ICCM2014
28-30th July, Cambridge,
England

DISCONTINUOUS GALERKIN FINITE VOLUME ELEMENT METHODS FOR ELLIPTIC OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, we have discuss a one parameter family of discontinuous Galerkin finite volume element methods for the approximation of the solution of distributed optimal control problems governed by a class of second order linear elliptic equations. In order to approximate the control problem, the method of variational discretization is used. By following the analysis of Kumar *et. al.* [Numer. Meth. Part. Diff. Equns. 25 (2009), pp. 1402–1424], optimal order of convergence in L^2-norm for state, costate and control variables are derived. Moreover, optimal order of convergence in broken H^1-norm are also derived for state and costate variables. Several numerical experiments are presented to validate the theoretical order of convergence.

Keywords: Optimal control; variational discretization; discontinuous Galerkin finite volume element methods; order of convergence; numerical experiments.

1. Introduction

This paper is concerned with the discontinuous Galerkin finite volume element (DGFVE) approximation of the elliptic optimal control problem of the following type : Find y, u such that

$$\min_{u\in U_{ad}} \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} \quad + \quad \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)}, \tag{1.1}$$

subject to

$$-\nabla.(K\nabla y) \;=\; Bu+f \quad in\,\Omega, \tag{1.2}$$

$$y \;=\; 0 \quad on\,\Gamma. \tag{1.3}$$

where, $\Omega\subset\mathbb{R}^2$ is a convex, bounded and polygonal domain and Γ is the boundary of Ω, λ is a positive number, f, $y_d\in L^2(\Omega)$ or $H^1(\Omega)$, $K=(k_{ij}(x))_{2\times2}$ denotes a real valued, symmetric and uniformly positive definite matrix in Ω, i.e., there exists a positive constant α_0 such that

$$\xi^T K\xi \geq \alpha_0\xi^T\xi \quad \forall\xi\in\mathbb{R}^2.$$

B is a bounded continuous linear operator and U_{ad} is denoted by

$$U_{ad}=[u\in L^2(\Omega): a\leqslant u(x)\leqslant b,\ a.e.\,in\,\Omega,\ a,b\in\mathbb{R}].$$

The numerical solutions of such kind of elliptic problems have been investigated by many researchers, since these problems have lots of applications in mathematical and physical problems. Finite element methods extensively used for the approximation of the control problems and for the error analysis of finite element methods (FEM) applied to elliptic control problems, we refer to [3, 4, 5, 6, 7, 15] and references therein. In most of these papers, the state and costate variables are discretized by continuous linear elements and control variable by piecewise constant or piecewise linear polynomials. More recently, Hinze given a new direction for approximating the control problem in which a new variational discretization approach is introduced for

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linear-quadratic optimal control problems whereas the control set is not discretized explicitly and obtained improved convergence order for optimal control, for more details, kindly see [8].

Because of local conservative property of the finite volume element (FVE) methods, these methods are very popular in computational fluid dynamics and (FVE) methods have also been used to solve fluid optimal control problems. In [12], the author has used the *optimize-then-discretize* approach and FVE discretizations to approximate elliptic optimal control problems.

It is well known the discontinuous Galerkin (DG) methods which was introduced by Arnold in [1] does not demand the inter element continuity criteria and has some attractive features such as: high order accuracy, localizabilty and suitable for parallel computing easily handle the boundary conditions. Keeping in mind the advantages of FVE methods and DG methods, in [16], Ye introduced discontinuous Galerkin finite volume element (DGFVE) methods for elliptic problems. Later Kumar *et. al.* [9] have discussed a one parameter family of DGFVE methods for the approximation of the elliptic problem. Recently, Kumar extended the analysis of [9] for approximation of miscible displacement problems, see [10].

In this paper, in order to obtain an optimal system, first we apply Lagrange multiplier method to the problem (1.1)-(1.3) and obtain an optimal system. Then we use DGFVE methods to discretize the state and adjoint equation of the system. For the optimal condition, we use variational discretization approach introduced in [8] to obtain the control.This paper is organized as follows: While the Section 1 is introductory, Section 2 is devoted to the DGFVE formulation for the optimal control problem. In Section 3, we discuss the convergence analysis of DGFVE in different norms and finally in Section 4, we present some numerical experiments to support the theoretical results obtained in Section 3.

2. Discontinuous Galerkin Finite Volume Element Formulation

We assume that our optimal control problem admits a unique control u, since U_{ad} is bounded, convex and closed. For the subsequent standard existence, uniqueness and first-order optimality results we refer to [14]. We can then write the first-order optimality condition in the following form:

$$(\lambda u + B^* p, v - u) \geq 0 \quad \forall v \in U_{ad}, \tag{2.1}$$

where the function p is called *adjoint state* (or *costate*) associated with u and solution of the *adjoint equation*

$$-\nabla.(K\nabla p) = y - y_d, \quad in\,\Omega \tag{2.2}$$

$$p = 0, \quad on\,\Gamma. \tag{2.3}$$

Let τ_h be a regular, quasi-uniform triangulation of $\bar{\Omega}$ into closed triangles T with $h = \max_{T\in\tau_h}(h_T)$, where h_T is the diameter of the triangle T. The dual partition τ_h^* of τ_h is constructed as follows: divide each triangle $T \in \tau_h$ into three triangles by joining the barycenter B and the vertices of T as shown in Figure 1. Let τ_h^* consists of all these triangles T_i^*. We define the finite dimensional Trial (V_h) and test space (W_h) associated with τ_h and τ_h^*, respectively as follows:

$$V_h = \{v_h \in L^2(\Omega) : v_h|_T \in P_1(T) \quad \forall T \in \tau_h\}$$

$$W_h = \{w_h \in L^2(\Omega) : w_h|_{T^*} \in P_0(T^*) \quad \forall T^* \in \tau_h^*\}.$$

where $P_m(T)$ or $P_m(T^*)$ denotes the space of all polynomials of degree less than or equal to m defined on T or T^*, respectively. Let $V(h) = V_h + H^2(\Omega) \cap H_0^1(\Omega)$. To connect the trial space and test space, we define a transfer operator $\gamma : V(h) \longrightarrow W_h$ as:

$$\gamma v|_{T^*} = \frac{1}{h_e}\int_e v|_{T^*} ds, \quad T^* \in \tau_h^*,$$

where e is an edge in T, T^* is the dual element in τ_h^* containing e, and h_e is the length of the edge e.

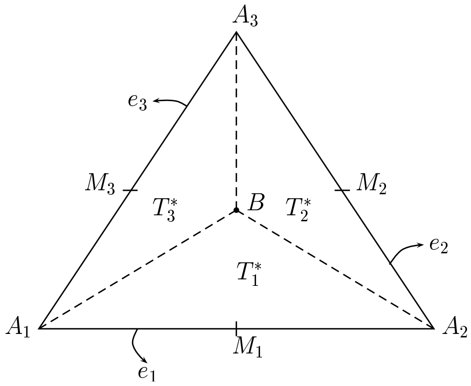

FIGURE 1. A triangular partition and its dual

Multiply (1.2) and (2.2) by γv_h, integrate over the control volumes and an application of Gauss divergence methods leads the following DGFVE formulation: Find $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ such that

$$A_h(y_h, w_h) = (Bu_h + f, \gamma w_h) \quad \forall w_h \in V_h, \tag{2.4}$$

$$A_h(p_h, q_h) = (y_h - y_d, \gamma q_h) \quad \forall q_h \in V_h, \tag{2.5}$$

$$(\lambda u_h + B^* p_h, v - u_h) \geq 0 \quad \forall v \in U_{ad}, \tag{2.6}$$

where the bilnear form $A_h(\cdot, \cdot)$ defined as

$$A_h(\Phi_h, \Psi_h) = -\sum_{T \in \tau_h} \sum_{j=1}^{3} \int_{A_{j+1}BA_j} (K\nabla\Phi_h.\mathbf{n})\gamma\Psi_h ds + \theta \sum_{e \in \Gamma} \int_e [\gamma\Phi_h].\langle K\nabla\Psi_h\rangle ds$$

$$-\sum_{e \in \Gamma} \int_e [\gamma\Psi_h].\langle K\nabla\Phi_h\rangle ds + \sum_{e \in \Gamma} \int_e \frac{\alpha}{h_e^\beta}[\Phi_h].[\Psi_h] ds \quad \forall \Phi_h, \Psi_h \in V_h.$$

Here, the symbols $[\cdot]$ and $\langle\cdot\rangle$ used for jump and average respectively and $\theta \in [-1, 1]$, α and β are penalty parameters, for more details kindly see [9]. Let $y_h(u)$ and $p_h(y)$ be the solutions of

$$A_h(y_h(u), w_h) = (Bu + f, \gamma w_h) \quad \forall w_h \in V_h, \tag{2.7}$$

and

$$A_h(p_h(y), q_h) = (y - y_d, \gamma q_h) \quad \forall q_h \in V_h, \tag{2.8}$$

respectively. A norm $\|\!|.\|\!|$ on V(h) is defined by

$$\|\!|v\|\!|^2 = |v|_{1,h}^2 + \sum_{e \in \Gamma} \frac{1}{h_e^\beta} \int_e [v]^2 ds,$$

where $|v|_{1,h}^2 = \sum_{T \in \tau_h} |\nabla v|_{0,T}^2$. Using the coercivity and boundedness of the bilinear form $A_h(\cdot, \cdot)$ which is proved in [9, pp. 1410–1413] and noting that $y_h = y_h(u_h)$ and $p_h = p_h(y_h)$ we have the following result.

Lemma 2.1. *Let $y_h(u)$ and $p_h(y)$ be the solutions of (2.7) and (2.8) respectively. Then the following results hold :*

$$\|\!|p_h(y) - p_h\|\!| \leq C \|y - y_h\| \quad \text{and} \quad \|\!|y_h(u) - y_h\|\!| \leq C \|u - u_h\|.$$

The result easily follows by using (Theorem 2.3, [9]) and Cauchy-Schwarz inequality.

We emphasis that throughout the article C is a generic positive constant (also appeared in Lemma 2.1) which is independent of the mesh size h but may depend on the bounds of f, u, y, p and size of the domain Ω.

3. Convergence Analysis

3.1. Convergence in L^2-norm.

Theorem 3.1. *Assume that $K \in W^{1,\infty}(\Omega)$ and $u, f, y_d \in L^2(\Omega)$. Let $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U_{ad}$ be the exact solutions and $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ be the solutions of (2.5)-(2.6). Then there exists an $h_0 > 0$ such that for all $0 < h \le h_0$*

$$\|u - u_h\| \le Ch. \tag{3.1}$$

Moreover, if $K \in W^{2,\infty}(\Omega)$ and $u, f, y_d \in H^1(\Omega)$, then there exists an $h_0 > 0$ such that for all $0 < h \le h_0$

$$\|u - u_h\| \le Ch^2. \tag{3.2}$$

The above theorem can be proved by using the variational inequalities (2.1) and (2.6) with the functions u and u_h, using (Lemma 2.4, Theorem 3.2, [9]) and Lemma 2.1. For more details, we refer to [11].

Now, using triangle inequality, (Theorem 3.2, [9]), Lemma 2.1 and Theorem 3.1, we have the following theorem.

Theorem 3.2. *Assume that $K \in W^{1,\infty}(\Omega)$ and $u, f, y_d \in L^2(\Omega)$. Let $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U_{ad}$ be the exact solutions and $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ be the solutions of (2.5)-(2.6). Then there exists an $h_0 > 0$ such that for all $0 < h \le h_0$*

$$\|y - y_h\| \le Ch, \quad \|p - p_h\| \le Ch. \tag{3.3}$$

Moreover, if $K \in W^{2,\infty}(\Omega)$ and $u, f, y_d \in H^1(\Omega)$, then there exists an $h_0 > 0$ such that for all $0 < h \le h_0$

$$\|y - y_h\| \le Ch^2, \quad \|p - p_h\| \le Ch^2. \tag{3.4}$$

Following the proof lines of (Theorem 3.1, [9]) and using Theorem 3.1, Theorem 3.2 together with Lemma 2.1, we can derive the following error estimates in the H^1-norm. For a detailed proof, we refer to [11].

3.2. Convergence in broken H^1-norm.

Theorem 3.3. *Assume that $K \in W^{1,\infty}(\Omega)$ and $u, f, y_d \in L^2(\Omega)$. Let $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U_{ad}$ be the exact solutions and $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ be the solutions of (2.5)-(2.6). Then there exists an $h_0 > 0$ such that for all $0 < h \le h_0$*

$$|||y - y_h||| \le Ch, \quad |||p - p_h||| \le Ch. \tag{3.5}$$

4. Numerical Experiments

In this section, we present two numerical examples in order to discuss the performance of the DGFVE for the approximation of the elliptic optimal control problem (1.1)-(1.3). The method holds true for any value of $\theta \in [-1, 1]$ but in particular, for the numerical experiments we take θ = -1,0,1, as these values of θ leads to different interesting schemes in the context of discontinuous finite element methods, kindly see [13]. We will investigate the order of convergence of state, costate and control variables in L^2-norm and order of convergence of state and costate variables in the broken norm $|||.|||$.

Example 1. We consider the following elliptic control problem with Dirichlet boundary value condition:

$$\min_{u \in U_{ad}} \frac{1}{2} \|y - y_d\|^2_{L^2(\Omega)} + \frac{1}{2} \|u\|^2_{L^2(\Omega)},$$

$$-\triangle y \;=\; u \quad in\, \Omega,$$

$$\begin{aligned} y &= 0 \quad in\,\Gamma, \\ u &\geq 0, \end{aligned}$$

where $\Omega = [(x_1, x_2) : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1]$, Γ denotes the boundary of Ω. The exact state y is $sin(\pi x_1)sin(\pi x_2)$, $y_d = (4\pi^4 + 1)sin(\pi x_1)sin(\pi x_2)$, $p = -2\pi^2 sin(\pi x_1)sin(\pi x_2)$ and $u = max(0, -p)$.

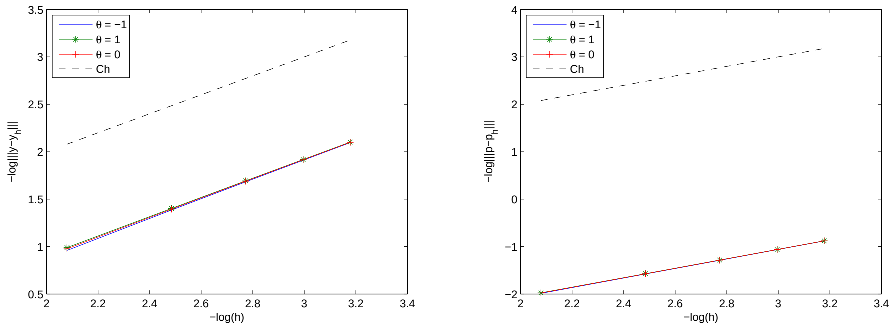

FIGURE 2. Order of convergence in broken H^1-norm for state and costate variables for Example 1.

$\|\|\|y - y_h\|\|\|$			$\|\|\|p - p_h\|\|\|$		
θ=-1	θ=0	θ=1	θ=-1	θ=0	θ=1
0.3719918	0.37650518	0.38334068	7.21776447	7.24618657	7.29030071
0.2460884	0.24772621	0.25033399	4.82156990	4.83286579	4.85124495
0.1839629	0.18473849	0.18600306	3.61641018	3.62227458	3.63199956
0.1469065	0.14733895	0.14805334	2.89231763	2.89585972	2.90178678
0.1222834	0.12255207	0.12299930	2.40948560	2.41184101	2.41580092

TABLE 1. Numerical results of broken H^1 error for θ=1, θ=-1 and θ=0 with β=1 for Example 1.

In the next example we take desired state y_d to be zero and include desired control u_0.
Example 2. We consider the following problem

$$\min_{u \in U_{ad}} \frac{1}{2} \|y - y_d\|^2_{L^2(\Omega)} + \frac{1}{2} \|u - u_0\|^2_{L^2(\Omega)},$$

$$\begin{aligned} -\triangle y &= u + f \quad in\,\Omega, \\ y &= 0 \quad in\,\Gamma, \\ u &\geq 1 \quad in\,\Omega. \end{aligned}$$

In this example we have,
$\Omega = [(x_1, x_2) : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1]$, $u_0 = 1 - sin(\pi x_1/2) - sin(\pi x_2/2) + s$, $y_d = 0$, $p = Z(x_1, x_2)$, $f = 4\pi^4 Z - u$, where $Z = sin(\pi x_1)sin(\pi x_2)$ and

$$s = \begin{cases} 0.5 & if\, x_1 + x_2 > 1.0 \\ 0.0 & if\, x_1 + x_2 \leq 1.0 \end{cases}$$

. The exact solution of this problem is $y = 2\pi^2 Z$, $u = max(u_0 - p, 1)$.

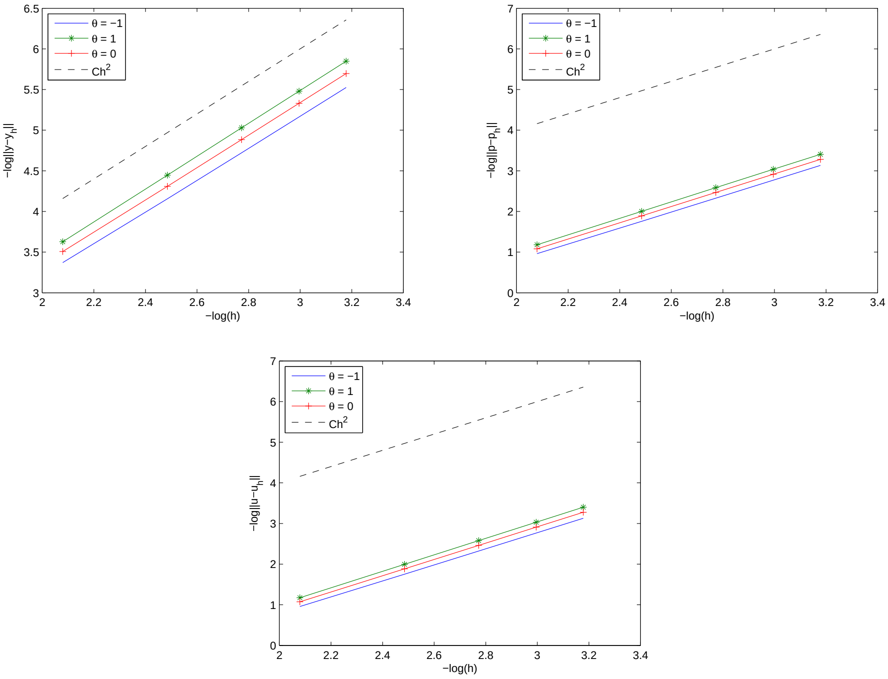

FIGURE 3. Order of convergence in L^2-norm for state and costate and control variables for Example 1.

$\|y - y_h\|$			
Dof	θ=-1	θ=0	θ=1
384	0.02657497	0.02999063	0.03437191
864	0.01172431	0.01344455	0.01568472
1536	0.00654811	0.00756756	0.00890502
2400	0.00416848	0.00483938	0.00572336
3456	0.00288324	0.00335721	0.00398345

$\|p - p_h\|$			
Dof	θ=-1	θ=0	θ=1
384	0.30726316	0.33952145	0.38187541
864	0.13500845	0.15109265	0.17258162
1536	0.07535372	0.08485681	0.09766827
2400	0.04797144	0.05421803	0.06268522
3456	0.03318749	0.03759811	0.04359833

$\|u - u_h\|$			
Dof	θ=-1	θ=0	θ=1
384	0.30968478	0.34176897	0.38392985
864	0.13604558	0.15206116	0.17347429
1536	0.07586822	0.08533866	0.09811434
2400	0.04825989	0.05448858	0.06293629
3456	0.03336438	0.03776417	0.04375267

TABLE 2. Numerical results of L^2 error for θ=1, θ=-1 and θ=0 with β=1 for Example 1.

The errors in broken H^1-norm for the DGFVEM solution of state and costate variables are presented in Tables 1 and 3 for examples 1 and 2, respectively whereas the errors in L^2-norm for the DGFVEM solution of state, costate and control variables for examples 1 and 2 are presented in Tables 2 and 4 respectively.

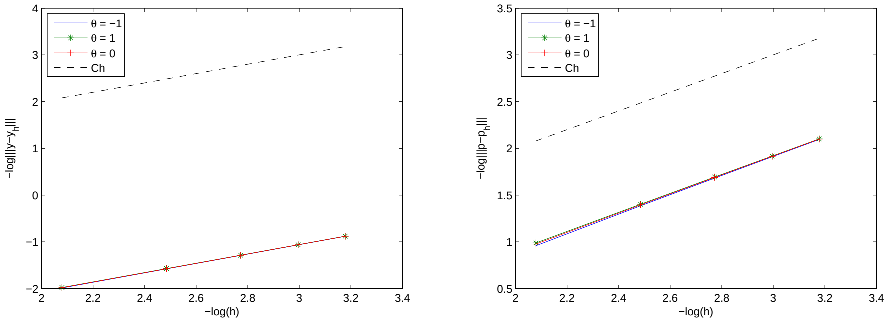

FIGURE 4. Order of convergence in broken H^1-norm for state and costate variables for Example 2.

$\|\|y - y_h\|\|$				$\|\|p - p_h\|\|$			
Dof	θ=-1	θ=0	θ=1	Dof	θ=-1	θ=0	θ=1
384	7.21797574	7.24658386	7.29098777	384	0.37204143	0.37657411	0.38343755
864	4.82162819	4.83298774	4.85147100	864	0.24610328	0.24774803	0.25036647
1536	3.61643379	3.62232663	3.63209930	1536	0.18396914	0.18474788	0.18601743
2400	2.89232940	2.89588651	2.90183911	2400	0.14690967	0.14734380	0.14806087
3456	2.40949230	2.41185655	2.41583166	3456	0.12228530	0.12255488	0.12300372

TABLE 3. Numerical results of broken H^1 error for θ=1, θ=-1 and θ=0 with β=1 for Example 2.

$\|y - y_h\|$				$\|p - p_h\|$			
Dof	θ=-1	θ=0	θ=1	Dof	θ=-1	θ=0	θ=1
384	0.30825840	0.34069258	0.38326672	384	0.02663156	0.03005540	0.03444685
864	0.13545263	0.15162708	0.17323066	864	0.01174998	0.01347458	0.01572027
1536	0.07560241	0.08515929	0.09803949	1536	0.00656257	0.00758466	0.00892548
2400	0.04812983	0.05441189	0.06292457	2400	0.00417772	0.00485037	0.00573660
3456	0.03329703	0.03773274	0.04376519	3456	0.00288965	0.00336485	0.00399269

$\|u - u_h\|$			
Dof	θ=-1	θ=0	θ=1
	$1.0e-003*$		
384	0.20013750	0.17824991	0.15447196
864	0.06389361	0.05698655	0.04942176
1536	0.02785402	0.02484840	0.02154210
2400	0.01452166	0.01295391	0.01122471
3456	0.00849969	0.00758109	0.00656606

TABLE 4. Numerical results of L^2 error for θ=1, θ=-1 and θ=0 with β=1 for Example 2.

Figures 2, 3 (for Example 1) and 4, 5 (for Example 2) indicate that the computed orders of convergence match the theoretical orders of convergence in L^2-norm and broken H^1-norm.

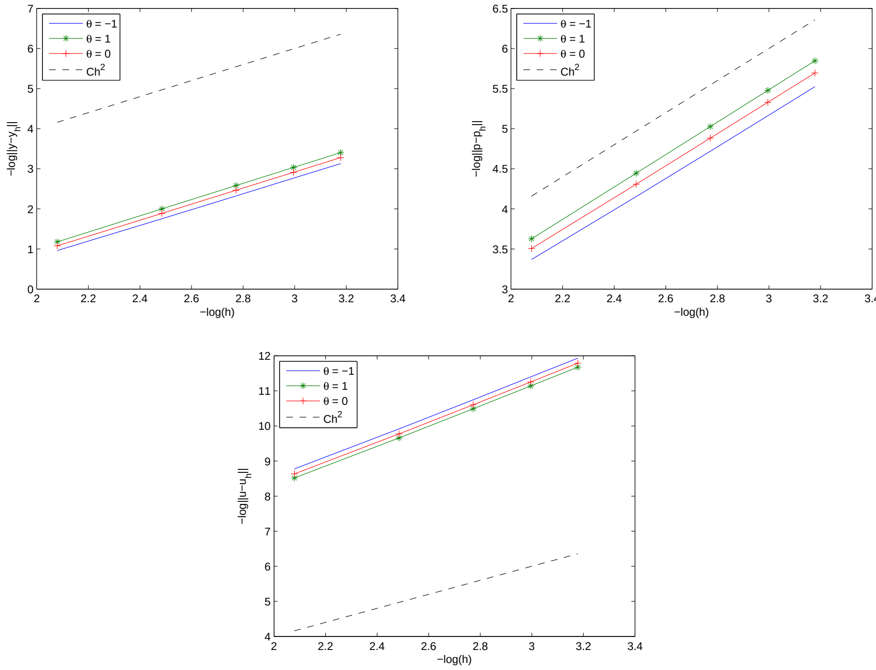

FIGURE 5. Order of convergence in L^2-norm for state and costate and control variables for Example 2.

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