# Eliminating the Pressure-Velocity Coupling from the Incompressible Navier-Stokes Equations Using Integral Transforms

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## Abstract

The present work proposes two methodologies using the Integral Transform Technique to solve the Poisson equation arising from the incompressible Navier-Stokes equations. The solution of this Poisson equation is very common in the formulations based on pressure-correction and are the main bottleneck of these approaches. The new formulation proposed in this work will allow the elimination of the pressure-velocity decomposition and also eliminate the sub-iterations of the usual pressure-correction methods. The results show a comparison in performance of both proposed approaches.

**Keywords:** Incompressible Navier-Stokes, Pressure Correction, Integral Transformation, Poisson Equation

x, y, z	Classical cartesian coordinates
L	Domain dimension in $x$ direction
Η	Domain dimension in <i>y</i> direction
t	Time
v	Vectorial velocity
и	Velocity component in x direction
V	Velocity component in <i>y</i> direction
f	Vectorial body force (in acceleration dimensions)
$f_x$	Body force component in $x$ direction
$f_y$	Body force component in <i>y</i> direction
р	Pressure
$\overline{p}$	Transformed pressure for CITT solution
ho	Fluid density
$\mu$	Dynamic viscosity
V	Kinematic viscosity
n	Index for CITT
n <sub>max</sub>	Truncation order for CITT
i, q, p	Indices for the position in mesh for $x$ direction
$i_{\rm max}$	Maximum mesh divisions in $x$ direction
j, r, s	Indices for the position in mesh for $y$ direction
$\dot{J}_{ m max}$	Maximum mesh divisions in y direction
l	Index for the time step
$X_n$	Eigenfunctions for the CITT solution
$\lambda_n$	Eigenvalues for the CITT solution

- $N_{\rm m}$  Norms of the eigenvalue problem
- y', y'' Dummy integration variables
- $c_1, c_2$  Integration constants
- V Domain volume
- $\Lambda$  General function
- $\epsilon$  Absolute error

## Introduction

In numerical simulations of incompressible flows, the main difficulty is the velocity and pressure coupling by the incompressibility constraint. The projection methods were developed to overcome this problem. These methods can be classified into three classes [Germond et al (2006)], namely pressure-correction methods, velocity-correction methods, and consistent splitting methods.

The most attractive feature of projection methods is that, at each time step, one only needs to solve a sequence of decoupled elliptic equations for the velocity and the pressure, making it very efficient for large scale numerical simulations. Although projection methods are widely used, many authors already drew attention to the fact that the decomposition used in this method is intrinsically second order accurate [Munz et al. (2003), Guermond et al. (2006)], preventing any approximation order higher than this. Pressure-correction schemes are time-marching techniques composed of two substeps for each time step: the pressure is treated explicitly or ignored in the first sub-step and is corrected in the second one. The linear momentum equations play the major role in determining the velocity components. Thus, it is left to the continuity equation to determine pressure, even if this variable does not appear explicitly in this equation. The most common methodology to determine an equation for the pressure-correction is to combine both equations by taking the divergence of the momentum equations and substituting the continuity equation where necessary, effectively generating a Poisson type equation for pressure. This makes it possible to obtain an equation for the pressure-correction, using the continuity equation. At this point, it is worth to highlight that the pressure-correction method is an iterative strategy which generate more accurate values at each iteration. The pressure-correction equation is an extrapolation to improve mass conservation at each iteration.

This procedure requires sub-iteration per time step, which is the major computational cost because at each sub-iteration, a Poisson equation for pressure must be solved. One could use Multigrid for the solution of the poisson equation to speed up the process, however, the sub-iteration are still required.

In the realm of analytical methods, the Integral Transform Technique, also known as the Classical Integral Transform Technique (CITT) [Mikhailov and Ozisik (1984)], has been playing a big role. It deals with expansions of the sought solution in terms of infinite orthogonal basis of eigenfunctions, keeping the solution process always within a continuous domain. The resulting system is generally composed of a set of uncoupled differential equations which can be solved analytically. However, a truncation error is involved since the infinite series must be truncated to obtain numerical results. This error decreases as the number of summation terms (truncation order) is increased, and the solution converges to a final value. Due to the series representation nature of the Integral Transform Technique, the estimated error can be easily obtained, which results in better global error control of the solution. The disadvantage associated with this approach is the need of a more elaborate analytical manipulation. This effort can be greatly minimized with the use of symbolic computation [Wolfram (2003)].

The present work proposes two methodologies using the Integral Transform Technique to solve the Poisson equation arising from the incompressible Navier-Stokes equations. The two proposed methodologies are: The single transformation and the double transformation. The new proposed formulation will allow the elimination of the pressure-velocity decomposition and also eliminate the sub-iterations of the usual pressure-correction methods.

Just a few works were concerned in mixing the Integral Transform Technique with other discrete schemes. Among these works, one could mention the work [Chalhub et al (2013)] that a new methodology for solving unsteady convective heat transfer problems via the generalized integral

transform technique was developed. The proposed scheme was based on writing the unknown potential in term of eigenfunction expansions; however, rather than transforming advection terms, an upwind approximation is used prior to the integral transformation. In the works [Guedes et al. (1994a; 1994b)], the authors analyzed the unsteady forced convection in laminar flow between parallel plates. This problem is solved using a hybrid scheme that combines the Generalized Integral Transform Technique with second-order finite differences. At [Cotta and Gerk (1994)], the integral transform method is employed in conjunction with second-order-accurate explicit finite-differences schemes, to handle accurately a class of parabolic-hyperbolic problems. In the work [Castelloes and Cotta (2006)], the solution is obtained using partial integral transformation strategy to solve the problem and the work of [Naveira et al. (2009)] showed a hybrid numeric-analytical solution for unsteady forced laminar convection between parallel plates.

## **Problem Formulation**

In order to solve a fluid flow problem, the conservation laws of physics are needed: the mass conservation, also known as the continuity equation and the momentum conservation. In addition to these equations, a constitutive equation is required, which the Newton's law of viscosity will be used. The flow is also considered to be incompressible, in other words, the fluid has a constant density.

Combining these equations [Kundu (1990)], one can arrive at the incompressible Navier-Stokes equations<sup>1</sup>:

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\frac{1}{\rho} \nabla p + v \nabla^2 v + f \quad \text{for} \quad x \in V \quad \text{and} \quad t \ge 0$$
(1)

$$\nabla \cdot v = 0 \quad \text{for} \quad x \in V \tag{2}$$

in which equation (1) is the momentum conservation equation and equation (2) is the mass conservation equation, also called the incompressibility constraint.

Based on the projection methods for incompressible flows [Guermond et al. (2006)], to obtain the Poisson equation for pressure, one should apply the divergence operator on equation (1) and use the continuity equation (2):

$$\frac{1}{\rho} \nabla^2 p = \nabla \cdot f - \nabla v : (\nabla v)^{\mathrm{T}} \quad \text{for} \quad x \in V$$
(3)

where  $\rho$  is the fluid density, f is the body force vector, v is the velocity vector, p is the pressure and V is a general domain volume.

Then, the continuity could be replaced at the Navier-Stokes equations (1) by the Poisson equation (3), resulting in the following system to be solved:

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\frac{1}{\rho} \nabla p + v \nabla^2 v + f \quad \text{for} \quad x \in V \quad \text{and} \quad t \ge 0$$
(4)

$$\frac{1}{\rho} \nabla^2 p = \nabla \cdot f - \nabla v : (\nabla v)^{\mathrm{T}} \quad \text{for} \quad x \in V$$
(5)

In this work, normal zero gradients for pressure at the boundaries will be used.

$$\left(\nabla p \cdot n\right)_{\partial V} = 0 \tag{6}$$

where  $\partial V$  is the boundary of the general domain volume. The problem can be simplified for cartesian domain:

$$\frac{\partial^2 p(x, y, t)}{\partial x^2} + \frac{\partial^2 p(x, y, t)}{\partial y^2} = \rho[h(x, y, t) - g(x, y, t)]$$
(7)

$$\left(\frac{\partial p(x, y, t)}{\partial y}\right)_{y=0} = 0 \qquad \left(\frac{\partial p(x, y, t)}{\partial y}\right)_{y=H} = 0 \tag{8}$$

<sup>&</sup>lt;sup>1</sup>Some authors refer to the Navier-Stokes equations as just the momentum equation.

$$\left(\frac{\partial p(x, y, t)}{\partial x}\right)_{x=0} = 0 \qquad \left(\frac{\partial p(x, y, t)}{\partial x}\right)_{x=L} = 0 \tag{9}$$

where:

$$g(x, y, t) = \left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial y}\right)^2$$
(10)

$$h(x, y, t) = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}$$
(11)

The main goal of this work is to develop the integral transformation technique to solve the Poisson equation (7) showed above.

## Single Transformation (ST)

In order to establish the transformation pair, the pressure field is written as function of an orthogonal eigenfunctions obtained from the following auxiliary eigenvalue problem known as the Helmholtz classical problem [Mikhailov and Osizik (1984)], where  $X_n(x)$  are the eigenfunctions and  $\lambda_n$  are the eigenvalues.

$$\frac{d^2 X_n(x)}{dx^2} + \lambda_n^2 X_n(x) = 0$$
(12)

$$X'(0) = 0 X'(L) = 0 (13)$$

which has the following solution:

$$X_n(x) = \cos\left(x\lambda_n\right) \tag{14}$$

$$\lambda_n = \frac{\pi n}{L} \quad \text{for} \quad n = 1, 2, 3, \dots \tag{15}$$

It should be noted that for these boundary conditions, one needs also to account for non-trivial solutions corresponding to  $\lambda_0 = 0$ .

$$X_0 = 1$$
 (16)

$$\lambda_0 = 0 \tag{17}$$

Now, the transformation pair can be defined:

Transformation 
$$\Rightarrow \overline{p}_n(y,t) = \int_0^L p(x,y,t) X(x)_n dx$$
 (18)

Inversion 
$$\Rightarrow p(x, y, t) = p_0(y, t) + \sum_{n=1}^{\infty} \frac{X_n(x)\overline{p}_n(y, t)}{N_n}$$
 (19)

where the norm  $N_n$  is defined by:

$$N_n = \int_0^L X_n^2 \, \mathrm{d}x = \frac{L}{2} \qquad \text{for} \qquad n \neq 0$$
 (20)

The final solution is given by two portions of the pressure: the average pressure in x direction  $p_{avg}$  and the modified pressure  $p_{mod}$ :

$$p(x, y, t) = p_{avg}(y, t) + p_{mod}(x, y, t)$$
(21)

where  $p_{avg}$  comes from the solution of the eigenproblem when  $\lambda = 0$  and  $p_{mod}$  comes from the solution when  $\lambda \neq 0$ , in other words:

$$p_{\text{avg}}(y,t) = p_0(y,t)$$
 (22)

$$p_{\text{mod}}(x, y, t) = \sum_{n=1}^{\infty} \frac{X_n(x)\overline{p}_n(y, t)}{N_n}$$
(23)

Solution for  $p_{\text{mod}} (\lambda \neq 0)$ 

The integral transformation of the governing differential equation is derived by applying the operator  $\int_0^L (\bullet) X_n dx$  on equation (7), obtaining the following transformed Poisson equation:

$$\frac{\partial^2 \overline{p}_n(y,t)}{\partial y^2} - \lambda_n^2 \overline{p}_n(y,t) = \rho \overline{h}_n(y,t) - \rho \overline{g}_n(y,t)$$
(24)

$$\left(\frac{\partial \overline{p}_{n}(y,t)}{\partial y}\right)_{y=0} = 0$$

$$\left(\frac{\partial \overline{p}_{n}(y,t)}{\partial y}\right)_{y=0} = 0$$
(25)

$$\left(\frac{\partial \overline{p}_n(y,t)}{\partial y}\right)_{y=H} = 0$$
(26)

where the transformation of the parameters g and h are given by:

$$\overline{g}_{n}(y,t) = \int_{0}^{L} g(x,y,t) X_{n} \,\mathrm{d}x \tag{27}$$

$$\overline{h}_{n}(y,t) = \int_{0}^{L} h(x,y,t) X_{n} \,\mathrm{d}x \tag{28}$$

The equation (24) has an analytical solution that is shown bellow:

$$\overline{p}_{n}(y,t) = e^{-y\lambda_{n}} \left[ -\frac{1}{2} \cosh\left(y\lambda_{n}\right) \operatorname{csch}\left(H\lambda_{n}\right) e^{(H+y)\lambda_{n}} \int_{0}^{H} \frac{\rho e^{-y\lambda_{n}}}{\lambda_{n}} \left(\overline{h}_{n}(y,t) - \overline{g}_{n}(y,t)\right) dy + \frac{1}{4} \left(e^{2y\lambda_{n}} + 1\right) \left( \coth\left(H\lambda_{n}\right) - 1\right) \int_{0}^{H} \frac{\rho e^{y\lambda_{n}}}{\lambda_{n}} \left(\overline{g}_{n}(y,t) - \overline{h}_{n}(y,t)\right) dy + \frac{1}{2} \int_{0}^{y} \frac{\rho e^{y'\lambda_{n}}}{\lambda_{n}} \left(\overline{g}_{n}(y',t) - \overline{h}_{n}(y',t)\right) dy' + \frac{1}{2} e^{2y\lambda_{n}} \int_{0}^{y} \frac{\rho e^{-y'\lambda_{n}}}{\lambda_{n}} \left(\overline{h}_{n}(y',t) - \overline{g}_{n}(y',t)\right) dy' \right)$$

$$(29)$$

To find the actual solution for modified pressure  $p_{mod}$ , the inversion formula is used, equation (22). By observing equations (29), (27) and (28), one can notice that there are integrals of the discrete variables u, v,  $f_x$  and  $f_y$ . In order to compute these integrals, the following integral separation is proposed:

$$\int_{0}^{L} \Lambda(u, v, f_{x}, f_{y}) dx = \sum_{q=1}^{i_{\max}} \int_{x_{q-1}}^{x_{q}} \Lambda(u, v, f_{x}, f_{y}) dx$$
(30)

where  $\Lambda$  is a general function of u, v,  $f_x$  and  $f_y$ .

Then, to compute the integrals analytically, a Taylor expansion is used to expand the variables u, v,  $f_x$  and  $f_y$  in each subdomain:

## Solution for $p_{avg}$ ( $\lambda = 0$ )

In order to obtain the transformed differential equation for  $\lambda = 0$ , a very similar process is done. The transformed equation is given bellow:

$$\frac{\partial^2 \overline{p}_0(y,t)}{\partial y^2} = \rho \overline{h}_0(y,t) - \rho \overline{g}_0(y,t)$$
(31)

$$\left(\frac{\partial \overline{p}_{0}(y,t)}{\partial y}\right)_{y=0} = 0$$
(32)

$$\left(\frac{\partial \overline{p}_{0}(y,t)}{\partial y}\right)_{y=H} = 0$$
(33)

where:

$$\overline{g}_0(y,t) = \int_0^L g(x,y,t) \,\mathrm{d}x \tag{34}$$

$$\overline{h}_{0}(y,t) = \int_{0}^{L} h(x, y, t) \,\mathrm{d}x$$
(35)

The previous equation admits analytical solution of the following form:

$$\overline{p}_{0}(y,t) = \int_{0}^{y} \int_{0}^{y'} \rho\left(\overline{h}_{0}(y',t) - \overline{g}_{0}(y',t)\right) dy' dy'' + c_{1}y + c_{2}$$
(36)

applying the boundary conditions, one arrives to the following system of equations:

$$c_1 = 0$$
 (37)

$$0 = \int_{0}^{H} \rho(\overline{h}_{0}(y,t) - \overline{g}_{0}(y,t)) dy + c_{1}$$
(38)

which tells that  $c_1$  must be zero and the integral also must be zero:

$$\int_{0}^{H} \rho\left(\overline{h}_{0}(y,t) - \overline{g}_{0}(y,t)\right) dy = 0$$
(39)

Equation (9) is know as the Poisson-Neumann compatibility condition [Abdallah (1987; 1988); Pozrikidis (2001)]. Knowing these information, the solution of the transformed differential equation is achieved:

$$\overline{p}_{0}(y,t) = \int_{0}^{y} \int_{0}^{y''} \rho\left(-\overline{g}_{0}(y',t) + \overline{h}_{0}(y',t)\right) dy' dy''$$
(40)

Then the same integral separation (equation (19)) and Taylor series expansions are used to derive analytically the coefficients  $\overline{h}_0$  and  $\overline{g}_0$ :

## **Double Transformation (DT)**

In a very similar manner from previous formulation, one first establishes the transformation pair. In order to obtain that for this approach, two eigenvalue problems are defined. The eigenvalue problem associated with the x direction is given by: Eigenvalue problem associated with the problem in x direction

$$\frac{d^2 X_n(x)}{dx^2} + \lambda_n^2 X_n(x) = 0$$
(41)

$$X'(0) = 0 X'(L) = 0 (42)$$

which has the following solution:

$$X_n(x) = \cos(x\lambda_n) \tag{43}$$

$$\lambda_n = \frac{\pi n}{L} \qquad \text{for} \qquad n = 1, 2, 3, \dots \tag{44}$$

The problem associated with the *y* direction is given by:

$$\frac{d^2 Y_n(y)}{dy^2} + \beta_m^2 Y_n(y) = 0$$
(45)

$$Y'(0) = 0$$
  $Y'(H) = 0$  (46)

which has the following solution:

$$Y_n(x) = \cos(y\beta_m) \tag{47}$$

$$\beta_m = \frac{\pi m}{H}$$
 for  $m = 1, 2, 3, ...$  (48)

The transformation pair can be defined:

Transformation 
$$\Rightarrow \overline{\overline{p}}_{n,m}(t) = \int_0^H \int_0^L p(x, y, t) X(x)_n Y(x)_m \, dx \, dy$$
 (49)

Inversion 
$$\Rightarrow p(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\overline{p}_n(y, t) X_n(x) Y_m(y)}{N x_n N y_m}$$
 (50)

where the norms  $N_{X_n}$  and  $Ny_m$  are defined by:

$$N_{X_n} = \int_0^L X_n^2 dx = \frac{L}{2}$$
 for  $n \neq 0$  (51)

$$Ny_m = \int_0^H Y_m^2 dx = \frac{H}{2} \quad \text{for} \quad m \neq 0$$
(52)

it is also known that  $N_{X_0} = L$  and  $Ny_0 = H$ 

Applying the operator  $\int_0^H \int_0^L (\bullet) X_n Y_m dx dy$  on the Poisson equation, the following transformed Poisson equation (algebraic) is obtained:

$$-(\lambda_n^2 + \beta_m^2)\overline{\overline{p}}_{n,m}(y,t) = \rho\overline{\overline{h}}_{n,m}(y,t) - \overline{\overline{g}}_{n,m}(y,t)$$
(53)

where the transformation of the parameters g and h are given by:

$$\overline{\overline{g}}_{n,m}(t) = \int_0^H \int_0^L g(x, y, t) X_n Y_m \,\mathrm{d}x \,\mathrm{d}y$$
(54)

$$\overline{\overline{h}}_{n,m}(t) = \int_0^H \int_0^L h(x, y, t) X_n Y_m \,\mathrm{d}x \,\mathrm{d}y$$
(55)

Which has a direct solution shown bellow:

$$\overline{\overline{p}}_{n,m}(t) = \frac{\rho(\overline{\overline{g}}_{n,m} - \overline{\overline{h}}_{n,m})}{\lambda_n^2 + \beta_m^2}$$
(56)

In order to obtain the final solution, the inversion formula needs to be applied, obtaining:

$$p(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\rho(\overline{g}_{n,m} - \overline{h}_{n,m})}{\lambda_n^2 + \beta_m^2} \frac{X_n(x)Y_m(y)}{Nx_n Ny_m}$$
(57)

which can also be rewritten in the form:

$$p(x, y, t) = \frac{\overline{\overline{p}}_{0,0}}{Nx_0 Ny_0} + \sum_{m=1}^{\infty} \frac{\rho(\overline{\overline{g}}_{0,m} - \overline{\overline{h}}_{0,m})}{\beta_m^2} \frac{Y_m(y)}{Nx_0 Ny_m} + \sum_{n=1}^{\infty} \frac{\rho(\overline{\overline{g}}_{n,0} - \overline{\overline{h}}_{n,0})}{\lambda_n^2} \frac{X_n(x)}{Nx_n Ny_0} + \sum_{n=1}^{\infty} \frac{\rho(\overline{\overline{g}}_{n,0} - \overline{\overline{h}}_{n,0})}{Nx_n Ny_0} + \sum_{n=1}^{\infty} \frac{\rho(\overline{\overline{g}}_{n,0} - \overline{\overline{h}_{n,0}})}{Nx_n Ny_0} + \sum_{n=1}^{\infty} \frac{\rho(\overline{\overline{g}}_{n,0} - \overline{\overline{h}_{n,0})}}{Nx_n Ny_0} + \sum_{n=1}^{\infty} \frac{\rho(\overline{\overline{g}}_{n,0} - \overline{\overline{h}_{n,0})}{Nx_n Ny_0} + \sum_{n=1}^{\infty} \frac{\rho(\overline{\overline{g}}_{n,0} - \overline{\overline{h}_{n,0})}{Nx_n Ny_0} + \sum_{n=1}^{\infty} \frac{\rho(\overline{\overline{g}}_{n,0} - \overline{\overline{h}_{n,0}})}{Nx_n Ny_0} + \sum_{n=1}^{\infty} \frac{\rho(\overline{\overline{g}}_{n,0} - \overline{\overline{h}_{n,0}})}{Nx_n Ny_0} + \sum_{n=1}^{\infty} \frac{\rho(\overline{\overline{g}}_{n,0}$$

where  $\overline{\overline{p}}_{0,0}$  is an arbitrary constant, which will be considered to be zero.

The greatest advantage of this approach is that it requires a lot less analytical effort and the final solution is more simple and compact. But the final solution has a double summation that can increase computational cost. In order to minimize this cost, one can use a reordering scheme, switching from the double summation to a single one.

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \Longrightarrow \sum_{k=1}^{\infty}$$
(59)

This can be done knowing the sum terms with higher magnitude and putting them in the beginning of the sum. This is achieved by taking (n,m) pairs that promote lowers  $(\lambda_n^2 + \beta_m^2)$ . By doing that, one can arrive at the expression:

$$p(x, y, t) = \sum_{m=1}^{\infty} \frac{\rho(\overline{\overline{g}}_{0,m} - \overline{\overline{h}}_{0,m})}{\beta_m^2} \frac{Y_m(y)}{Nx_0 Ny_m} + \sum_{n=1}^{\infty} \frac{\rho(\overline{\overline{g}}_{n,0} - \overline{\overline{h}}_{n,0})}{\lambda_n^2} \frac{X_n(x)}{Nx_n Ny_0} + \sum_{k=1}^{\infty} \frac{\rho(\overline{\overline{g}}_{n(k),m(k)} - \overline{\overline{h}}_{n(k),m(k)})}{\lambda_{n(k)}^2 + \beta_{m(k)}^2} \frac{X_{n(k)}(x)Y_{m(k)}(y)}{Nx_{n(k)} Ny_{m(k)}}$$
(60)

#### **Discrete Derivatives**

In order to solve the pressure problem, the discrete derivatives of u, v,  $f_x$  and  $f_y$  must be calculated. In this work, a second order central differencing scheme is used inside the domain and second-order the backward/forward (depending of the boundary) differencing scheme is used at the boundaries.

#### **Results**

For all cases presented in this chapter, L=1, H=1, and  $\rho=1$  were used. The chosen source term of the Poisson equation (4) satisfies the compatibility condition (9) and it is of the following form:

$$\left[\rho(h(x, y, t) - g(x, y, t))\right]_{i,j} = \rho\left(x_i^4 + x_i^3 + x_i^2 + x_i - \frac{77}{60}\right)\left(y_j^4 + y_j^3 + y_j^2 + y_j\right)$$
(61)

A comparison of computational cost is done for both techniques presented in this work. In order to compare the CITT performance, a fixed mesh is used and many truncation orders for the summations are computed, so only the CITT error is captured. The CITT error is calculated using the following formula:

$$\epsilon_{i,j}^{\text{CITT}}(n_{\max}) = \text{abs}[p_{i,j}(n_{\max}) - p_{i,j}(n_{\max} + 5)]$$
(62)

The mesh error for the x-mesh and the y-mesh is calculated using the following formulations respectively:

$$\epsilon_{i,1024}^{x}(n_{\max}) = \operatorname{abs}[p_{i,1024}(n_{\max}) - p_{2i,1024}(n_{\max})]$$
(63)

$$\epsilon_{1024, i}^{y}(n_{\max}) = \operatorname{abs}[p_{1024, i}(n_{\max}) - p_{1024, 2i}(n_{\max})]$$
(64)

The codes were compiled and ran using GFORTRAN and the flags -O3 and -fopenmp and in a 8 core CPU machine.

In order to illustrate the convergence of the solution with the variation of  $\Delta x$ , figure 1 shows a graphic of the maximum absolute error with the variation of the mesh size  $\Delta x$ . As one can observe the convergence order is about 2, which was expected since all approximation made in the mathematical formulation were of this order.

Figure 2 shows the convergence of the absolute error with  $\Delta y$ . Although it seems to have a higher convergence order for the poorer refined meshes, the order stabilizes at 2 when more refined meshes are implemented.



Figure 1. Convergence for the mesh in x-direction,  $n_{\text{max}} = 15$  and using  $j_{\text{max}} = 1024$ .



Figure 2. Convergence for the mesh in *y*-direction,  $n_{\text{max}} = 15$  and using  $i_{\text{max}} = 1024$ .

In order to illustrate the problem, tables 1 and 2 show the value of the pressure for different points of the domain. It is clear that both methodologies converge to the same value, even though the values are not fully converged with six digits accuracy. One can see that the mesh convergence is very similar for both methodologies.

Table 1. Convergence of the mesh for some points of the domain for y = 0.25 and  $n_{max} = 30$ .

$i_{\max} \times j_{\max}$	x = 0.25		x = 0.75	
	ST	DT	ST	DT
16×16	0.0801190	0.0793513	-0.0799973	-0.0809956
32×32	0.0811750	0.0810473	-0.0807687	-0.0811251
64×64	0.0814272	0.0814572	-0.0809571	-0.0811557
128×128	0.0814885	0.0815574	-0.0810034	-0.0811630

Table 2. Convergence of the mesh for some points of the domain for y = 0.75 and  $n_{\text{max}} = 30$ .

$i_{\max} \times j_{\max}$	x = 0.25		x = 0.75	
	ST	DT	ST	DT
16×16	0.183004	0.186430	-0.185268	-0.181505
32×32	0.186532	0.187220	-0.184883	-0.183862
64×64	0.187368	0.187422	-0.184825	-0.184438
128×128	0.187571	0.187473	-0.184815	-0.184580

Figure 3 shows a comparison of the computational cost of both methodologies for a mesh of  $32 \times 32$ . One can clearly see that the Double Transformation (DT) needs more time to obtain the same error. This effect is due to the introduction of a double summation needed to solve the problem by the DT which requires a bigger effort to compute the solution. Even with the implementation of the reordering scheme, it is not enough to beat the computational cost of the ST. The same effect can be seen on figure 4, which shows the computational cost for a mesh of  $128 \times 128$ .



Figure 3. Comparison of the computational cost of CITT using single transformation and CITT using double transformation for a mesh  $i_{max} = 32$  and  $j_{max} = 32$ .



Figure 4. Comparison of the computational cost of CITT using single transformation and CITT using double transformation for a mesh  $i_{max} = 128$  and  $j_{max} = 128$ .

#### Conclusion

This work presented results of the solution of the Poisson equation arising from the incompressible Navier-Stokes equations. The main motivation of the proposed work is the implementation of this semi-analytical formulation for the Poisson equation in the momentum equation and thus solve it using a numerical technique for initial value problem.

The solution of the Poisson equation using this semi-analytical approach was accomplished using two different schemes: CITT single transformation (ST) and CITT double transformation (DT). Both techniques presented a very similar convergence behavior and results, showing that the formulations proposed are consistent. The comparison between both schemes showed that the double transformation has poorer performance in comparison with the single transformation scheme. Even though the performance of the double transformations was not so good, this formulation has a simpler analytical solution, which might be more interesting when implementing the pressure solution in the momentum equation, and so being possible a fully explicit time marching method for time.

The proposed schemes are very good for smooth pressure fields, but it might have some convergence problems with discontinuous pressure fields, which arise in phase-change problems, due to the Gibbs phenomenon in series truncation. This issue must be further investigated in order to compute the real impact of the implementation of the proposed formulations in these type of problems.

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