Legendre-collocation method for nonlinear Volterra integral equations of the second kinds

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Abstract

In this paper, we propose an efficient numerical method for Volterra-type nonlinear integral equations, based on Legendre-Gauss-Radau interpolation, which is easy to be implemented and possesses the spectral accuracy. We also develop a multi-step version of this approach. Numerical results demonstrate the effectiveness of these approaches.

Keywords: Legendre collocation method, Volterra integral equation, Nonlinear

Introduction

We are interested in numerically solving Volterra integral equation of the second kind in the following form:

$$u(t) = g(t) + \int_0^t K(t,\tau)F(u(\tau))d\tau, \qquad t > 0 \tag{1}$$

with $u(0) = g(0)$, where the source function $g(t)$ and the kernel function $K(t,\tau)$ are given, and $u(t)$ is the unknown function to be determined. $F()$ is a nonlinear function with certain smoothness. Let $D = \{(t,\tau) \mid 0 \le t \le \tau \le T\}$. If $K(t,\tau) \in C^\infty(D)$ and $F(u) = u$, then the solution to (1) exists, is unique,and belongs to $C^\infty(0,T)$ (Theorems 2.1.2 and 2.1.3 in [Brunner(2004)]). For the existence and uniqueness of nonlinear integral equation as (1), one can consult with [Guo and Sun(1987); Wazwaz(2011)]. We will consider the case where the solutions of (1) are sufficiently smooth -- in this case it is necessary to consider very high-order numerical methods such as spectral methods for approximating the solutions.

As we know, the spectral method employs global orthogonal polynomials as trial functions. It often provides exceedingly accurate numerical results with relatively less degree of freedoms, and thus has been widely used for scientific computation[Bernardi and Maday(1997); Boyd(2000); Guo(1998); Shen and Tang(2006); Shen et al.(2011)]. For spectral methods of (1), the first try may be Chebyshev spectral methods in [Elnagar and Kazemi(1996)]. In [Fujiwara(2006)], Chebyshev spectral methods are investigated for Fredholm integral equations of the first kind under multiple-precision arithmetic. However, no theoretical analysis is provided to justify the high accuracy obtained. In [Tang et al.(2008)], authors developed successfully spectral method and conducted the convergence analysis. Later in [Chen and Tang(2009): Wan et al.(2009); Tao et al.(2011); Xie et al.(2012)], various spectral methods proposed for integral equation with spectral accuracy. Recently, a Legendre-Gauss-Radau collocation method was proposed for initial value problems of ordinary differential equations [Wang and Guo(2012)]. Motivated by the idea in [Wang and Guo(2012)], we propose a Legendre-Gauss-Radau collocation method for a Volterra integral equation (1).

Legendre-Gauss-Radau collocation method and its implementation

Legendre-Gauss-Radau collocation method

Let $L_l(x)$ be the standard Legendre polynomial of degree l. The shifted Legendre polynomials $L_{T,l}(x)$ are defined by $L_{T,l}(t) = L_l(2t/T - 1), l = 0,1,2,....$ The nodes of the standard Legendre-Gauss-Radau interpolation on the interval $[-1,1)$ are $\xi_j (0 \le j \le N)$ and the corresponding Christoffel numbers are $\rho_j (0 \le j \le N)$. The nodes of the shifted Legendre-Gauss-Radau interpolation on the interval $[0,T)$ are $t_{T,j}^N (0 \le j \le N)$ and the corresponding Christoffel numbers are $\omega_{T,j}^N (0 \le j \le N)$. Clearly, we have links $t_{T,j}^N = \frac{T}{2}\left(\xi_j + 1\right)$ and $\omega_{T,j}^N = \frac{T}{2}\rho_j$ for $0 \le j \le N$. Let $P_N(0,T)$ be the set of polynomials of degree at most N. For any $v \in C[0,T)$, the shifted Legendre-Gauss-Radau interpolation $\Pi_N v \in P_N(0,T)$ is determined uniquely by $\Pi_N v(t_{T,j}^N) = v(t_{T,j}^N)$ ($0 \le j \le N$). For any $K(t,s) \in C[0,T)^2$, notation $\tilde{\Pi}_N^k$ means the shifted Legendre-Gauss-Radau interpolation on $[0, t_{T,k}^N)$ with respect to s. The corresponding nodes and Christoffel numbers are $t_{t_k,j}^N$ and $\omega_{t_k,j}^N$, respectively.

The Legendre-Gauss-Radau collocation method for (1) is to seek $u^N(t) \in P_N(0,T)$ such that

$$u(t_{T,k}^N) = g(t_{T,k}^N) + \sum_{j=0}^{N} K(t_{T,k}^N, t_{t_k,j}^N)[\Pi_N F(u^N)](t_{t_k,j}^N)\omega_{t_k,j}^N, \qquad 1 \le k \le N \tag{2}$$

with $u(t_{T,0}^N) = g(t_{T,0}^N)$. Because of exactness of the Legendre-Gauss-Radau quadrature for polynomials of degree at most $2N$ and $\tilde{\Pi}_N^k K(t_{T,j}^N, \tau)\Pi_N F(u^N(\tau)) \in P_{2N}(0, t_{T,k}^N)$, scheme (2) is equivalent to

$$u(t_{T,k}^N) = g(t_{T,k}^N) + \int_0^{t_{T,k}^N} \tilde{\Pi}_N^k K(t_{T,k}^N, \tau)\Pi_N F(u^N(\tau))d\tau, \qquad 1 \le k \le N. \tag{3}$$

Implementation of the scheme (2)

We first express the approximate solution as

$$u^N(t) = \sum_{l=0}^{N} \hat{u}_l L_{T,l}(t) \tag{4}$$

with $\hat{u}_l = \frac{2l+1}{T}\sum_{j=0}^{N} u^N(t_{T,j}^N)L_{T,l}(t_{T,j}^N)\omega_{T,j}^N$ $(0 \le l \le N)$. Then the nonlinear term under the integral in (3) is expressed as

$$\begin{aligned}\Pi_N F(u^N(t)) = &\frac{1}{T}\sum_{j=1}^{N}\sum_{l=0}^{N}(2l+1)L_{T,l}(t_{T,j}^N)F(u^N(t_{T,j}^N))\omega_{T,j}^N L_{T,l}(t) \\ &+ \frac{1}{T}\sum_{l=0}^{N}(2l+1)L_{T,l}(t_{T,0}^N)F(u^N(t_{T,0}^N))\omega_{T,0}^N L_{T,l}(t).\end{aligned} \tag{5}$$

Inserting (5) into (2), we have

$$\begin{aligned}u(t_{T,k}^N) = g(t_{T,k}^N) &+ \sum_{l=0}^{N}\left(\sum_{i=0}^{N}\tilde{\Pi}_N^k K(t_{T,k}^N, t_{t_k,i}^N)L_{T,l}(t_{t_k,i}^N)\omega_{t_k,i}^N\right)\left(\frac{2l+1}{T}L_{T,l}(t_{T,0}^N)\omega_{T,0}^N\right)F(g(t_{T,0}^N)) \\ &+ \sum_{j=1}^{N}\left[\sum_{l=0}^{N}\left(\sum_{i=0}^{N}\tilde{\Pi}_N^k K(t_{T,k}^N, t_{t_k,i}^N)L_{T,l}(t_{t_k,i}^N)\omega_{t_k,i}^N\right)\left(\frac{2l+1}{T}L_{T,l}(t_{T,j}^N)\omega_{T,j}^N\right)\right]F(u^N(t_{T,j}^N)).\end{aligned} \tag{6}$$

Further, we set

$$\mathbf{u} = [u^N(t_{T,1}^N), u^N(t_{T,2}^N), \ldots, u^N(t_{T,N}^N)]^T, \quad \mathbf{g} = [g(t_{T,1}^N), g(t_{T,2}^N), \ldots, g(t_{T,N}^N)]^T,$$

$$\mathbf{F(u)} = [F(u^N(t_{T,1}^N)), F(u^N(t_{T,2}^N)), \ldots, F(u^N(t_{T,N}^N))]^T,$$

$$\mathbf{B} = (b_{lj})_{(N+1)\times N} = \left(\frac{2l+1}{T} L_{T,l}(t_{T,j}^N)\omega_{T,j}^N \right)_{1\le j\le N;0\le l\le N},$$

$$\mathbf{C} = (c_{kl})_{N\times(N+1)} = \left(\sum_{j=0}^{N} K(t_{T,k}^N, t_{t_k,j}^N) L_{T,l}(t_{t_k,j}^N)\omega_{t_k,j}^N \right)_{1\le k\le N;0\le l\le N},$$

$$\mathbf{h} = (h_l)_{(N+1)\times 1} = \left(\frac{2l+1}{T} L_{T,l}(t_{T,0}^N)\omega_{T,0}^N \right)_{0\le l\le N}.$$

Then we can rewrite (6) as the following compact matrix form

$$\mathbf{u} = \mathbf{g} + \mathbf{CBF(u)} + \mathbf{Ch}F(g(t_{T,0}^N)). \tag{7}$$

In actual computation, we first use (7) to evaluate $u^N(t_{T,k}^N), 1 \le k \le N$. Then we use (4) to obtain

$$u^N(T) = \sum_{l=0}^{N} \left(\frac{2l+1}{T} \sum_{j=0}^{N} u^N(t_{T,j}^N) L_{T,l}(t_{T,j}^N)\omega_{T,j}^N \right) L_{T,l}(T). \tag{8}$$

The expression above can be in matrix –vector form as

$$u^N(T) = \mathbf{bDv} \tag{9}$$

where

$$\mathbf{b} = (b_l)_{1\times(N+1)} = \left(\frac{2l+1}{T} L_{T,l}(T) \right)_{0\le l\le N}, \quad \mathbf{D} = (d_{jl})_{(N+1)\times(N+1)} = \left(L_{T,l}(t_{T,j}^N) \right)_{0\le j\le N;0\le l\le N},$$

$$\mathbf{v} = (v_j)_{(N+1)\times 1} = \left(u^N(t_{T,j}^N)\omega_{T,j}^N \right)_{0\le j\le N}.$$

Multi-step version of the collocation method

Let M be a positive integer number, and $N_m (1 \le m \le M)$ be positive integer numbers. We divide the interval $[0,T]$ as $0 = t_0 < t_1 < ... < t_{m-1} < t_m < ... < t_M = T$. Set $\tau_m = t_m - t_{m-1}$. Replacing T and N by τ_1 and N_1 in (2) we obtain the local numerical solution $u_1^{N_1}(t) \in P_{N_1}(t_0, t_1)$ with the initial value $u_1^{N_1}(0) = g(0)$. Next we evaluate the local numerical solution $u_m^{N_m}(t) \in P_{N_m}(t_{m-1}, t_m)$ for $m = 2,3,...,M$ by

$$u_m^{N_m}(t_{\tau_m,k}^{N_m}) = g(t_{\tau_m,k}^{N_m}) + \int_{t_{n-1}}^{t_{\tau_m,k}^{N_m}} \widetilde{\Pi}_N^k K(t_{\tau_m,k}^{N_m}, \tau)\Pi_N F(u_m^{N_m}(\tau))d\tau$$
$$+ \sum_{s=1}^{m-1} \int_{t_{s-1}}^{t_s} \widetilde{\Pi}_N^k K(t_{\tau_m,k}^{N_m}, \tau)\Pi_N F(u_s^{N_s}(\tau))d\tau, \quad 1 \le k \le N_m, \tag{10}$$

with $u_m^{N_m}(t_{\tau_m,0}^{N_m}) = u_{m-1}^{N_{m-1}}(t_{m-1})$. Notation $t_{\tau_m,k}^{N_m}$ in (10) denote the shifted Legendre-Gauss-Radau nodes in the interval $[t_{m-1}, t_m)$ and $t_{\tau_m,k}^{N_m} = \frac{\tau_m}{2}(\xi_k + 1) + t_{m-1}$ (the corresponding Christoffel numbers are $\omega_{\tau_m,k}^{N_m} = \frac{\tau_m}{2}\rho_k (0 \le k \le N_m)$). Now we set

$$\mathbf{u}^{\tau_k} = [u_k^{N_k}(t_{\tau_k,1}^{N_k}), u_k^{N_k}(t_{\tau_k,2}^{N_k}), ..., u_k^{N_k}(t_{\tau_k,N_k}^{N_k})]^T, \quad \mathbf{g}^{\tau_k} = [g(t_{\tau_k,1}^{N_k}), g(t_{\tau_k,2}^{N_k}), ..., g(t_{\tau_k,N_k}^{N_k})]^T,$$

$$\mathbf{F}(\mathbf{u}^{\tau_k}) = [F(u_k^{N_k}(t_{\tau_k,1}^{N_k})), F(u_k^{N_k}(t_{\tau_k,2}^{N_k})), ..., F(u_k^{N_k}(t_{\tau_k,N_k}^{N_k}))]^T,$$

$$\mathbf{B}^{\tau_k} = (b_{lj})_{(N_k+1)\times N_k} = \left(\frac{2l+1}{\tau_k} L_{\tau_k,l}(t_{\tau_k,j}^{N_k})\omega_{\tau_k,j}^{N_k}\right)_{1\le j\le N_k;0\le l\le N_k},$$

$$\mathbf{C}^{(\tau_k,\tau_n)} = (c_{kl})_{N_k\times(N_n+1)} = \left(\int_{t_{k-1}}^{t_{\tau_k,m}^{N_k}} K(t_{\tau_k,m}^{N_k},\tau)L_{\tau_n,l}(\tau)d\tau\right)_{1\le m\le N_k;0\le l\le N_n},$$

$$\mathbf{h}^{\tau_k} = (h_l)_{(N_k+1)\times 1} = \left(\frac{2l+1}{\tau_k} L_{\tau_k,l}(t_{\tau_k,0}^{N_k})\omega_{\tau_k,0}^{N_k}\right)_{0\le l\le N_k},$$

$$\mathbf{b}^{\tau_k} = (b_l)_{1\times(N_k+1)} = \left(\frac{2l+1}{\tau_k} L_{\tau_k,l}(t_k)\right)_{0\le l\le N_k},$$

$$\mathbf{D}^{\tau_k} = (d_{jl})_{(N_k+1)\times(N_k+1)} = \left(L_{\tau_k,l}(t_{\tau_k,j}^{N_k})\right)_{0\le j\le N_k;0\le l\le N_k},$$

$$\mathbf{v}^{\tau_k} = (v_j)_{(N_k+1)\times 1} = \left(u_k^{N_k}(t_{\tau_k,j}^{N_k})\omega_{\tau_k,j}^{N_k}\right)_{0\le j\le N_k}.$$

Then the matrix form of the multi-step version is, first we solve the system

$$\begin{cases} \mathbf{u}^{\tau_1} = \mathbf{g}^{\tau_1} + \mathbf{C}^{(\tau_1,\tau_1)}\mathbf{B}^{\tau_1}\mathbf{F}(\mathbf{u}^{\tau_1}) + \mathbf{C}^{(\tau_1,\tau_1)}\mathbf{h}^{\tau_1}F(g(0)) \\ \qquad\qquad u_1^{N_1}(t_1) = \mathbf{b}^{\tau_1}\mathbf{D}^{\tau_1}\mathbf{v}^{\tau_1} \end{cases}$$

Next, step by step, the local numerical solution $u_m^{N_m}(t) \in P_{N_m}(t_{m-1},t_m)$ $(m>1)$ can be obtained by

$$\begin{cases} \mathbf{u}^{\tau_m} = \mathbf{g}^{\tau_m} + \mathbf{C}^{(\tau_m,\tau_m)}\mathbf{B}^{\tau_m}\mathbf{F}(\mathbf{u}^{\tau_m}) + \mathbf{C}^{(\tau_m,\tau_m)}\mathbf{h}^{\tau_m}F(g(t_{\tau_m,0}^{N_m})) \\ \qquad + \sum_{k=1}^{m-1}\left(\mathbf{C}^{(\tau_m,\tau_k)}\mathbf{B}^{\tau_k}\mathbf{F}(\mathbf{u}^{\tau_k}) + \mathbf{C}^{(\tau_m,\tau_k)}\mathbf{h}^{\tau_k}F(g(t_{\tau_k,0}^{N_k}))\right) \\ \qquad\qquad u_m^{N_m}(t_m) = \mathbf{b}^{\tau_m}\mathbf{D}^{\tau_m}\mathbf{v}^{\tau_m} \end{cases}.$$

Noting that $t_{\tau_k,0}^{N_k} = t_{k-1}$, the process above can be done without any gap.

Numerical experiments

The first issue in performing the proposed method is how to solve the nonlinear system (7). We can use a simple iterate scheme as follow:

$$\mathbf{u}^{k+1} = \mathbf{g} + \mathbf{CBF}(\mathbf{u}^k) + \mathbf{Ch}F(g(0)), \quad k = 0,1,2,.... \tag{11}$$

If the sequence $\mathbf{u}^k$ converges, we can obtain a good approximation solution to (7).

In the following examples, because the exact solutions are known, we can compute the error between numerical solution and the corresponding exact ones. The errors in L^∞ and L^2 are defined by

$$\text{Errinf} = \max_{0\le k\le N} |u^N(t_{T,k}^N) - u(t_{T,k}^N)|, \quad \text{Err2} = \sqrt{\sum_{k=0}^{N} |u^N(t_{T,k}^N) - u(t_{T,k}^N)|^2 \ \omega_{T,k}^N}.$$

where $u^N(t)$ and $u(t)$ are numerical and exact solution, respectively. Another absolute error at $t=T$ is defined by $\text{Err}(T) = |u(T) - u^N(T)|$.

Example 1 The first example is concerned with a linear problem. Consider Volterra integral equation (1) with $g(t) = e^{2t} - \dfrac{e^{t(t+2)}-1}{t+2}$, $K(t,\tau) = e^{t\tau}$ and $F(u) = u$. The exact solution is $u(t) = e^{2t}$.

With the same exact solution and kernel as above, authors test a Legendre collocation(LCM) in [Tang et al.(2008)] and also spectral Jacobi-Galerkin method (spectral Legendre-Galerkin method (SLGM) and spectral Chebyshev-Galerkin method(SCGM)) and pseudo-spectral Jacobi-Galerkin method (pseudo-spectral Legendre-Galerkin method (PSLGM) and pseudo-spectral Chebyshev-

Galerkin method(PSCGM)) in [Xie et al.(2012)] for a slightly different problem. We compare the results in table 1 for L^∞-errors. Our results are slightly better than the ones in [Tang et al.(2008)] and [Xie et al.(2012)].

Table 1. Comparison of L^∞-errors for example 1.

N	4	6	8	10	12	14
LCM	-	3.66e-01	1.88e-02	6.57e-04	1.65e-05	3.11e-07
SLGM	5.243e-02	1.262e-03	1.753e-05	1.572e-07	9.779e-10	4.618e-12
SCGM	2.915e-02	5.696e-04	7.276e-06	5.751e-08	3.950e-10	1.737e-12
PSLGM	6.007e-03	9.386e-05	8.710e-07	6.378e-09	3.322e-11	1.323e-13
PSCGM	7.113e-03	1.003e-04	9.958e-07	6.995e-09	3.638e-11	1.492e-13
Our method	3.137e-04	1.411e-06	3.833e-09	7.665e-12	1.245e-12	1.373e-12

In table 2, we present the L^2 error and the absolute errors at $T=1$ with different N. The results show "spectral accuracy".

Table 2. Errors with different N for example 1.

N	4	6	8	10	12
Err2	2.192e-04	9.098e-07	2.433e-09	4.534e-12	4.406e-13
Err(1)	5.508e-03	3.897e-05	1.539e-07	3.809e-10	8.383e-12

Example 2 The second example is concerned with a nonlinear problem. Consider Volterra integral equation (1) with $g(t)=\sin t+\frac{1-\cos t}{2}+\frac{\sin 2t-2t}{8}+\frac{\cos t-e^{-2t}-2\sin t}{10}+\frac{2-e^{-2t}-\sin 2t-\cos 2t}{16}$, $K(t,\tau)=-\frac{1-e^{-2(t-\tau)}}{2}$ and $F(u)=u-u^2$. The exact solution is $u(t)=\sin t$. For solving the nonlinear system (7), the simple iterate method (11) is employed with tolerate $\varepsilon=10^{-16}$.

In table 3, we present the errors with different N. The results also show "spectral accuracy".

Table 3. Errors with different N for example 2.

N	4	6	8	10	12
Errinf	1.042e-06	2.254e-09	3.566e-12	4.385e-15	2.220e-16
Err2	7.466e-07	1.572e-09	2.521e-12	3.079e-15	1.118e-16
Err(1)	5.584e-05	9.846e-08	9.688e-11	3.948e-13	1.929e-13

Example 3 Consider Volterra equation (1) with

$$g(t)=\frac{6\pi\sin(6\pi t)-\cos(6\pi t)+36\pi^2-36\pi^2e^t+1}{2(1+36\pi^2)},\ K(t,\tau)=e^{t-3\tau},\ F(u)=u^2.$$

The exact solution is $u(t) = e^t \sin(3\pi t)$. In table 4, we present the errors with different N. The results also show "spectral accuracy".

Table 4. Errors with different N for example 3.

N	8	12	16	20	24	28
Errinf	2.498e-02	8.997e-04	7.892e-06	2.202e-08	2.180e-11	4.219e-14
Err2	1.667e-02	6.681e-04	6.065e-06	1.712e-08	1.639e-11	1.581e-14
Err(1)	7.825e-02	3.265e-03	3.677e-05	1.240e-07	1.404e-10	2.412e-13

Conclusions

In this paper, we proposed a Legendre-Gauss-Radau collocation method for solving Volterra-type integral equations of second kind. This method is easy to be implemented for nonlinear problems. In particular, benefiting from the rapid convergence of the Legendre-Gauss-Radau interpolation, this method possesses spectral accuracy.

We also provided a multi-step version of Legendre-Gauss-Radau collocation method. We could use this process with moderate mode N to evaluate the numerical solution, step by step. This simplifies actual computation and saves work essentially. In the derivation of the existing collocation method, one could use the Lagrange interpolation which is unstable for large N. Whereas we used the Gauss-type interpolation as in [Guo and Wang(2007;2009;2010)], which makes our methods much more stable for large N. This is also confirmed by the numerical results.

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