimal model (27).

$$\begin{cases} find \quad z \\ make \, \Phi(z) \to \max \\ st. \quad z \ge 0 \end{cases}$$

$$\Phi(\mathbf{x}, z) = \min_{\underline{t_i} \le t_i \le 1} (L(\mathbf{x}, z)), L(\mathbf{x}, z) = \sum_{i=1}^{N} (a_i x_i^2 + b_i x_i) + \sum_{j=1}^{J} z_j \sum_{i=1}^{N} (\overline{u'_j} x_i - \phi_{ji}).$$

$$(28)$$

From Kuhn-Tucker condition, we can get the standard quadratic programming. Then solve it and update design variables until the convergence condition of structural weight is satisfied. In this paper, a precision of convergence is prescribed to be 0.001.

Numerical example

Take T300/4211 as base structure of topology optimization, the modules of which is as fellow: E1=126GPa, E2=8GPa, G12=3.7GPa, $\rho = 15600$ kg/m³. The damping coefficient of the structure is 0.02 and structure size is $80 \times 50 \times 10$ mm. The exciting force is 1000N located bottom right corner, the frequency of which is 1000Hz, Divide 80×50 meshes. The angle of material coordinate system and geometric coordinate system are respectively 0 °, 30 °, 45 °, 60 ° and 90 °.



Fig.3 Base structure











Fig.6 Iteration history of response amplitude

The optimized topology configurations for different ply angles are shown in Fig.4. Iteration history of structural weight and response amplitude with different ply angles are given in Fig.5-Fig.6. As a result, we find that the optimized topology configurations for different ply angles are different. With increasing the ply angle, the structural weight is increasing. But the optimal results are all satisfied with the response amplitude.

Conclusions

Based on ICM topology optimization method, the minimum weight subject to dynamic amplitude response with anisotropic material is established. The logarithmic type filter functions are introduced to build up the anisotropic structure topology optimization model. By using the dual quadratic programming and sequential approximation method, the mathematical model is solved. Numerical example shows that the method of this paper can effectively solve the problem of dynamic response topology optimization of anisotropic material.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (11072009, 11172013)

References

Allahdadian S, Boroomand B, Barekatein A R(2012). Towards optimal design of bracing system of multi-story structures under harmonic base excitation through a topology optimization scheme. *Finite Elements in Analysis and Design*. 61: 60-74.

Kang Z, Zhang X, Jiang S, et al(2012). On topology optimization of damping layer in shell structures under harmonic excitations. *Structural and Multidisciplinary Optimization*. 46(1): 51-67.

Rong J H, Xie Y M, Yang X Y, et al(2000). Topology optimization of structures under dynamic response constraints. *Journal of Sound and Vibration* 234(2): 177-189.

Zhang Qiao, Zhang Weihong, Zhu Jihong. (2010) Topology Optimization of Structures under Dynamic Response Constraints, Journal of Mechanical Engineering ,(15):45-51

Motamarri P, Ramani A, Kaushik A(2012). Structural topology synthesis with dynamics and nonlinearities using equivalent linear systems. *Structural and Multidisciplinary Optimization*. 45(4): 545-558.

Sui Yunkang(1996). Modeling, Tansformation and Optimization: New Developments of Structural Synthesis Method. *Dalian: Dalian University of Technology Press*.92-102.

Sui Yunkang, Ye Hongling(2013). Continum topology optimization meyhods ICM. Science Press.