Free vibration analysis of the functionally graded coated and undercoated plates

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Abstract

In this paper, the free vibration behaviors of the functionally graded (FG) coated and undercoated plates are studied by a meshfree boundary-domain integral equation method. Based on the twodimensional elasticity theory, the boundary-domain integral equations for each single layer of these coating-substrate plates are derived initially by using elastostatic fundamental solutions. Employ the radial integration method to transform the domain integrals into boundary integrals and achieve a meshfree scheme. By applying the multi-region boundary element method, obtain the generalized eigenvalue system of the whole plate, which involves system matrices with boundary integrals only and the complete solutions for natural frequency and vibration modes are rigidly resolved. A comparative study of FG versus homogeneous coating is conducted. The influences of material composition, material gradient, coating thickness ratio, plate aspect ratio and the boundary conditions on the natural frequencies of the FG coated and undercoated plates are evaluated and discussed.

Key words: free vibration, FG coated and undercoated plates, boundary-domain integral equations, meshfree method, multi-region boundary element method

Introduction

In many applications, especially in the space industry, energy industry and electronic industry, structures or part of structures are exposed to high temperature or high temperature gradients. Conventional metallic materials, such as carbon steels or stainless steels cannot resist such high temperature. In order to improve the resistance of metallic structures against extreme temperature conditions, without suppressing their strength and toughness, a thin layer of appropriate ceramic is generally used to cover the surface of the structures. For those structures which are subjected to constantly rolling, sliding contacts or abrasive wear, additional hardening process should be carried out within the outer surface of the materials. These two techniques are all forming the coatingsubstrate system, where a functional material is coated on the substrate material to increase the durability and reliability of the structures. However, due to the discontinuous of the material properties of these two or more materials, severe residual and working stresses discontinuity at the material interfaces usually cause damage to the coating, or failure due to delamination. As a remedy to the aforementioned disadvantages in coating-substrate system, a concept of functionally graded (FG) coating are proposed, where a smooth spatial gradation of the material properties are introduced from coating to substrate in order to eliminated the effect of the suddenly change of the material properties, such that stress and strain discontinuous can be mitigated in the coatingsubstrate system.

Due to the superiorly properties, a world-wide requirements of the application of the FG coatingsubstrate system triggered a series of research activities. The incorporation of functionally graded materials (FGMs) into coating design can help eliminate the mismatch of mechanical and thermomechanical properties between the metal plates and coating layers. Thus a number of studies existed in the literature for analyzing of the mechanical and thermo-mechanical behavior of homogeneous plates coated by an FG layer. An FG coated elastic solid under thermomechanical loading was carried out by Shodja and Ghahremaninejad [2006]. Three-dimensional elastic deformation of a functionally graded coating/substrate system was investigated by Kashtalyan et al [2007]. Chung and Chen [2007] analyzed the bending behavior of the thin plates coated by FG layer. Several researches have addressed contact response of FG coatings. Saizonou et al. [2002] studied the subsurface stress distribution of an FG-coated elastic solid under normal and sliding contact loading by the boundary element method (BEM). Contact mechanics of the FG coated solids was analyzed by Guler and Erdogan [2004]. It should be noted that in all above studies the properties the FGM were all assumed to vary exponentially through the thickness.

Theoretical modeling of FG coatings has been focused predominantly on prediction of their fracture behaviours. Chen and Erdogan [2003] studied the interface cracks for a FG coating medium. A crack in the FG coating surface and its expansion into the substrate along the direction perpendicular to the interface between the coating and the substrate was presented by Chi and Chuang [1996]. Pindera et al. [2002, 2005] examined fracture mechanisms in thermal barrier coatings with FG bond coats under uniform cyclic thermal loading. However, the dynamic analyses of the FG coatings are very rare in the literature. Liew et al. [2006] investigated linear and non-linear vibrations of a coating-FGM-substrate cylindrical panel subjected to a temperature gradient, which were based on the first order shear deformation theory and von-Karman geometric nonlinearity. Hosseini-Hashemi et al. [2012] presented the exact closed-form solutions for both in-plane and out-of-plane free vibration of the simply supported rectangular plates coated by a FG layer, based on three-dimensional elasticity theory.

In this paper, attention is focused on investigating the free vibration behaviors of two FG coatingsubstrate structures. The first one involves a two-layer plate, namely an FG layer coated on a homogeneous substrate which is simply called the FG coated plate, the other involves a three-layer plate in which an FGM is employed for the inter-medium layer and different homogeneous materials are in the top and bottom layers, this is called an FG undercoated plate [Chung (2007)]. For each single layer of these plates, the boundary-domain integral equation formulations are derived initially by using the elastostatic fundamental solutions which is based on the twodimensional elasticity theory. A meshfree scheme is achieved to apply the radial integration method to transform the domain integrals arising from the material inhomogeneous and the inertial effects to the boundary integrals. Finally, an eigenvalue system involving system matrices with boundary integrals only is obtained through assembling all the sub-layer integral equations together by employing the multi-region BEM. By the harmonious combination of this meshfree boundarydomain integral equation method and the multi-region BEM, a comparative study of FG coating versus homogeneous coating is conducted. Extensive numerical results are presented to demonstrate the influences of FG coating thickness ratio, plate aspect ratio, as well as boundary condition on the vibration characteristics of the FG coated and the FG uncoated plates.

Material properties of the coating-substrate structures

Three considered coating-substrate plates, namely, the homogeneous coated, FG coated, as well as the FG undercoated plates are schematic depicted in Fig. 1. Assume the layers of these coating-substrate plates are perfected bonded to each other. The total length and height of these coating-substrate structures are denoted by L and h_t . h_i represents the thickness of each layer. The coating and the substrate of the homogeneous coated plate as well as the top and the bottom layers of the FG undercoated plate are composed by pure ceramic and pure steel, respectively, there material parameters are described in Table 1. For the FG layer existing in FG coated and FG undercoated plates, assuming the top is ceramic rich and the bottom is steel rich, the Young's modulus and the mass density are varying continuously in the transverse direction according to an exponential function described in Eqs. (1) and (2), while the Poisson ratio is constant.

$$E(x_2) = E_b e^{\beta x_2} \quad \text{where} \quad \beta = \frac{1}{h_f} \ln(\frac{E_t}{E_b}) , \qquad (1)$$

$$\rho(x_2) = \rho_b e^{\gamma x_2} \quad \text{where} \quad \gamma = \frac{1}{h_f} \ln(\frac{\rho_t}{\rho_b}),$$
(2)

where E_t , ρ_t are the Young's modulus and mass density for the top face constituent of the FG layer, and E_b , ρ_b are for the bottom face constituent. FGM gradation parameters are represented by β and γ for Young's modulus and mass density respectively. x_2 denotes the Cartesian coordinates variable in the transvers direction and h_f is the thickness of the FG layer. The through thickness variation of the Young's modulus for the three considered coating-substrate plates is shown in Fig. 2.

Table 1. Material properties of the homogeneous ceramic and steel



Figure 1. Coordinates and geometry of the coating-substrate plates (a) homogeneous coated plate; (b) FG coated plate; (c) FG undercoated plate



Figure 2. Variation of Young's modulus of (a) homogeneous coated plate; (b) FG coated plate; (c) FG undercoated plate

Problem formulation

The fulfillment of the free vibration analyses of the FG coated and FG undercoated plates as well as the homogeneous coated plates are by the harmonious combination of the developed meshfree boundary-domain integral equation method and the multi-region BEM.

The meshfree boundary-domain integral equation method

For each single layer of the coating-substrate plates, the governing differential equations of the steady-state elastodynamics without damping is expressed in terms of the frequency ω as

$$\sigma_{ij,j}(\boldsymbol{x}) + \omega^2 \rho u_i(\boldsymbol{x}) = 0.$$
(3)

Which is based on the two-dimensional elasticity theory and the stress tensor σ_{ij} , mass density ρ , displacement u_i are quantities for each layer. A comma after a quantity represents spatial derivatives and repeated indexes denote summation.

The elasticity tensor c_{ijkl} is described in the form of

$$c_{ijkl}(\boldsymbol{x}) = \mu(\boldsymbol{x})c_{ijkl}^{0} \quad \text{where} \quad c_{ijkl}^{0} = \frac{2\nu}{1-2\nu}\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}, \quad (4a, b)$$

where c_{ijkl}^0 represents the elastic tensor of the reference homogenous material, which is a "fictitious" homogeneous material with $\mu=1$. $\mu(\mathbf{x}) = E(\mathbf{x})/2(1+\nu)$ is the shear modulus. For the FG layer, $\mu(\mathbf{x})$ varies gradationally according to the coordinates, while it keeps a constant for the homogeneous layer. δ_{ij} is the Kronecker delta. By taking the elastostatic displacement fundamental solutions $U_{ij}(\mathbf{x}, \mathbf{y})$ as the weight function, the weak-form of the equilibrium Eq. (3) can be obtained as

$$\int_{\Omega} [\sigma_{jk,k} + \rho \omega^2 u_j] \cdot U_{ij} d\Omega = 0.$$
⁽⁵⁾

Application of the generalized Hooke's law $\sigma_{ij} = c_{ijkl}u_{k,l} = \mu(\mathbf{x})c_{ijkl}^0u_{k,l}$ and the Gauss's divergence theorem yields the following boundary-domain integral equations

$$\tilde{u}_{i}(\mathbf{y}) = \int_{\Gamma} U_{ij}(\mathbf{x}, \mathbf{y}) t_{j}(\mathbf{x}) d\Gamma - \int_{\Gamma} T_{ij}(\mathbf{x}, \mathbf{y}) \tilde{u}_{j}(\mathbf{x}) d\Gamma + \int_{\Omega} V_{ij}(\mathbf{x}, \mathbf{y}) \tilde{u}_{j}(\mathbf{x}) d\Omega + \omega^{2} \int_{\Omega} \frac{\rho(\mathbf{x})}{\mu(\mathbf{x})} U_{ij}(\mathbf{x}, \mathbf{y}) \tilde{u}_{j}(\mathbf{x}) d\Omega$$
(6)

In Eq. (6), the traction vector $t_i = \sigma_{ij}n_j$, n_j is the components of the outward unit normal to the boundary Γ of the considered domain Ω . \tilde{u}_i is recognized as the normalized displacement vector correlating with the normalized shear modulus $\tilde{\mu}$, which are defined by [Gao (2008)]

$$\tilde{\mu}_i(\boldsymbol{x}) = \mu(\boldsymbol{x})u_i(\boldsymbol{x}), \qquad \qquad \tilde{\mu}(\boldsymbol{x}) = \ln[\mu(\boldsymbol{x})].$$
(7a, b)

The fundamental solutions arising in equation (6) can be expressed as following, where $U_{ij}(x, y)$ and $T_{ij}(x, y)$ are chosen as the elastostatic displacement fundamental solutions for homogeneous, isotropic and linear elastic solids with $\mu=1$ [Gao and Davies (2002)].

$$U_{ij} = \frac{-1}{8\pi(1-\nu)} [(3-4\nu)\delta_{ij}\ln(r) - r_{,i}r_{,j}],$$
(8)

$$\Sigma_{ijl} = c_{rsjl}^0 U_{ir,s} = \frac{-1}{4\pi (1-\nu)r} [(1-2\nu)(\delta_{il}r_{,j} + \delta_{ij}r_{,l} - \delta_{jl}r_{,i}) + 2r_{,i}r_{,j}r_{,l}],$$
(9)

$$T_{ij} = \Sigma_{ijl} n_l = \frac{-1}{4\pi (1-v)r} [(1-2v)(n_i r_{,j} - n_j r_{,i}) + ((1-2v)\delta_{ij} + 2r_{,i} r_{,j})r_{,l} n_l],$$
(10)

$$V_{ij} = \Sigma_{ijl} \tilde{\mu}_{,l} = \frac{-1}{4\pi(1-\nu)r} [(1-2\nu)(\tilde{\mu}_{,i}r_{,j} - \tilde{\mu}_{,j}r_{,i}) + ((1-2\nu)\delta_{ij} + 2r_{,i}r_{,j})r_{,l}\tilde{\mu}_{,l}], \quad (11)$$

where r=|x-y| is the distance from the field point x to the source point y. Boundary-domain integral equations for boundary points can be obtained by letting y to the boundary Γ in Eq. (6). There are two domain integrals emerged in the Eq. (6), the first one is due to the material inhomogeneous and the other arises from the inertial effect.

Homogeneous layer

Respect to the homogeneous layer, in the virtue of the shear modulus is a constant through the medium, therefore, $\tilde{\mu}_{,l}$ appears a zero value in Eq. (11), which leads to the integral kernel V_{ij} inside the first domain integral of Eq. (6) vanish. Only the domain integral arises from the inertial effect

left with a constant ρ_h/μ_h (ρ_h , μ_h are mass density and shear modulus for the homogenous layer), which can be extracted out of the domain integration.

FG layer

For the FG layer, the emerged two domain integrals are all remained. In the first domain integral, $\tilde{\mu}_{,l}$ is no longer a zero value. However, in the case of an exponential law for the Young's modulus or shear modulus such as those used in this analysis, it can be seen from Eq. (7b) that $\tilde{\mu}_{,l}$ is constant and $V_{ij}(\mathbf{x}, \mathbf{y})$ thus becomes very simple for integration. While the material properties ratio $\rho(\mathbf{x})/\mu(\mathbf{x})$ in the second domain integral still need to be consideration inside the domain integration due to the material properties are varying according with the coordinates.

In order to treat the domain integrals in the Eq. (6), the radial integration method (RIM) proposed by Gao [2002] is employed to transform the domain integrals into the boundary integrals over the global boundary. In the RIM, \tilde{u}_i in the domain integrals of Eq. (6) are approximated by a combination of the radial basis function and the polynomials of the global coordinates as

$$\tilde{u}_{i}(\boldsymbol{x}) = \sum_{A} \alpha_{i}^{A} \phi^{A}(R) + a_{i}^{k} x_{k} + a_{i}^{0}, \quad \sum_{A} \alpha_{i}^{A} = 0, \quad \sum_{A} \alpha_{i}^{A} x_{j}^{A} = 0.$$
(12a, b, c)

In this analysis, the 4th order spline-type radial basis function $\phi^A(R)$ is applied. By taking all the boundary nodes (N_b) and some internal nodes (N_i) to constitute the application points A (N_t=N_b+N_i), and substituting the coordinates of the field points x (x_k) and the application point A (x_j^A) into Eqs. (12), if with no two coincide noes, the unknown coefficient vectors can be calculated by a set of linear algebraic equations as

$$\tilde{\boldsymbol{u}} = \boldsymbol{\phi} \cdot \boldsymbol{\alpha}$$
, and $\boldsymbol{\alpha} = \boldsymbol{\phi}^{-1} \cdot \tilde{\boldsymbol{u}}$. (13a, b)

Subsequently determining the coefficients α_i^A , a_i^k and a_i^0 , substitute Eq. (12a) into the domain integrals of Eq. (6) and apply the RIM, the two domain integrals are transformed into the boundary integrals in the form of [Yang et al. (2014)]

$$\int_{\Omega} V_{ij} \tilde{u}_j d\Omega = \alpha_j^A \int_{\Gamma} \frac{1}{r} \frac{\partial r}{\partial n} F_{ij}^{A2} d\Gamma + a_j^k \int_{\Gamma} \frac{r_{,k}}{r} \frac{\partial r}{\partial n} F_{ij}^{12} d\Gamma + (a_j^k y_k + a_j^0) \int_{\Gamma} \frac{1}{r} \frac{\partial r}{\partial n} F_{ij}^{02} d\Gamma, \qquad (14)$$

$$\omega^{2} \int_{\Omega} \frac{\rho}{\mu} U_{ij} \tilde{u}_{j} d\Omega = \omega^{2} \frac{\rho_{b}}{\mu_{b}} e^{(\gamma - \beta)y_{2}} [\alpha_{j}^{A} \int_{\Gamma} \frac{1}{r} \frac{\partial r}{\partial n} P_{ij}^{A2} d\Gamma + a_{j}^{k} \int_{\Gamma} \frac{r_{k}}{r} \frac{\partial r}{\partial n} P_{ij}^{12} d\Gamma + (a_{j}^{k} y_{k} + a_{j}^{0}) \int_{\Gamma} \frac{1}{r} \frac{\partial r}{\partial n} P_{ij}^{02} d\Gamma]$$

$$(15)$$

where the relation $x_i = y_i + r_i r$ is used to relate x with r. By rewriting Eq. (11) with $V_{ij} = \overline{V}_{ij} / r$, the integral functions in Eqs. (14) and (15) can be expressed as [Yang et al. (2014)]

$$F_{ij}^{A2} = \int_{0}^{r} r V_{ij} \phi^{A2} dr = \overline{V}_{ij} \int_{0}^{r} \phi^{A2} dr , \quad F_{ij}^{12} = \int_{0}^{r} r^{2} V_{ij} dr = \frac{1}{2} r^{2} \overline{V}_{ij} , \quad F_{ij}^{02} = \int_{0}^{r} r V_{ij} dr = r \overline{V}_{ij} , \quad (16a,b,c)$$

$$P_{ij}^{A2} = \int_{0}^{r} r U_{ij} \phi^{A} e^{(\gamma - \beta)r_{2}r} dr , \quad P_{ij}^{12} = \int_{0}^{r} r^{2} U_{ij} e^{(\gamma - \beta)r_{2}r} dr , \quad P_{ij}^{02} = \int_{0}^{r} r U_{ij} e^{(\gamma - \beta)r_{2}r} dr . \quad (17a,b,c)$$

Since $r_{,i}$ in the above radial integrals is constant, then Eqs. (16b, c) can be evaluated analytically and other integrals are calculated by standard Gaussian quadrature formula [Yang et al. (2014)]. Therefore the displacement boundary integral equations with only boundary integrals are obtained.

$$c_{ij}\tilde{u}_{j} = \int_{\Gamma} U_{ij}t_{j}d\Gamma - \int_{\Gamma} T_{ij}\tilde{u}_{j}d\Gamma + [\alpha_{j}^{A}\int_{\Gamma} \frac{1}{r}\frac{\partial r}{\partial n}F_{ij}^{A2}d\Gamma + a_{j}^{k}\int_{\Gamma} \frac{r_{,k}}{r}\frac{\partial r}{\partial n}F_{ij}^{12}d\Gamma + (a_{j}^{k}y_{k} + a_{j}^{0})\int_{\Gamma} \frac{1}{r}\frac{\partial r}{\partial n}F_{ij}^{02}d\Gamma] + \omega^{2}\frac{\rho_{b}}{\mu_{b}}e^{(\gamma-\beta)y_{2}}[\alpha_{j}^{A}\int_{\Gamma} \frac{1}{r}\frac{\partial r}{\partial n}P_{ij}^{A2}d\Gamma + a_{j}^{k}\int_{\Gamma} \frac{r_{,k}}{r}\frac{\partial r}{\partial n}P_{ij}^{12}d\Gamma .$$
(18)
$$+(a_{j}^{k}y_{k} + a_{j}^{0})\int_{\Gamma} \frac{1}{r}\frac{\partial r}{\partial n}P_{ij}^{02}d\Gamma]$$

After the spatial discretization of the each layer boundary into quadratic boundary elements with N_b boundary nodes, collocating the resulting boundary integral equations at the N_t boundary and internal nodes, two sets of discretized boundary integral equations are obtained, which can be expressed in the matrix from as

$$([\boldsymbol{H}_{\boldsymbol{b}} \quad \boldsymbol{\theta}] - [\boldsymbol{V}_{\boldsymbol{b}}])_{2N_{\boldsymbol{b}} \times 2N_{t}} \left\{ \begin{array}{c} \tilde{\boldsymbol{u}}_{\boldsymbol{b}} \\ \tilde{\boldsymbol{u}}_{i} \end{array} \right\}_{2N_{t} \times 1} - \omega^{2} [\boldsymbol{P}_{\boldsymbol{b}}]_{2N_{\boldsymbol{b}} \times 2N_{t}} \left\{ \tilde{\boldsymbol{u}} \right\}_{2N_{t} \times 1} = [\boldsymbol{G}_{\boldsymbol{b}}]_{2N_{\boldsymbol{b}} \times 2N_{b}} \left\{ \boldsymbol{t}_{\boldsymbol{b}} \right\}_{2N_{\boldsymbol{b}} \times 1}, \quad (19a)$$

$$([\boldsymbol{H}_{i} \quad \boldsymbol{I}] - [\boldsymbol{V}_{i}])_{2N_{i} \times 2N_{i}} \begin{cases} \tilde{\boldsymbol{u}}_{b} \\ \tilde{\boldsymbol{u}}_{i} \end{cases}_{2N_{i} \times 1} - \boldsymbol{\omega}^{2} [\boldsymbol{P}_{i}]_{2N_{i} \times 2N_{i}} \{ \tilde{\boldsymbol{u}} \}_{2N_{i} \times 1} = [\boldsymbol{G}_{i}]_{2N_{i} \times 2N_{b}} \{ \boldsymbol{t}_{b} \}_{2N_{b} \times 1}, \quad (19b)$$

where I is the identity matrix with the size of $2N_i \ge 2N_i$. { \tilde{u}_b }, { \tilde{u}_i } and { \tilde{u} } are the displacement vectors of the boundary nodes, internal nodes and applications points respectively. By considering the boundary conditions, the sub-columns of the coefficient matrices respect to the known displacements nodes should be interchanged with that respected to the tractions, so do the displacements and the tractions vectors. Meanwhile, it is noticed that the sub-columns of the matrices [V_b], [P_b] and [V_i], [P_i] corresponding to the known boundary displacement nodes should be taken as zero. Then Eqs. (19) lead to the following system of linear algebraic equations

$$\begin{bmatrix} A_b \end{bmatrix} \begin{cases} \overline{x}_b \\ \widetilde{u}_i \end{cases} - \omega^2 \begin{bmatrix} P_b \end{bmatrix} \{ \widetilde{u} \} = \begin{bmatrix} B_b \end{bmatrix} \{ \overline{y}_b \}$$
 for boundary nodes, (20a)

$$\begin{bmatrix} \boldsymbol{A}_i \end{bmatrix} \begin{cases} \overline{\boldsymbol{x}}_b \\ \widetilde{\boldsymbol{u}}_i \end{cases} - \omega^2 \begin{bmatrix} \boldsymbol{P}_i \end{bmatrix} \{ \widetilde{\boldsymbol{u}} \} = \begin{bmatrix} \boldsymbol{B}_i \end{bmatrix} \{ \overline{\boldsymbol{y}}_b \}$$
 for internal nodes. (20b)

It is convenient to find that the traction vector { \overline{y}_b } in Eq. (20b) which is for the internal nodes is the same with that for the boundary nodes, such that the boundary nodes traction vector can be expressed in the terms of the coefficient matrices of Eq. (20a) by multiply the $[B_b]^{-1}$, and a new relationship can be set up by the equations

$$\begin{bmatrix} \bar{A}_b \end{bmatrix} \begin{cases} \bar{x}_b \\ \tilde{u}_i \end{cases} - \omega^2 \begin{bmatrix} \bar{P}_b \end{bmatrix} \{ \tilde{u} \} = \{ \bar{y}_b \}$$
 for boundary nodes, (21a)

$$\begin{bmatrix} \bar{A}_i \end{bmatrix} \begin{cases} \bar{x}_b \\ \tilde{u}_i \end{cases} - \omega^2 \begin{bmatrix} \bar{P}_i \end{bmatrix} \cdot \{ \tilde{u} \} = 0 \qquad \text{for internal nodes,} \qquad (21b)$$

where

$$\begin{bmatrix} \bar{\boldsymbol{A}}_b \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_b \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{A}_b \end{bmatrix} \qquad \begin{bmatrix} \bar{\boldsymbol{P}}_b \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_b \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{P}_b \end{bmatrix}, \qquad (22a)$$

$$\begin{bmatrix} \bar{\boldsymbol{A}}_i \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_i \end{bmatrix} - \begin{bmatrix} \boldsymbol{B}_i \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_b \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{A}_b \end{bmatrix} \qquad \begin{bmatrix} \bar{\boldsymbol{P}}_i \end{bmatrix} = \begin{bmatrix} \boldsymbol{P}_i \end{bmatrix} - \begin{bmatrix} \boldsymbol{B}_i \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_b \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{P}_b \end{bmatrix}.$$
(22b)

In each single region, all the nodes could be divided into three sets as shown in Fig 3. The first set includes the boundary nodes solely associated with a single region. This set nodes are denoted by 's'. The remaining boundary nodes reside on region-to-region interfaces belong to the second set which are denoted by 'c'. The third set is formed by the internal nodes which are denoted by 'i'. Then Eqs. (21) can be rewritten in the form of these three sets nodes as

$$\begin{bmatrix} \overline{A}_{bss} & \overline{A}_{bsc} & \overline{A}_{bsi} \\ \overline{A}_{bcs} & \overline{A}_{bcc} & \overline{A}_{bci} \\ \overline{A}_{is} & \overline{A}_{ic} & \overline{A}_{ii} \end{bmatrix} \begin{bmatrix} \overline{x}_{bs} \\ \widetilde{u}_{bc} \\ \widetilde{u}_{i} \end{bmatrix} - \omega^{2} \begin{bmatrix} \overline{P}_{bss} & \overline{P}_{bsc} & \overline{P}_{bsi} \\ \overline{P}_{bcs} & \overline{P}_{bcc} & \overline{P}_{bci} \\ \overline{P}_{is} & \overline{P}_{ic} & \overline{P}_{ii} \end{bmatrix} \begin{bmatrix} \overline{x}_{bs} \\ \widetilde{u}_{bc} \\ \widetilde{u}_{i} \end{bmatrix} = \begin{bmatrix} \overline{y}_{bs} \\ t_{bc} \\ 0 \end{bmatrix}.$$
(23)

In Eq. (23), \bar{x}_{bs} contains the unknown normalized displacements and the unknown traction vectors for the first set boundary nodes, while \bar{y}_{bs} contains all the known vectors. In the free vibration analysis, only the homogeneous system of the linear algebraic equations is needed, which can be obtained by taking the vectors \bar{y}_{bs} containing the known normalized boundary displacements as well as the known boundary tractions to be zero. However, the normalized displacements \tilde{u}_{bc} and traction t_{bc} for the set two nodes are all unknown.



Figure 3. Node sets of the coting-substrate plates

Assemble the system of equations by the multi-region BEM

After obtaining the system linear algebraic equations for the each single layer separately, the multiregion BEM is then employed to assemble the stiffness matrix and mass matrix for the whole coating-substrate plates. Taking a two-layer FG coated plate as an example. The divided boundaries are described in Fig. 4. The boundary of the each layer is discretized into two sub-boundaries Γ_1 , Γ_2 and Γ_3 , Γ_4 , where Γ_2 and Γ_3 are the common interface. Let u_i and t_i denote the nodal displacement and the traction vectors on boundary Γ_i respectively. The boundary integral equations can be written together in the matrix form

$$\begin{bmatrix} \bar{A}_{b11} & \bar{A}_{b12} & \bar{A}_{b1i1} \\ \bar{A}_{b21} & \bar{A}_{b22} & \bar{A}_{b2i1} \\ \bar{A}_{i11} & \bar{A}_{i12} & \bar{A}_{i1i} \end{bmatrix} \begin{bmatrix} \bar{x}_{b1} \\ \tilde{u}_{b2} \\ \tilde{u}_{i1} \end{bmatrix} - \omega^{2} \begin{bmatrix} \bar{P}_{b11} & \bar{P}_{b12} & \bar{P}_{b1i1} \\ \bar{P}_{b21} & \bar{P}_{b22} & \bar{P}_{b2i1} \\ \bar{P}_{i11} & \bar{P}_{i12} & \bar{P}_{i1i} \end{bmatrix} \begin{bmatrix} \bar{x}_{b1} \\ \tilde{u}_{b2} \\ \tilde{u}_{i1} \end{bmatrix} = \begin{bmatrix} 0 \\ t_{b2} \\ 0 \end{bmatrix} \quad \text{for } \Omega_{1}, \text{ m}=1,2, \quad (24a)$$

$$\begin{bmatrix} \bar{A}_{b33} & \bar{A}_{b34} & \bar{A}_{b3i2} \\ \bar{A}_{b43} & \bar{A}_{b44} & \bar{A}_{b4i2} \\ \bar{A}_{i33} & \bar{A}_{i34} & \bar{A}_{i2i} \end{bmatrix} \begin{bmatrix} \tilde{u}_{b3} \\ \bar{x}_{b4} \\ \tilde{u}_{i2} \end{bmatrix} - \omega^{2} \begin{bmatrix} \bar{P}_{b33} & \bar{P}_{b34} & \bar{P}_{b3i2} \\ \bar{P}_{b33} & \bar{P}_{b44} & \bar{P}_{b4i2} \\ \bar{P}_{i33} & \bar{P}_{i34} & \bar{P}_{i2i} \end{bmatrix} \begin{bmatrix} \tilde{u}_{b3} \\ \bar{x}_{b4} \\ \tilde{u}_{i2} \end{bmatrix} = \begin{bmatrix} t_{b3} \\ 0 \\ 0 \end{bmatrix} \quad \text{for } \Omega_{2}, \text{ m}=3,4 \quad (24b)$$

Then takes into account the interface equilibrium and compatibility conditions for the tractions and displacements shared by Ω_1 and Ω_2

$$\boldsymbol{t}_{b2} = -\boldsymbol{t}_{b3}, \qquad \tilde{\boldsymbol{u}}_{b2} = \tilde{\boldsymbol{u}}_{b3} \qquad \text{between } \Omega_1 \text{ and } \Omega_2$$
 (25)

Finally yields a 2N x 2N (N contains all the boundary nodes and the internal nodes for the whole coating-substrate plate) generalized eigenvalue system.

$$[\mathbf{K}]{\mathbf{X}} = \omega^2[\mathbf{M}]{\mathbf{X}}, \qquad (26)$$

where

$$\begin{bmatrix} K \end{bmatrix} = \begin{bmatrix} \bar{A}_{b11} & \bar{A}_{b12} & 0 & \bar{A}_{b111} & 0 \\ \bar{A}_{b21} & \bar{A}_{b22} + \bar{A}_{b33} & \bar{A}_{b34} & \bar{A}_{b211} & \bar{A}_{b322} \\ 0 & \bar{A}_{b43} & \bar{A}_{b44} & 0 & \bar{A}_{b422} \\ \bar{A}_{i11} & \bar{A}_{i12} & 0 & \bar{A}_{i11} & 0 \\ 0 & \bar{A}_{i33} & \bar{A}_{i34} & 0 & \bar{A}_{i2i} \end{bmatrix} \begin{bmatrix} M \end{bmatrix} = \begin{bmatrix} \bar{P}_{b11} & \bar{P}_{b12} & 0 & \bar{P}_{b13} & \bar{P}_{b34} & \bar{P}_{b312} & \bar{P}_{b322} \\ \bar{P}_{b21} & \bar{P}_{b22} + \bar{P}_{b33} & \bar{P}_{b44} & 0 & \bar{P}_{b422} \\ 0 & \bar{P}_{b43} & \bar{P}_{b44} & 0 & \bar{P}_{b422} \\ \bar{P}_{i11} & \bar{P}_{i12} & 0 & \bar{P}_{i1i} & 0 \\ 0 & \bar{P}_{i33} & \bar{P}_{i34} & 0 & \bar{P}_{i2i} \end{bmatrix}$$
(27ab)
$$\{X\} = \{\bar{x}_{b1} & \tilde{u}_{b2} & \bar{x}_{b4} & \tilde{u}_{i1} & \tilde{u}_{i2}\}^{T}.$$
(27c)

Figure 4. Boundary discretization of a two-layered FG coated plates

By resolving this general eigenvalue equation, the eigenvalue ω and the eigenvector $\{X\}$ for the coating-substrate plates can be obtained numerically.

Numerical analysis and discussion

Two numerical analysis examples make up this section. The first one is conducted by the comparative study of the 2D homogeneous coated and the FG coated plates. The other investigates the free vibration behaviors of the FG undercoated plates. The boundary conditions for these three plates are notated by the combination of four edges boundary situations moving counter clockwise starting from the edges x_2 =0. The simply supported (S), fixed (C) and free (F) boundary conditions are imposed as below and shown in Fig. 5.

S: $t_{x1}=0$, v=0, on $x_1=0$; C: u=v=0, on $x_1=0$; F: $t_{x1}=t_{x2}=0$, on $x_1=0$.

A developed FORTRAN program [Yang et al. (2014)] is using to fulfill this numerical evaluation and plane-strain condition is considered throughout this study. The natural frequencies are all normalized by

$$\boldsymbol{\varpi} = \omega h_t \sqrt{\rho_{Al} / E_{Al}} \,, \tag{28}$$

where h_t is the total thickness of the analyzed plate.



Figure 5. Different supports for coating-substrate plates (a) simply supported; (b) fixed; (c) free

Verify the accuracy of the results

In order to verify the accuracy of the present method, the results evaluated by the developed meshfree boundary-domain integral equation method are used to compare with that calculated by the traditional finite element method (FEM). The first ten normalized natural frequencies of the SSSS supported homogeneous coated plate (HCP), FG coated plate (FCP) and the FG undercoated plate (FUCP) with $L/h_t=1$ are shown in Table 2. In this study, the coating thickness ratio considered for HCP is $h_3/h_t=0.5$, for FCP is $h_2/h_t=0.5$, and for FUCP is $h_c/h_t=0.5$, $h_2/h_t=0.6$. From the Table 2, it can be seen that the results of the present methods have a great agree with that of the FEM, even for the high frequencies.

coated plate and the FG undercoated plate									
HCP	FEM	Error(%)	FCP	FEM	Error(%)	FUCP	FEM	Error(%)	
1.8456	1.8408	0.26	1.8057	1.7999	0.32	1.8070	1.8021	0.28	
1.9856	1.9772	0.42	1.9378	1.9270	0.56	1.9042	1.8946	0.50	
2.9478	2.9427	0.17	3.0070	2.9999	0.23	3.1112	3.0580	1.74	
3.8201	3.8151	0.13	3.9569	3.9179	1.00	3.9257	3.8885	0.96	
3.9252	3.9161	0.23	4.0271	3.9850	1.06	4.1889	4.1533	0.86	
4.2923	4.2861	0.14	4.4794	4.4641	0.34	4.5159	4.5055	0.23	
4.4885	4.4773	0.25	4.6930	4.6709	0.47	4.7561	4.7356	0.43	
5.2895	5.2786	0.21	5.4499	5.4207	0.54	5.4261	5.4075	0.34	
5.5976	5.5928	0.09	5.6657	5.6671	0.02	5.5483	5.5491	0.01	
5.8012	5.7909	0.18	5.9679	5.8794	1.51	5.8907	5.8831	0.13	

Table 2. Comparison the normalized frequencies of the homogeneous coated plate, FGcoated plate and the FG undercoated plate

Comparative study for the homogeneous coated plates and the FG coated plates

For the sake of understanding the free vibration behaviors of the FG coated plates in a more comprehensive view, the free vibration of the homogeneous coated plates is also analyzed to do the comparative study. The square and the rectangular homogeneous coated and the FG coated plates with five coating thickness ratios ($h_i/h_i=0.1, 0.2, 0.3, 0.4$ and 0.5) as well as six boundary conditions are investigated in details. The square coating-substrate plates with six different boundary conditions are described in Fig. 6.



Figure 6. Boundary conditions for the FG coated plates (a) CFSF; (B)CCSS; (C)CFFF; (D)SSSS; (E)FSCS; (F)SFCS

The normalized fundamental frequency versus the coating thickness ratio of the homogeneous coated and the FG coated plates with $L/h_t=1$ and 2 are drawing in Fig. 7 and 8, respectively. From figures 7(a) and 8(a), It can be seen that, with increasing the coating thickness ratio, the normalized fundamental frequencies of the CCSS and CFFF coated plates are increased, but it is decreased for the FSCS and SFCS coated plates, while, for the other two coated plates it varies in a parabolic tendency, which is for the square coated plates. Nevertheless, for the rectangular coated plates, the variation trend of the normalized frequency according with the coating thickness ratio is the similar with that of the square one, except for the SSSS coated plates, which decrease with increasing of h_i/h_t . The plate aspect ratio effects the normalized frequencies of the coated plates in a way like that, for the CFSF and CFFF coated plates, increase the normalized frequencies with increasing the plates aspect ratios, however, it effects the CCSS and SSSS coated plates in an opposite tendency. What is more, the normalized fundamental frequencies for the FSCS coated plates makes no difference with that of the SFCS coated plates, then it can be concluded that the plate aspect ratio effects less for the SFCS and FSCS coated plates.



Figure 7. The normalize fundamental frequency versus coating thickness ratio of the homogeneous coated plates with different boundary conditions (a) $L/h_t=1$; (b) $L/h_t=2$



Figure 8. The normalized fundament frequency versus coating thickness ratio of the FG coated plates with different boundary conditions (a) $L/h_t=1$; (b) $L/h_t=2$

From Fig 7 and 8, it can be seen that the variation trend of the homogeneous coated plates and the FG coated plates with different aspect ratios, coating thickness ratios and boundary conditions almost in an identical way, which shed lights on that the free vibration behaviors of the coating-substrate plates will be determined based on the associated effects of the different kinds of the variables, and contrast to the others, the material properties play a weaker role. But compare the

free vibration behaviors of these two coating-substrate plates in a more detail, it will be found that, with the variation of the important parameters, the changing of the normalized natural frequencies for the FG coated plates are more temperately, that can be seen from Fig. 9. Then it can be concluded that, the coating thickness ratio, plate aspect ratio and the boundary conditions have a less effectiveness on the FG coated plates than the homogeneous one.



Figure 9. Normalized fundamental frequencies versus coating thickness ratio (a) CFSF coated plates; (b) CCSS coated plates

Free vibration behaviors of the FG undercoated plates

In this section, the three-layered FG undercoated plates are investigated. For the FG undercoated plates, the thickness of the whole coating is denoted by h_c , in which the top homogeneous coating has the thickness h_3 , and the bottom FG layer has the thickness h_2 . In this study, fixes the coating thickness ratio, $h_c/h_t=0.5$, meanwhile, six variational $h_2/h_c=0$, 0.2, 0.4, 0.6, 0.8 and 1 are considered in the parametric study. Respect to the $h_2/h_c=0$, that is the FG layer thickness is changing to zero which refers to the homogenous coated plate and when the $h_2/h_c=1$, it refers to the FG coated plates. Two plate aspect ratios and six different boundary conditions are still used to simulate the free vibration behaviors of this FG undercoated plates. The normalized fundamental frequency versus h_2/h_c of the considered FG undercoated plates are plotted in Fig. 10.



Figure 10. The normalized fundamental frequency versus h₂/h_c of the FG undercoated plates

It is important to be noted that, the parameter h_2/h_c changing from 0 to 1, represents the thickness of the FG layer changing from 0 to h_c . In the meantime, the rising of this parameter makes the covered proportion of the steel constituent enlarged, which directly leads to the grown up of the young's modulus and mass density for the entire FG undercoated plates. It illustrates that the larger the

parameter h_2/h_c , the stiffer of the FG undercoated plates. Then it can be obtained from the Fig 10 that, with the increasing of the h_2/h_t , decrease the normalized fundamental frequencies of the CFSF, CCSS and CFFF FG undercoated plates, while increase that of the FSCS and SFCS FG undercoated plates, these characters are fitting for both of the square and the rectangular FG undercoated plates. However, the SSSS FG undercoated plates is a special case, that is for the $L/h_t=1$ the normalized fundamental frequency is in an upward trend and for the $L/h_t=2$, it plays an opposite trend.

Conclusions

In this paper, the free vibration of the FG coated and the FG undercoated plates are analyzed by the developed meshfree boundary-domain integral equation method. The homogeneous coated plates are also considered to do the comparative study. These numerical analyses demonstrate that the present method is accuracy and efficiency. Based on the parametric studies, it obtained that, the free vibration behaviors of the FG coated plates and the FG undercoated plates are influenced by the associated effects of the different kinds of the important parameters, and these parameters affect the homogeneous coated plates a lot.

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