Accurate Transient Response Analysis of Non-Classically Damped Systems

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Abstract

It is difficult, or even unnecessary, to obtain all the modes of a large-scaled model. Thus, the modal truncation error is generally introduced and the quality of the responses may be adversely affected. Based on the Neumann series and the FFT technique, an accurate modal superposition method is presented to calculate the transient response of non-classically damped systems. The presented method maintains original-space without having to involve the state-space formula. The method is convergent if and only if all the complex modes whose resonant frequencies are less than the maximal sampling frequency of the FFT must be available. Finally, the applicability of the method is investigated using a simple numerical example with non-classical damping.

Keywords: Transient response analysis, Non-classically damped systems, FFT, Modal truncation error, Mode superposition method, Krylov subspace

Introduction

The purpose of a transient response analysis is to calculate the behaviour of a structural or mechanical system subjected to a time-varying forces. The inclusion of damping in the dynamic analyses of structural or mechanical systems has become an integral part of many design methodologies, including predicting vibration levels, transient responses, transmissibility and design problems dominated by energy dissipation.

In general, two different methods are used for the transient response analysis: direct transient response method (DTRM) and modal transient response method. The DTRM calculate dynamic responses by performing a direct numerical integration on complete coupled equations of motion at discrete times, typically with a fixed time step. In the most likely case for many engineering applications, the DTRM should be implemented for many time steps and large-scaled problems. Under such circumstance, it may be more effective by using reduced basis technique. Often the modes are used as the reduced basis vectors (known as the mode superposition method). The mode superposition method allows us to treat the equations of motion as a reduced-order form so that the step-by-step solution is less costly. It is should be noted that the quality of the responses depends on the number of modes involved. Although the accuracy of the calculated responses can be improved by increasing the number of modes, the convergence rate is very slow. Note that the eigenvalue solution is very computationally expensive, or even impossible, especially for large-scaled problems. Many approaches (Huang et al., 1997; Palmeri and Lombardo, 2011; D'Aveni and Muscolino, 2001; Besselink et al., 2013; Qu, 2007) were presented to deal with the modal truncation problems. However, these correction approaches are only restricted to the case of undamped or proportionally damped systems. In general, proportional damping means that energy dissipation is almost uniformly distributed throughout the mechanical system. There is no any physical reason why the proportional damping must be satisfied. In practice, mechanical systems with significantly different levels of energy dissipation are frequently encountered in dynamic designs. As can be seen from experimental data, physical system produces complex modes and therefore no physical system is strictly proportionally damped system. To this end, the concern of this study is the non-classically damped system. Several approximation techniques are developed to efficiently calculate the responses of non-classically damped systems. Among these approximation methods, the most common method is so-called the proportional approximation method (PAM), which is simply to ignore the off-diagonal (coupling) elements of the transformed damping matrix. It was shown that light damping, diagonal dominance of the transformed damping matrix and good separation of the normal modes (these conditions are once believed to produce small errors) are not sufficient conditions for the accuracy of the PAM. Although the PAM is a powerful approximate method, the results of the PAM are not always with acceptable accuracy. The accurate responses may be obtained by using the complex modal superposition method. Although these correction approaches used in undamped systems can be extended to non-proportionally damped systems based on the state-space formula, these state-space approaches are usually time-consuming since its size is two times the size of the original space and lack the physical insight provided by the superposition of the modes of the equation of motion in physical space. Note that the complex modes may be also efficiently calculated using the computational methods in the original space (Fischer, 2000; Holz et al., 2004; Adhikari, 2011; Rajakumar, 1993; Lee et al., 1998). Some of these original space based approaches have been programmed in famous softwares [see, e.g., Nastran (Komzsik, 2001)]. Some works (Li et al., 2014c; Li et al., 2014b; Li et al., 2013) were therefore developed to eliminate the complex modal truncation error of the frequency responses of damped systems without having to involve the state-space formula. It is shown (Li et al., 2014c) that the complex modal truncation error can be exactly expressed as a power-series expansion in terms of the available modes and system matrices and a hybrid expansion method (HEM) is presented to calculate the frequency responses of non-classically damped systems. Complex modes can be also recently shown to transform any viscously damped system with N DOF into N independent second-order equations [see, e.g., (Kawano et al., 2013; Morzfeld et al., 2011; Ma et al., 2010) for details].

This paper presents an accurate method to calculate the transient response of non-classically damped systems based on the Neumann series and the FFT technique. The method maintains original-space without having to involve the state-space formula so that it is efficient in computational effort and storage capacity. The method is convergent if and only if all the complex modes whose resonant frequencies are less than the maximal sampling frequency of the FFT must be available. Finally, it will be shown by a numerical example that, the proposed method can show a good agreement with the exact responses.

Transient response analysis

The equation of motion of an N DOF linear damped system with zero initial condition appears as the following matrix form

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{f}(t) \tag{1}$$

Here **M**, **C** and **K** are real mass, viscous damping and stiffness matrices, respectively. $\mathbf{x}(t)$ $\dot{\mathbf{x}}(t)$ $\ddot{\mathbf{x}}(t)$ and $\mathbf{f}(t)$ are displacement, velocity, acceleration and force, respectively. In this paper, assume that **K** is a positive definite symmetric matrix, **M** and **C** are symmetric matrices.

The time domain equation of motion may be cast into a frequency domain form by using the Fourier transform technique

$$\left(-\omega^2\mathbf{M}+\mathrm{i}\omega\mathbf{C}+\mathbf{K}\right)\mathbf{X}(\omega)=\mathbf{F}(\omega) \tag{2}$$

where

$$\mathbf{F}(\omega)=F\left[\mathbf{f}(t)\right] \text{ and } \mathbf{X}(\omega)=F\left[\mathbf{x}(t)\right] \tag{3}$$

Here $F[\ ]$ denote the Fourier transform and ω is the circular (angle) frequency. The form can be given under the assumption that the complex input forcing can be interpolated by trigonometric polynomials. In practice, we usually need to find the frequency spectra of excitation by using the Fourier transform.

The transient response can be then obtained by using the inverse Fourier transform, that is

$$\mathbf{x}(t)=F^{-1}\left[\mathbf{X}(\omega)\right] \tag{4}$$

In practice, a general analytical loading function cannot be easily obtained. It means that the discrete Fourier transform and inverse discrete Fourier transform algorithms given below should be used.

$$\begin{cases} \mathbf{X}_k = \displaystyle\sum_{n=0}^{N_{FT}-1} \mathbf{x}_n \exp\left(\frac{-2\pi ikn}{N_{FT}}\right) & \forall k=0,1,2,\ldots,N_{\mathrm{FT}}-1 \\ \mathbf{x}_n = \dfrac{1}{N_{FT}}\displaystyle\sum_{k=0}^{N_{FT}-1} \mathbf{X}_k \exp\left(\frac{2\pi ikn}{N_{FT}}\right) & \forall n=0,1,2,\ldots,N_{\mathrm{FT}}-1 \end{cases} \tag{5}$$

Here N_{FT} is a number of sample points. $\mathbf{x}_n$ are the elements of discrete time displacements and $\mathbf{X}_k$ are the elements of frequency spectrums of the discrete time series $\{\mathbf{x}_n\}$. Note the forcing samples should be obtained using the discrete Fourier transform [Equation (3)] and the transient response [Equation (4)] can be obtained by using inverse discrete Fourier transform once the frequency spectrums are calculated by solving Equation (2). The inverse Fourier transform procedure is defined using a positive sign in the exponential term and can be efficiently calculated using the *inverse fast Fourier transform* (IFFT) algorithm, which has been developed into a mature technology applied successfully to calculate the displacement both in the frequency and time domain [for detail discussions on this aspect can be found in (Barkanov et al., 2003; Brigham, 1988; Duhamel and Vetterli, 1990)].

Accurate calculation of frequency spectrums

The eigenvalue problem can be written in matrix form as

$$\left(\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K}\right)\boldsymbol{\varphi}_j=\mathbf{0}\quad\forall\, j=1,\ 2,\ \ldots,2N \tag{6}$$

Here λ_j and $\boldsymbol{\varphi}_j$ denote the jth eigenvalue and eigenvector (complex mode shape). Suppose these eigenvalues are ordered following increasing magnitude of imaginary parts. Assume these eigenvalues are distinct, the frequency spectrums can be calculated using the complex mode superposition method as

$$\mathbf{X}(\omega)=\sum_{j=1}^{2N}\frac{\boldsymbol{\varphi}_j^T\mathbf{F}(\omega)\boldsymbol{\varphi}_j}{\left(\mathrm{i}\omega-\lambda_j\right)\theta_j}\ \text{with}\ \boldsymbol{\varphi}_i^T\left(2\lambda_i\mathbf{M}+\mathbf{C}\right)\boldsymbol{\varphi}_i=\theta_j \tag{7}$$

Note the parameter θ_j can be chosen to be unity by normalizing eigenvectors. The method requires that all the modes should be available to obtain an exact response. Often only a few lower modes are considered in practical analysis. Suppose the lower L pairs of complex modes are available, the frequency spectrums are usually calculated in the following way with a modal truncation error involved

$$\mathbf{X}(\omega)=\mathbf{X}_{\mathrm{MDM}}(\omega)+\mathbf{E}_{\mathrm{MDM}}(\omega) \tag{8}$$

in which

$$\mathbf{X}_{\mathrm{MDM}}(\omega)=\sum_{j=1}^{2L}\frac{\boldsymbol{\varphi}_j^T\mathbf{F}(\omega)\boldsymbol{\varphi}_j}{\theta_j\left(\mathrm{i}\omega-\lambda_j\right)} \tag{9}$$

$$\mathbf{E}_{\mathrm{MDM}}(\omega)=\sum_{j=2L+1}^{2N}\frac{\boldsymbol{\varphi}_j^T\mathbf{F}(\omega)\boldsymbol{\varphi}_j}{\theta_j\left(\mathrm{i}\omega-\lambda_j\right)} \tag{10}$$

Equation (10) is known as the mode displacement method (MDM). Although the MDM is an efficient approximate method, the results of the MDM are not always with acceptable accuracy since the modal truncation error given by Equation (11) is introduced. Sometimes, the MDM will lead to misleading results [see, e.g., in (Li et al., 2014c; Li et al., 2014a)]. Recently, some reduced basis techniques were developed for the frequency response analysis [see e.g., (Hetmaniuk et al., 2012; Freund, 2003; Bai and Su, 2005; Hetmaniuk et al., 2013; Rumpler et al., 2014) for details].

Next, we present an accurate mode superposition method to calculate the frequency spectrums.

Theorem. *Suppose the damped system (1) only has distinct eigenvalues and the sample frequencies ω satisfy the convergence condition*

$$|\omega|<|\lambda_{2L+1}| \tag{11}$$

Then the frequency spectrums can be given by

$$\mathbf{X}(\omega)=\tilde{\mathbf{X}}(\omega)+\tilde{\mathbf{E}}(\omega) \tag{12}$$

where

$$\tilde{\mathbf{X}}(\omega)=\sum_{j=1}^{2L}\frac{\boldsymbol{\varphi}_j^T\mathbf{F}(\omega)\boldsymbol{\varphi}_j}{\theta_j\left(\mathrm{i}\omega-\lambda_j\right)}+\sum_{r=1}^{h}\left(\mathrm{i}\omega\right)^{r-1}\left[\mathbf{E}_r(\omega)+\sum_{j=1}^{2L}\frac{\boldsymbol{\varphi}_j^T\mathbf{F}(\omega)\boldsymbol{\varphi}_j}{\theta_j\lambda_j^r}\right] \tag{13}$$

$$\tilde{\mathbf{E}}(\omega)=\sum_{j=2L+1}^{2N}\left(\frac{\mathrm{i}\omega}{\lambda_j}\right)^h\frac{\boldsymbol{\varphi}_j^T\mathbf{F}(\omega)\boldsymbol{\varphi}_j}{\theta_j\left(\mathrm{i}\omega-\lambda_j\right)} \tag{14}$$

with

$$\begin{cases}\mathbf{E}_1(\omega)=\mathbf{K}^{-1}\mathbf{F}(\omega)\\ \mathbf{E}_2(\omega)=-\mathbf{K}^{-1}\left[\mathbf{C}\mathbf{E}_1(\omega)\right]\\ \mathbf{E}_r(\omega)=-\mathbf{K}^{-1}\left[\mathbf{C}\mathbf{E}_{r-1}(\omega)+\mathbf{M}\mathbf{E}_{r-2}(\omega)\right]\quad\forall r>2\end{cases} \tag{15}$$

Proof. The frequency responses can be exactly expressed as the lower available modes and system matrices in terms of Neumann series expansion (Li et al., 2014c)

$$\mathbf{X}(\omega)=\sum_{j=1}^{2L}\frac{\boldsymbol{\varphi}_j^T\mathbf{F}(\omega)\boldsymbol{\varphi}_j}{\theta_j\left(\mathrm{i}\omega-\lambda_j\right)}+\sum_{r=1}^{\infty}\left(\mathrm{i}\omega\right)^{r-1}\left[\mathbf{E}_r(\omega)+\sum_{j=1}^{2L}\frac{\boldsymbol{\varphi}_j^T\mathbf{F}(\omega)\boldsymbol{\varphi}_j}{\theta_j\lambda_j^r}\right] \tag{16}$$

In practice, a few terms in power-series expressed by the second term on the right-hand of Equation (17) are considered for suitable accuracy requirements. Suppose the first h power-series terms are retained, the frequency response can be expressed as

$$\mathbf{X}(\omega)\approx\tilde{\mathbf{X}}(\omega)=\sum_{j=1}^{2L}\frac{\boldsymbol{\varphi}_j^T\mathbf{F}(\omega)\boldsymbol{\varphi}_j}{\theta_j\left(\mathrm{i}\omega-\lambda_j\right)}+\sum_{r=1}^{h}\left(\mathrm{i}\omega\right)^{r-1}\left[\mathbf{E}_r(\omega)+\sum_{j=1}^{2L}\frac{\boldsymbol{\varphi}_j^T\mathbf{F}(\omega)\boldsymbol{\varphi}_j}{\theta_j\lambda_j^r}\right] \tag{17}$$

The series-truncation error of Equation (18) is introduced and given by

$$\tilde{\mathbf{E}}(\omega)=\mathbf{X}(\omega)-\tilde{\mathbf{X}}(\omega)=\sum_{r=h+1}^{\infty}\left(\mathrm{i}\omega\right)^{r-1}\left[\mathbf{E}_r(\omega)+\sum_{j=1}^{2L}\frac{\boldsymbol{\varphi}_j^T\mathbf{F}(\omega)\boldsymbol{\varphi}_j}{\theta_j\lambda_j^r}\right] \tag{18}$$

Here we introduce an important property between system matrices and complex modes (Li et al., 2014c)

$$\mathbf{E}_r(\omega)+\sum_{j=1}^{2L}\frac{\boldsymbol{\varphi}_j^T\mathbf{F}(\omega)\boldsymbol{\varphi}_j}{\theta_j\lambda_j^r}=-\sum_{j=2L+1}^{2N}\frac{\boldsymbol{\varphi}_j^T\mathbf{F}(\omega)\boldsymbol{\varphi}_j}{\theta_j\lambda_j^r} \tag{19}$$

Substituting Equation (20) into Equation (19) yields

$$\tilde{\mathbf{E}}(\omega)=-\sum_{j=2L+1}^{2N}\sum_{r=h+1}^{\infty}\left(\frac{\mathrm{i}\omega}{\lambda_j}\right)^r\frac{\boldsymbol{\varphi}_j^T\mathbf{F}(\omega)\boldsymbol{\varphi}_j}{\mathrm{i}\omega\theta_j} \tag{20}$$

If the convergence condition (12) is satisfied, one obtains

$$|\omega| < |\lambda_j| \quad \forall j = 2L+1,\ 2L+2,\ ...,\ 2N \tag{21}$$

Then the series-truncation error can be given by Equation (15) by using the theory of the geometric sequence, and the theorem is proved.

Remark 1. The method is convergent if and only if all the complex modes whose eigenvalues are less than the maximal sampling frequency of the FFT are available. Equation (14) can be used in the dynamic analysis of practical problems since the number *h* of correct terms is usually very small. Since the power-series expansion is truncated, the series-truncation error given by Equation (15) is introduced. When the convergence condition is satisfied, the errors can be decreased with the number *h* is increased. By comparing Equations (11) and (15), it is clearly shown that the frequency response obtained by Equation (14) can improve the accuracy of the response calculated by MDM if the sample frequency is at 0-$|\lambda_{L+1}|$ rad/s.

Remark 2. One of the most robust approaches to obtain these vectors $\mathbf{E}_r(\omega)$ (known as the Krylov vectors) is to firstly compute a matrix factorization (e.g., $\mathbf{LDL}^T$ factorization) of the sparse stiffness matrix **K**. Note it only need to be obtained once for different sample frequencies. Then vectors $\mathbf{E}_r(\omega)$ can be determined by an iteration process in terms of forward and backward substitutions. In general, the computational cost of forward and backward substitutions is much smaller than the matrix factorization.

Criteria. The number *h* for any simple frequency ω can be determined if the following inequality is satisfied.

$$\|\mathbf{S}(h)\| + \|\mathbf{S}(h-1)\| < \varepsilon \quad \forall h > 1 \tag{22}$$

in which

$$\mathbf{S}(h) = (\mathrm{i}\omega)^{h-1}\left[\mathbf{E}_h(\omega) + \sum_{j=1}^{2L} \frac{\boldsymbol{\varphi}_j^T \mathbf{F}(\omega) \boldsymbol{\varphi}_j}{\theta_j \lambda_j^h}\right] \tag{23}$$

Here the parameter ε is the given accuracy tolerance.

Numerical Example

To illustrate this new method, we consider a simple there DOF damped system with the mass, damping and stiffness matrices given by

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 10^2 & 0 \\ 0 & 0 & 10^4 \end{pmatrix},\ \mathbf{C} = 10^3 \times \begin{pmatrix} 5 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix},\ \mathbf{K} = 10^8 \times \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 9 \end{pmatrix}$$

The complex eigenvalues are obtained as: −0.2475±222.46i, −5.0526±1421.4i and −2499.8±13920i. The excitation point is located at the first DOF and the force is shown in Figure 1. The time step is chosen as Δt=0.005 seconds, which means that the maximum simple frequency is 400π rad/s. In view of the convergence condition (12), the first pair of complex modes must be included to calculate the transient responses. The MDM is also considered to calculate the responses by using the first pair of complex modes. Figure 2 shows the responses at the first DOF (here N_{FT}=512). In this case, the responses calculated by the MDM is misleading and meaningless. The proposed method can show a good agreement with the exact responses by considering a few numbers of the correction terms.

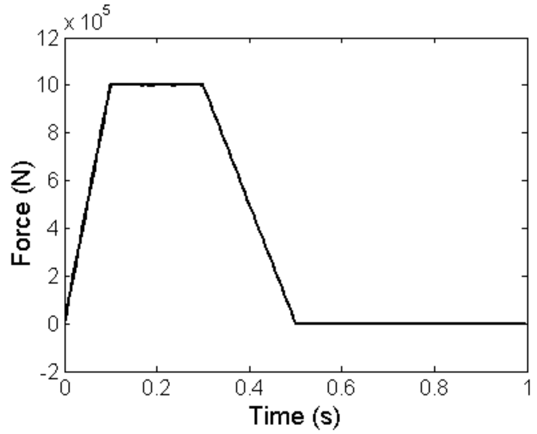

Figure 1. Applied force.

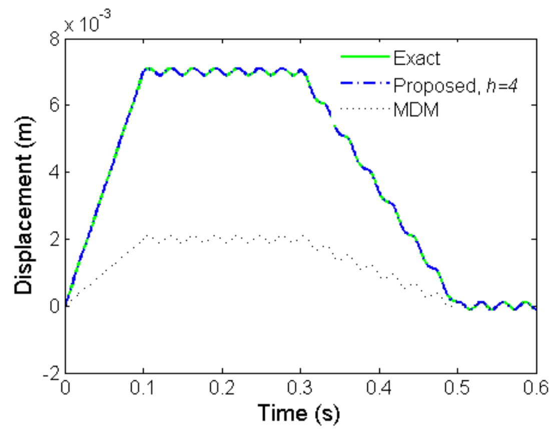

Figure 2. Transient response.

Conclusions

This paper consider the transient response analysis of non-classically damped systems. When the mode superposition method is used to calculate transient response, the modal truncation error is generally introduced since it is difficult, or even unnecessary, to obtain all the modes of a large-scaled model. An accurate modal superposition method is presented to calculate the transient response of non-classically damped systems based on the Neumann series and the FFT technique. The method maintains original-space without having to involve the state-space formula. The method can converge to exact results if and only if all the complex modes whose resonant frequencies are less than the maximal sampling frequency of the FFT must be available. The applicability of the method is investigated using a simple numerical example with non-classical damping. It is shown that, the responses calculated by the MDM is misleading and meaningless and the proposed method can show a good agreement with the exact responses.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (51375184).

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