# **Generalized Integral Transform Solution of Extended Graetz Problems with**

# **Axial Diffusion**

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## Abstract

A methodology for obtaining fully analytical solutions for the extended Graetz-Brinkman problem including the effects of axial conduction in infinite and semi-infinite domains has been proposed. The infinite domain formulation consists of a preparation (unheated) region (x < 0) followed by a heated one (x > 0) such that back diffusion effects can have an influence on the temperature fields. The solution methodology is based on the Generalized Integral Transform Technique, in which eigenfunction expansions in terms of orthogonal bases are employed. A simple eigenfunction basis in terms of Helmholtz problems are used to maintain the calculation of integral coefficients and the solution of the involved eigenproblems themselves analytical and simple. With the exception of matrix eigenvalues calculation the solution process is mainly analytical.

Keywords: Generalized integral transform, Laminar flow, Forced convection, Duct flow.

# Introduction

Graetz problems have been studied for a long time, since early works that include those of Graetz himself [Graetz, 1882]. Initially, simpler versions of the problem, having negligible axial diffusion, simple wall heating conditions (isothermal and isoflux), simple geometry cross-section (either parallel plates or circular channels), and no fluid flow heating effects were investigated, which can be generally denoted the Classical Graetz Problem. This problem is basically of parabolic nature, such that its solutions is given by from a known inlet condition, and extents infinitely in the direction of the flow. Over the course of the decades that followed, a number of extensions to the problem were presented. Since the Graetz problem itself, regardless of the proposed extension, is generally linear, many analytical solutions were proposed, but numerical solution schemes are also found in the literature for the more complicated extensions of the problem.

A common extension to the Graetz is the introduction of axial diffusion into the problem, which complicates matters notably due to the back-diffusion effects that is increased for smaller values of the Péclet number. For this situation, the solution domain must be altered as a preparation unheated (uncooled) region must be considered prior to the heated (or cooled) section. This is usually carried-out by considering an infinite domain that extends from  $-\infty$  to  $\infty$  in the flow direction, with a step change in the wall boundary condition at the center of the domain.

Hsu [1968] studied a Graetz problem with axial diffusion in a circular tube, using a semi- infinite domain formulation with a specified inlet condition, while Michelsen and Villadsen [1974] analyzed the effects of axial diffusion in a infinite domain formed by a insulated preparation region followed by an isothermal wall. Both studies used a numerical scheme to complete the solution of

the problem.

Vick et al. [1980] considered a similar Graetz problem with an insulated preparation region followed by a finite isoflux region, and presents an approximate (lowest order analysis) analytical solution in terms of eigenfunctions that arise from the solution of case without axial diffusion. A similar approximate solution procedure was carried out by Bayazitoglu and Ozisik[1980], which solved the axial diffusion Graetz problem with convective boundary conditions at the duct walls and internal energy sources (i.e. flow heating effects). Further on, Vick and Ozisik [1981] and Vick et al. [1983] analyzed similar problems (isoflux walls and convective wall, respectively), but proposed a solution in terms of an alternative eigenfunction basis arising from a problem that does not belong to the traditional Sturm-Liouville class.

Ku and Hatziavramidis [1984] also solved the extend problem subjected to a step-change in wall temperature such that the infinite domain is divided in two portions having an isothermal wall conditions, but with different temperatures. A numerical solution by means of expansions using Chebyshev polynomials was carried out.

Laohakul et al. [1985] and Najjar and Laohakul [1986] considered the infinite domain formulation with two different wall boundary condition arrangements: a heated isothermal wall region surrounded by two cooled isothermal walls, and a heated isoflux wall region surrounded by two insulated regions, respectively. Both studies presented approximate analytical solutions for both large and small Péclet number values in terms using integral transforms with orthogonal eigenfunction expansions.

Ebadian and Zhang [1989, 1990] presented an alternate analytical solution to the extended Graetz problem, writing the temperature field in the infinite domain in terms of a Fourier integral. Although the authors presented a closed-form solution for the temperature field, a Runge-Kutta method was required for performing the required integration to obtain the temperature distribution and Nusselt values.

Johnston [1991] investigated the Graetz problem with axial diffusion in a semi-infinite domain with a prescribed inlet condition, and also used an integral transform solution technique with an orthogonal eigenfunction basis arising from the simpler version of the problem without axial diffusion. However, the author was able to calculate the complete solution of the system (beyond the lower order approximation) in an analytical fashion, by rewriting the transformed ODE system in a first-order differential form.

Min et al. [1997] presented a solution for a Graetz problem with axial diffusion and flow heat- ing effects in a semi-infinite domain with a given inlet condition, for the velocity profile obtained for a Bingham plastic. The solution is expressed in terms of an eigenseries expansion using the same non-Sturm-Liouville basis employed in previous investigations. Olek[1998] also presented a solution for similar extended Graetz problem with a non-Newtonian velocity profile; nevertheless, a solution method similar to that of [Johnston, 1991] was employed.

Çetin et al. [2008] considered a Graetz-type problem for a slip-flow regime with viscous dissipation heating and included the effects of axial diffusion. The authors considered a semi-infinite domain with a prescribed inlet condition, in which a coordinate transformation was employed for arriving at a finite domain. The resulting problem was numerically solved via finite differences. The same authors Çetin et al. [2009] proposed a solution to a similar problem, using a different wall heating condition (step-change in wall temperature), and considering an eigenfunction expansion approach

in terms of a non-Sturm-Liouville eigenfunction basis. Sharma and Chakraborty [2008] also adopted a similar solution technique for tackling an extended Graetz problem with a velocity profile arising from combined pressure and electroosmotic flow driving mechanisms.

In this study, an extended version of a Graetz problem with axial diffusion in an infinite domain is considered. Two heating conditions are analyzed, these being a step change in wall temperature, and a step change in wall heat flux (the lower value corresponding to an insulated condition). The adopted solution strategy is based on orthogonal Sturm-Liouville eigenfunctions expansions of the sought solutions, following the formalism of the nowadays called Generalized Integral Transform Technique [Cotta, 1993]. Closed-form analytical solutions are obtained, and the solution process involves a single numerical methods step, which involves the calculation of numerical matrix eigenvalues and eigenvectors.

## **Problem Formulation**

The studied convective heat transfer problem is an extension of the Graetz problem, including the effects of axial diffusion, for an infinite circular channel with a sudden change in boundary condition at the wall at the origin of the axial coordinate. The problem formulation is given by two sets of equations, valid for different regions. For the upstream region ( $\xi \leq 0$ ) the governing equation and boundary conditions in the transversal direction are given by:

$$u^* \frac{\partial \Phi}{\partial \xi} = \operatorname{Pe}^{-2} + \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{1}{\eta} \frac{\partial}{\partial \eta} \left( \eta \frac{\partial \Phi}{\partial \eta} \right), \quad \text{for } \xi \le 0,$$
(1a)

$$\beta \left(\frac{\partial \Phi}{\partial \eta}\right)_{n=1} + (1-\beta) \Phi(\xi, 1) = 0, \quad \text{for } \xi \le 0$$
<sup>(1b)</sup>

$$\left(\frac{\partial \Phi}{\partial \eta}\right)_{\eta=0} = 0, \quad \text{for } \xi \le 0, \tag{1c}$$

whereas for the downstream region ( $\xi \ge 0$ ), the equations are given by:

$$u^* \frac{\partial \Theta}{\partial \xi} = \operatorname{Pe}^{-2} + \frac{\partial^2 \Theta}{\partial \xi^2} + \frac{1}{\eta} \frac{\partial}{\partial \eta} \left( \eta \frac{\partial \Theta}{\partial \eta} \right), \quad \text{for } \xi \ge 0,$$
(2a)

$$\beta \left(\frac{\partial \Theta}{\partial \eta}\right)_{\eta=1} + (1-\beta) \Theta(\xi, 1) = 0, \quad \text{for } \xi \ge 0$$
(2b)

$$\left(\frac{\partial \Theta}{\partial \eta}\right)_{\eta=0} = 0, \quad \text{for } \xi \ge 0, \tag{2c}$$

where  $\Theta$  and  $\Phi$  represent the dimensionless temperatures in the two different regions. The  $\beta$  coefficient will lead to different wall heating conditions:  $\beta = 0$  for isothermal wall and  $\beta = 1$  for constant heat flux.

Part of the required boundary conditions in the axial direction are the inlet and outlet conditions at infinity, defined as:

$$\left| \frac{\partial \Phi}{\partial \xi} \right|_{\xi \to \infty} < \infty, \qquad \left| \frac{\partial \Theta}{\partial \xi} \right|_{\xi \to \infty} < \infty, \qquad (3a)$$

which reflect the fact that the temperature gradient cannot increase indefinitely for a large value of  $|\xi|$ . This form for the boundary is chosen since both prescribed wall temperature and wall flux conditions are used. If only a prescribed wall temperature was used, the actual temperature at  $|\xi| \rightarrow \infty$  become bounded. Nonetheless, for the employed solution strategy the currently adopted form suffices. The remaining boundary conditions in this direction are coupling conditions between the two regions, at the axial coordinate origin ( $\xi = 0$ ):

$$\lim_{\xi \to 0^-} \Phi(\xi, \eta) = \lim_{\xi \to 0^+} \Theta(\xi, \eta)$$
(3b)

$$\lim_{\xi \to 0^-} \frac{\partial \Phi}{\partial \xi} = \lim_{\xi \to 0^+} \frac{\partial \Theta}{\partial \xi}$$
(3b)

The employed dimensionless variables are defined by the following equations:

$$\Phi(\xi,\eta) = \frac{T(x,y) - T_0}{\Delta T} \quad \text{for } x \le 0, \qquad \Theta(\xi,\eta) = \frac{T(x,y) - T_0}{\Delta T} \quad \text{for } x \ge 0 \tag{4a}$$

$$\eta = \frac{y}{D/2}, \quad \xi = \frac{x}{D/2 \operatorname{Pe}} \quad \text{with} \quad \operatorname{Pe} = \frac{\overline{u} D/2}{\alpha}, \tag{4b}$$

where *D* is the channel diameter and  $T_0$  is a reference temperature, and the temperature difference  $\Delta T$  can have two different definitions according to the wall heating condition in the downstream region:

$$\Delta T = T_w - T_0 \quad \text{for} \quad \beta = 0, \qquad \Delta T = \frac{D}{2} \frac{\dot{q}_w''}{k} \quad \text{for} \quad \beta = 1 \tag{5}$$

where  $T_{w}$  and  $\dot{q}''_{w}$  represent the wall temperature and wall heat flux in the downstream region.

The presented governing equations allow for a variety of dynamically-developed velocity profiles. However, for illustration purposes, a Hagen-Poiseuille profile is used:

$$\boldsymbol{u}^* = 2 \left( 1 - \boldsymbol{\eta}^2 \right) \tag{6}$$

In spite of this choice, as it will be seen, the solution methodology is practically independent of the choice of  $u^*$ , such that any other expressions for  $u^*$  as a function of  $\eta$  can be used.

Finally, the Nusselt number in the downstream region can be calculated from In the upstream region, special cases exist:

$$Nu = \frac{h D}{k} = \frac{2}{\Theta_w - \Theta_m}, \quad \text{for} \quad \beta = 1,$$
(7a)

$$\operatorname{Nu} = \frac{h D}{k} = \frac{2(\partial \Theta / \partial \eta)_{\eta=1}}{1 - \Theta_m}, \quad \text{for} \quad \beta = 0,$$
<sup>(7b)</sup>

where  $\Theta_{w}$  is the dimensionless wall temperature, and the dimensionless bulk temperature  $\Theta_{m}$  is calculated form:

$$\Theta_m = 2 \int_0^1 u^* \Theta \ \eta \ \mathrm{d}\eta, \tag{7c}$$

### **Proposed Solution Scheme**

The adopted solution methodology is based on the Generalized Integral Transform Technique (GITT). As usual among this type of methodology, filter problems are proposed for removing non-homogenities from the original system. Since the upstream portion of the problem is homogeneous, only the downstream formulation needs to be filtered, which is carried out based on the following solution separation:

$$\Theta(\xi,\eta) = \theta(\xi,\eta) + F(\xi,\eta) \tag{8}$$

in which  $\theta$  represents the filtered variables, whereas F represent the filter function, which is commonly obtained from simpler versions of the original problem.

For the isothermal walls condition ( $\beta = 0$ ), the selected downstream filter function is simply a constant:

$$F(\xi,\eta) = 1, \tag{9a}$$

 $\langle \mathbf{n} \rangle$ 

whereas for the isoflux wall condition ( $\beta = 1$ ) a polynomial filter is used:

$$F(\xi,\eta) = 2\xi - \frac{7}{24} + \eta^2 - \frac{\eta^4}{4}$$
(9b)

whereas, both filters correspond to the solution of the problem as  $\xi \to \infty$ .

If the previously presented filters are employed, the resulting filtered problem for the downstream region is given by:

$$\boldsymbol{u}^{*} \frac{\partial \theta}{\partial \xi} = \operatorname{Pe}^{-2} + \frac{\partial^{2} \theta}{\partial \xi^{2}} + \frac{1}{\eta} \frac{\partial}{\partial \eta} \left( \eta \frac{\partial \theta}{\partial \eta} \right), \quad \text{for } \xi \ge 0,$$
(10a)

$$\beta \left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=1} + \left(1 - \beta\right) \theta \left(\xi, 1\right) = 0, \quad \text{for } \xi \ge 0, \tag{10b}$$

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} = 0, \quad \text{for } \xi \ge 0, \tag{10c}$$

and the boundary and coupling conditions in the axial direction are given by:

$$\left. \frac{\partial \theta}{\partial \xi} \right|_{\xi \to \infty} < \infty, \tag{11a}$$

$$\Phi(0,\eta) = \theta(0,\eta) + F(0,\eta)$$
(11b)

$$\frac{\partial \Phi}{\partial \xi}(0,\eta) = \frac{\partial \theta}{\partial \xi}(0,\eta) + \frac{\partial F}{\partial \xi}(0,\eta)$$
(11c)

#### Integral Transformation

The solution of the considered problem is accomplished employing the Generalized Integral Transform Technique []. The solution process is started by defining the transformation pairs:

$$Transform \Rightarrow \overline{\phi}_n = \int_{-1}^{1} \eta \, \Phi(\xi, \eta) \Lambda_n(\eta) \, d\eta$$

$$Inversion \Rightarrow \Phi(\xi, \eta) = \sum_{n=1}^{\infty} \frac{\phi_n(\xi) \Lambda_n(\eta)}{\phi_n(\xi)}$$
(12a)

$$T_{n=0} \qquad N_n \qquad (12b)$$

$$Transform \Rightarrow \theta_n = \int_0^{\infty} \eta \ \theta(\xi, \eta) \Lambda_n(\eta) \ d\eta,$$
(12c)

Inversion 
$$\Rightarrow \theta(\xi, \eta) = \sum_{n=0}^{\infty} \frac{\theta_n(\xi) \Lambda_n(\eta)}{N_n},$$
 (12d)

where  $\Lambda_n$  s are orthogonal solution of a Sturm-Liouville problem. For the current application, a one-dimensional Helmholtz problem is selected:

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}(\eta \Lambda_n'(\eta)) + \lambda_n^2 \Lambda_n(\eta) = 0, \text{ for } 0 \le \eta \le 1,$$
(13a)

$$\Lambda'_{n}(0) = 0, \ \beta \ \Lambda'_{n}(1) + (1 - \beta)\Lambda_{n}(1) = 0, \tag{13b}$$

which leads to infinite nontrivial solutions in the form:

$$\Lambda_n(\eta) = J_0(\lambda_n \eta), \tag{14a}$$

for =1,2,...,∞. For the constant heat flux condition ( $\beta$  = 1) problem (13) admits a non-trivial constant solution for  $\lambda_0$  = 0. As a result, one must also include definitions for this situation:

$$\Lambda_{_{0}}(\eta) = \beta, \tag{14b}$$

(1 41 )

such that for isothermal wall conditions ( $\beta = 0$ ) trivial solutions are obtained.

The eigenvalues are calculated numerically, being obtained from the roots of the following equations:

$$J_{\rho}(\lambda_{r}) = 0, \text{ for } \beta = 0, \tag{14c}$$

$$J_1(\lambda_n) = 0, \text{ for } \beta = 1, \tag{14d}$$

Finally, the norms are obtained from:

$$N_n = \int_0^1 \eta \Lambda_n^2(\eta) \,\mathrm{d}\eta, \text{ for } \mathbf{n} = 0, 1, 2, \underline{\prec},$$
(15)

noting that for isothermal walls there is no need for calculating  $N_0$  as the trivial solution is obtained for n = 0.

The transformation of the given problem is accomplished by multiplying equations (1a) and (10a) by  $\Lambda_n \eta$ , integrating within  $0 \le \eta \le 1$ , and applying the inversion formulas (12b) and (12d) to the non-transformable terms. This process yields the following coupled system of ODEs:

$$\frac{\mathrm{d}^{2}\phi_{n}}{\mathrm{d}\xi^{2}} - \sum_{m=0}^{\infty} A_{n,m} \frac{\mathrm{d}\phi_{m}}{\mathrm{d}\xi} - \mathrm{Pe}^{2}\lambda_{n}^{2}\overline{\phi}_{n}(\xi) = 0, \text{ for } \xi \leq 0,$$
(16a)

$$\frac{\mathrm{d}^{2}\overline{\theta}_{n}}{\mathrm{d}\xi^{2}} - \sum_{m=0}^{\infty} A_{n,m} \frac{\mathrm{d}\overline{\theta}_{m}}{\mathrm{d}\xi} - \mathrm{Pe}^{2}\lambda_{n}^{2}\overline{\theta}_{n}(\xi) = 0, \text{ for } \xi \ge 0,$$
(16b)

$$\left| \frac{\mathrm{d}\bar{\phi}_n}{\mathrm{d}\xi} \right|_{\xi \to \infty} < \infty, \qquad \left| \frac{\mathrm{d}\bar{\theta}_n}{\mathrm{d}\xi} \right|_{\xi \to \infty} < \infty, \qquad (16c)$$

$$\overline{\phi}_n(0) = \overline{\theta}_n(0) + b_n, \tag{16d}$$

$$\frac{\mathrm{d}\phi_n}{\mathrm{d}\xi}(0) = \frac{\mathrm{d}\theta_n}{\mathrm{d}\xi}(0) + d_n,\tag{16e}$$

for  $n = 0, 1, 2, ..., \infty$ . However, one should note that for  $\beta = 0$  there is no need to calculate  $\overline{\phi_0}$  and the summations in equations (16a) and (16b) should start from m = 1, such that there is no need for an equation for n = 0.

The coefficients  $A_{n,m}, b_n$ , and  $d_n$  are given by:

$$A_{n,m} = \frac{\operatorname{Pe}^2}{N_{ml}} \int_{0}^{1} \eta \, u^*(\eta) \Lambda_m(\eta) \Lambda_n(\eta) \, \mathrm{d}\eta, \tag{17a}$$

$$b_n = \int_{\mathbf{d}}^{\mathbf{m}} \eta \ F(0,\eta) \ \Lambda_n(\eta) \ \mathrm{d}\eta, \tag{17b}$$

$$d_n = \int_0^0 \eta \left(\frac{\partial F}{\partial \xi}\right)_{\xi=0} \Lambda_n(\eta) \, \mathrm{d}\eta, \qquad (17c)$$

In order to solve system (16), the infinite system representation must be truncated to a finite number of terms, which is denoted the truncation order. Once truncated, the resulting system can be solved numerically using a commercially or publicly available ODE system solver. Nevertheless, an analytical alternative to the ODE integration can be achieved if the truncated systems are reduced to first order forms by writing it in terms of new unknown vectors x and y:

$$x(\xi) = (\overline{\phi}_0, \overline{\phi}_1, \overline{\phi}_2, \underline{\neg}, \overline{\phi}_{\eta_{\max}}, \overline{\phi}_0', \overline{\phi}_1', \overline{\phi}_2', \underline{\neg}, \overline{\phi}_{\eta_{\max}}', \overline{\phi}_1', \overline{\phi}_2', \underline{\neg}, \overline{\phi}_{\eta_{\max}}', (\xi)),$$
(18a)

$$y(\xi) = (\overline{\theta}_0, \overline{\theta}_1, \overline{\theta}_2, \underline{\neg}, \overline{\theta}_{\eta_{\max}}, \overline{\theta}_0', \overline{\theta}_1', \overline{\theta}_2', \underline{\neg}, \overline{\theta}_{\eta_{\max}}'(\xi)),$$
(18b)

Where a condition of a prescribed wall flux in both upstream and downstream regions has been considered. For isothermal walls x needs not include  $\overline{\phi}_0$  nor  $\overline{\phi}_0'$ , and y needs not include  $\overline{\theta}_0$  nor  $\overline{\theta}_0'$ .

By employing the new unknown vectors, equations (16a) and (16b) are rewritten in the following forms:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\boldsymbol{\xi}} = \boldsymbol{M} \ \boldsymbol{x}, \quad \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\boldsymbol{\xi}} = \boldsymbol{M} \ \boldsymbol{y} \tag{19}$$

Where *M* is a block matrix defined as:

$$\boldsymbol{M} = \left(\frac{\boldsymbol{\theta}}{\boldsymbol{D}} \middle| \frac{\boldsymbol{I}}{\boldsymbol{A}}\right),\tag{20}$$

in which the sub-matrix **A** is given by the coefficients  $A_{n,m}$  and **D** is given by:

$$D_{nn} = \operatorname{Pe}^2 \lambda_n^2. \tag{21}$$

Once the transformed ODEs have been written in the modified form given by equations (19), a solution can be obtained using analytical integration. If the eigenvalues and eigenvectors of M are calculated, the solution for the components of x and y can be written in the following form:

$$x_{k}(\xi) = \sum_{l=1}^{k_{\text{max}}} Q_{kl} c_{l}^{-} \exp(\mu_{l} \xi), \quad \text{for} \quad l = 1, 2, \underline{\rightarrow}, k_{\text{maxx}}, \quad (22a)$$

$$y_{k}(\xi) = \sum_{l=1}^{k_{\text{max}}} Q_{k,l} c_{l}^{+} \exp(\mu_{l} \xi), \quad \text{for} \quad l = 1, 2, , , k_{\text{maxx}}, \quad (22b)$$

in which  $k_{max} = 2 n_{max}$  for constant wall temperature and  $k_{max} = (2 n_{max} + 1)$  for constant wall flux. The coefficients  $Q_{k,l}$  originate from matrices Q that contain the eigenvectors of M as columns. The coefficients  $\mu_l$  are the eigenvalues of M, while  $c_l^-$  and  $c_l^+$  are arbitrary constants, to be determined from boundary conditions. The matrix M yields about  $n_{max}$  positive eigenvalues and  $n_{max}$  negative eigenvalues, such that roughly half of the arbitrary coefficients can be directly eliminated to satisfy the boundary conditions at  $\xi \to \infty$  and  $\xi \to -\infty$ :

$$c_k^+ = 0$$
 if  $\mu_k > 0$ , for  $k = 1, 2 \xrightarrow{} k_{\text{max}}$  (23a)

$$c_k^- = 0$$
 if  $\mu_k < 0$ , for  $k = 1, 2 \xrightarrow{\sim} k_{\text{max}}$  (23b)

The remaining  $c_k^-$  and  $c_k^+$  values are calculated directly from solving the linear algebraic system that stems from the coupling conditions at  $\xi = 0$ . These equations can be directly combined into a single vector equation:

$$\boldsymbol{Q}\,\boldsymbol{c}^{-} = \boldsymbol{Q}\,\boldsymbol{c}^{+} + \boldsymbol{h},\tag{24}$$

where the vectors  $c^-$  and  $c^-$  are given by the coefficients  $c_n^-$  and  $c_n^+$ , the vector **h** is defined as:

$$\boldsymbol{h} = \left(\boldsymbol{b}_0, \boldsymbol{b}_1, \boldsymbol{b}_2, \overleftarrow{\rightarrow}, \boldsymbol{b}_{\max}, \boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_2, \overleftarrow{\rightarrow}, \boldsymbol{d}_{\max}\right) \tag{25}$$

(05)

After using equations (23), equation (24) is directly solved to yield the remaining unknown coefficients, and the solution of the transformed potentials in both upstream and downstream regions is complete. Once  $\overline{\phi}$  and  $\overline{\theta}$  are determined, the temperature profiles are obtained directly from the inversion formulas (12b) and (12d) combined with the separation formula defined by equation (8).

### **Results and Discussion**

After describing the problem formulation and solution methodology, numerical results are presented. The first set of results are dedicated to examining the solution convergence for the two-different boundary condition cases. Table 1 presents the calculated values of the Nusselt number for different axial positions (all within the downstream region) and different values of the Péclet number for the isothermal walls configuration. As can be seen, the convergence behavior follows a pattern in which notably better convergence rates are obtained as one moves downwards with the flow. For positions in this region, 40 to 60 terms (depending on the value of Péclet) are sufficient for ensuring a Nusselt values with six converged digits. For positions near the boundary condition discontinuity ( $\xi = 0$ ) a worse convergence rate is seen, and the convergence becomes highly dependent on the Pe value. For Pe = 1 and  $\xi = 10-3$  not even a single converged digit is obtained with 100 terms; however, for Pe = 100, 100 terms in the series leads to four converged digits.

Also on this table, the converged solution of the case with no axial diffusion (labeled as  $Pe = \infty$ ) is presented for verification purposes. As one can observe from these values, it is clear that as Pe is increased, the Nu values gradually approach the solution with no axial diffusion.

The next table (Tab. 2) presents similar Nusselt number convergence results for the isoflux walls situation. As one can observe from the presented results, again, better convergence rates are seen in regions further downstream; however, when compared to the isothermal walls case, much better convergence rates are obtained. In fact, in several occasions, as much as five terms in the series yield six converged Nu-digits. When looking into the convergence dependence on the Péclet number, one notices that the convergence in positions near the channel entrance is notably less dependent on Pe than for the case with isothermal walls. Finally, when comparing the Nusselt number values with axial diffusion with the case with no axial diffusion, one notices that with increasing Pe number, the converged Nusselt values approach the Pe =  $\infty$  case, as expected.

The last set of results examine the behavior of the Nusselt number for different values of the Péclet number. Fig. 1 shows the distribution of Nu in the thermal developing region for different Péclet values. As can be seen, the Péclet number has a much more pronounced effect on the Nusselt values for positions near the channel entrance. In fact, although hard to see in the presented scale, different Péclet values also yield different Nu values for the isothermal walls conditions (as expected) in the thermal developed region ( $\xi > 1$ ), which can be seen from the data in table 1.



Subsequently, figure 2 depicts the Nu distribution in the thermally developing region for different Péclet values, for the case with an insulated downstream and a uniform heated upstream. As one can infer from the presented results, a somewhat different behavior is seen when compared to the previous case. Firstly, the Nu values are independent of the Péclet number in the developed region (as expected, and seen in table 2). Secondly, although the dependence on the Péclet number becomes very notable as one moves further into the thermal entrance region, the Nu values have a general tendency of decreasing with increasing Péclet for this case, rather than increasing (which was observed for the isothermal walls case).



Table 1           Isothermal walls										
										n <sub>max</sub>
Pe=2										
10	28.8358	22.875	7.07583	3.8087	3.80203					
20	54.1035	34.0065	7.07748	3.80826	3.8016					
40	100.116	43.107	7.07682	3.8082	3.80154					
60	140.698	45.6955	7.07676	3.8082	3.80153					
80	176.489	46.4317	7.07674	3.8082	3.80153					
100	208.054	46.641	7.07674	3.8082	3.80153					
conv.	443.795	46.7239	7.07673	3.8082	3.80153					
Pe=5										
10	24.7794	15.09	4.61058	3.69582	3.69581					
20	43.1881	17.4615	4.60843	3.69527	3.69526					
40	72.3622	18.0455	4.60815	3.6952	3.69519					
60	93.6745	18.0689	4.60813	3.6952	3.69518					
80	109.242	18.0694	4.60812	3.69519	3.69518					
100	120.613	18.0693	4.60812	3.69519	3.69518					
conv.	151.419	18.0691	4.60811	3.69519	3.69518					
Pe=10										
10	23.1915	10.8647	4.16302	3.66821	3.66821					
20	36.6831	11.025	4.16151	3.66763	3.66763					
40	53.7184	11.0232	4.16131	3.66756	3.66755					
60	62.8062	11.0223	4.16129	3.66755	3.66755					
80	67.6537	11.0221	4.16129	3.66755	3.66755					
100	70.2391	11.022	4.16128	3.66754	3.66754					
conv.	73.1923	11.0219	4.16128	3.66754	3.66754					
Pe=50										
10	20.7269	7.67478	4.01245	3.65792	3.65792					
20	21.8546	7.64915	4.01115	3.65733	3.65733					
40	22.0204	7.64572	4.01098	3.65725	3.65725					
60	22.0114	7.64537	4.01096	3.65724	3.65724					
80	22.0071	7.64529	4.01096	3.65724	3.65724					
100	22.0055	7.64525	4.01096	3.65724	3.65724					
conv.	22.0039	7.64522	4.01096	3.65724	3.65724					
Pe=100										
10	19.8799	7.54188	4.0077	3.65758	3.65758					
20	18.1528	7.51788	4.0064	3.65699	3.65699					
40	17.9536	7.51471	4.00623	3.65692	3.65692					
60	17.9428	7.51438	4.00622	3.65691	3.65691					
80	17.9404	7.5143	4.00621	3.65691	3.65691					
100	17.9395	7.51427	4.00621	3.65691	3.65691					
conv.	17.9386	7.51424	4.00621	3.65691	3.65691					
Pe= oo										
conv.	16.264	7.47038	4.00463	3.65679	3.65679					

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Isoflux walls										
n <sub>max</sub>	ξ=0.001	ξ=0.01	ξ=0.1	ξ=1	ξ=10					
Pe=2										
10	7.9516	7.44095	5.50366	4.36881	4.36364					
20	8.10429	7.5016	5.50319	4.36881	4.36364					
40	8.17579	7.51492	5.50314	4.36881	4.36364					
60	8.19683	7.5161	5.50313	4.36881	4.36364					
80	8.20611	7.51626	5.50313	4.36881	4.36364					
100	8.21101	7.51628	5.50313	4.36881	4.36364					
conv.	8.22006	7.51629	5.50313	4.36881	4.36364					
Pe=5										
5	9.40303	8.15929	5.1761	4.36377	4.36364					
10	9.77981	8.23411	5.17148	4.36377	4.36364					
20	9.97406	8.24913	5.17099	4.36377	4.36364					
40	10.0564	8.24983	5.17093	4.36377	4.36364					
60	10.0764	8.24977	5.17092	4.36377	4.36364					
80	10.0837	8.24975	5.17092	4.36377	4.36364					
100	10.0869	8.24975	5.17092	4.36377	4.36364					
conv.	10.0903	8.24974	5.17092	4.36377	4.36364					
		Pe	=10							
5	11.819	8.9259	5.04433	4.36365	4.36364					
10	12.2099	8.86054	5.04022	4.36365	4.36364					
20	12.4298	8.85146	5.03973	4.36365	4.36364					
40	12.5093	8.85019	5.03968	4.36365	4.36364					
60	12.5235	8.85007	5.03967	4.36365	4.36364					
80	12.5272	8.85004	5.03967	4.36365	4.36364					
100	12.5284	8.85003	5.03967	4.36365	4.36364					
conv.	12.529	8.85002	5.03967	4.36365	4.36364					
		Pe	=50							
5	20.9235	9.53499	4.97964	4.36364	4.36364					
10	18.9231	9.30816	4.97572	4.36364	4.36364					
20	18.6708	9.29342	4.97524	4.36364	4.36364					
40	18.6412	9.2916	4.97519	4.36364	4.36364					
60	18.6382	9.29142	4.97518	4.36364	4.36364					
80	18.6375	9.29138	4.97518	4.36364	4.36364					
100	18.6372	9.29136	4.97518	4.36364	4.36364					
conv.	19.6625	9.29551	4.97283	4.36364	4.36364					
Pe=100										
5	24.5256	9.53521	4.97728	4.36364	4.36364					
10	20.2961	9.31245	4.97337	4.36364	4.36364					
20	19.7092	9.2976	4.9729	4.36364	4.36364					
40	19.6679	9.29577	4.97284	4.36364	4.36364					
60	19.6641	9.29559	4.97284	4.36364	4.36364					
80	19.6631	9.29554	4.97283	4.36364	4.36364					
100	19.6628	9.29553	4.97283	4.36364	4.36364					
conv.	19.6625	9.29551	4.97283	4.36364	4.36364					
Pe=∞										
conv.	19.9868	9.29502	4.97205	4.36364	4.36364					

#### Conclusions

This paper presented an analytical solution to the extended Graetz problem with axial heat diffusion in an infinite domain configuration with a step change in wall boundary condition at the axial coordinate origin. The channel geometry comprises circular channels (tubes or pipes), and both isothermal walls and insulated/isoflux walls were analyzed as boundary conditions. The solution methodology was based on the Generalized Integral Transform Technique using simple eigenfunctions as a basis for the sough solution. The eigenfunctions are obtained from a Helmholtz problem, and all integral coefficients can be calculated analytically. In fact, the entire solution procedure is analytical, except for the numerical calculation of matrix eigenvalues and eigenvectors. The solution method was verified to lead to converged values which are in accordance with physically expected results. The results also match the values without axial diffusion as the Péclet number is increased. After demonstrating the convergence of the solution, the Nusselt number distribution for different Péclet values was analyzed, and the results are also in accordance with expected literature values. As final comments one should mention that the same solution procedure can be used for any dynamically developed velocity profile, as it occurs in many other occasions. Also, the methodology can be easily extended to other configurations such as other channel geometries, different wall heating conditions, and vicious and other flow heating effects.

#### Acknowledgements

The authors would like to acknowledge the financial support provided by the Brazilian Government research funding agencies CNPq, CAPES and FAPERJ.

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