

On mixed mode stress intensity factor evaluation using Interaction Integral Method for the tetrahedral finite element (for cracks with kinks)

*Y. Wakashima¹ and H. Okada²

¹Department of Mechanical Engineering, Graduate School of Science and Technology, Tokyo University of Science

²Department of Mechanical Engineering, Faculty of Science and Technology, Tokyo University of Science

*Corresponding author: young_island_126@yahoo.co.jp

Abstract

We have been developing an interaction integral method for tetrahedral finite element, as a way to calculate the stress intensity factors for mixed mode crack problem. If cracks involve kinks, interaction integral method must have the surface integrals at the crack faces. The surface integral terms add complexities to the calculation procedures. In this paper, a method that can obviate the needs for the surface integral terms is proposed. Results that are computed by the proposed method are compared with there by the virtual crack closure-integral method. It is shown that the results of the proposed technique are more accurate than those of the conventional way.

Keywords: J-integral, Stress Intensity Factors, Virtual Crack Extension, Interaction Integral Method

Introduction

We need to calculate the stress intensity factors for the predictions of crack propagation behavior using the fracture mechanics concept. Finite element method is used for calculating the stress intensity factors. The computations of the stress intensity factors generally require the use of hexahedral finite element. However, the generations of finite element models for the cracks, consisting of the hexahedral finite elements require a large amount of manual labor and man time. On the other hand, the meshing can fully be automated if we use the tetrahedral finite elements. The tetrahedral finite elements have been commonly used in three-dimensional solid analysis in recent years.

We have been developing an interaction integral method for the quadratic tetrahedral finite element, as a way to calculate the stress intensity factors for mixed mode crack problem. And we showed that the method can calculate the stress intensity factors for mixed mode crack problem accurately, with the correction terms proposed by Daimon et al. (2014).

If cracks involve kinks, the interaction integral method has the surface integral terms at the crack faces. The surface integral terms add complexities to the calculation procedures. In this paper, a method that can obviate the needs for the surface integral terms is proposed as an extension of the interaction integral method with the correction terms. Results that are computed by the proposed method are compared with those by the virtual crack closure-integral method of Okada et al. (2008). It is shown that the results, that are obtained by proposed technique, are more accurate than those by the conventional way.

Calculation of mixed mode stress intensity factors by the interaction integral method

The J-integral can be evaluated by the domain integral method

The interaction integral method is based on the domain integral method of Nikishkov et al. (1987) that calculates energy release rate under linear elastic and small scale yielding conditions. The

formulation and the numerical evaluations of the domain integral method are briefly described, as follows.

The J-integral that was proposed by Rice (1968) evaluates the energy release rate in two-dimensional crack problems. The J-integral can also be defined in three-dimensional problems as presented by Shivakumar et al. (1992). It is defined as a surface integral on an infinitesimally small pipe S_0 surrounding the crack front, as depicted in Fig. 1. The size of Δ and ε/Δ are set to be 0 in a limit, and we can write the three-dimensional J-integral, as:

$$J = \lim_{\substack{\varepsilon/\Delta \rightarrow 0 \\ \Delta \rightarrow 0}} \frac{1}{\Delta} \int_{S_0} \left(W n_1^\varepsilon - n_j^\varepsilon \sigma_{ij} \frac{\partial u_j}{\partial x_1} \right) dS_0 \quad (1)$$

Here, W is the strain energy density, n_i are the unit outward normal vector, u_i are the displacements, and σ_{ji} are the stress tensor.

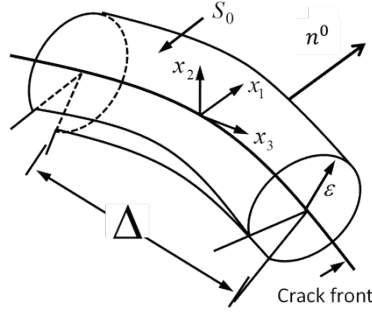


Figure 1. Definition of three-dimensional J-Integral

The J-integral can be evaluated by the domain integral method. Continuous and piecewise differentiable vector function q_i are introduced. They represent the virtual crack extension vector at the crack front and δA is the area of virtual crack extension. The three-dimensional J-integral can be represented by its domain integral form, as:

$$J = \frac{1}{\delta A} \lim_{\substack{\varepsilon/\Delta \rightarrow 0 \\ \Delta \rightarrow 0}} \int_{S_0} \left(W n_k^0 - n_i^0 \sigma_{ij} \frac{\partial u_j}{\partial x_k} \right) q_k dS_0 \quad (2)$$

The surface integral version of J-integral expression as shown in Eq. (2) is converted to the domain integral version by applying the Gauss divergence theorem. The domain of integration is as shown in Fig. 2. Here q_i are the components of vector function that are continuous and piece-wise differentiable in the domain of integration, the J-integral can be expressed by the following equation that is called the domain integral method.

$$\begin{aligned} J &= \frac{1}{\delta A} \lim_{\substack{\varepsilon/\Delta \rightarrow 0 \\ \Delta \rightarrow 0}} \int_{S_0 + S^+ + S^- + S_R + S_L + S_\varepsilon} \left(W n_k^0 - n_i^0 \sigma_{ji} \frac{\partial u_j}{\partial x_k} \right) q_k dS \\ &= -\frac{1}{\delta A} \left[\int_V \left(W \delta_{ik} - \sigma_{ij} \frac{\partial u_j}{\partial x_k} \right) \frac{\partial q_k}{\partial x_i} dV + \int_{S^+ + S^-} \left(W n_k^\varepsilon - n_i^\varepsilon \sigma_{ji} \frac{\partial u_j}{\partial x_k} \right) q_k dS \right] \end{aligned} \quad (3)$$

If the crack were planer, the crack surface integral terms vanish because $n_k q_k$ equals 0. But if cracks had kinks, $n_k q_k$ does not vanish, Eq. (3) contains a crack face integral.

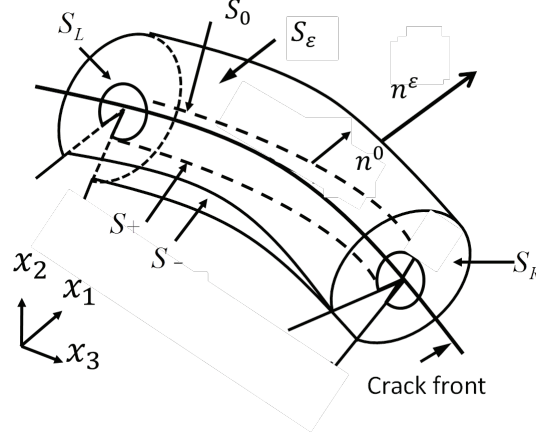
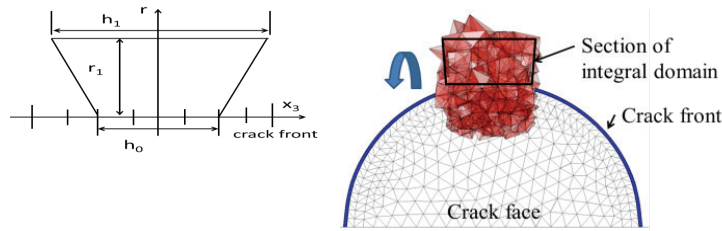


Figure 2. Definition of the domain integral method

Domain of integral method and distribution of value of q -function

To evaluate the domain integral we need to set the domain of integral and the variation of q -function. The domain is specified by using the q -function. In proposed domain integral method we set q -function to nodes by using the information of nodal locations. We set q -function values at each node by using the function $q = q(r, x_3)$ of distance r from the crack front and the coordinate x_3 in the tangential direction along the crack front. Here, r and x_3 are defined as shown in Fig. 2 (a). The q -function value linearly decreases as the distance r increases and q vanishes at the outer surface of the domain of integral. We define a trapezoid at a crack front node, as shown in Fig. 2(a) and rotate it around the crack front. The q -function values $q = q(r, x_3)$ are assigned to nodes inside the rotating body. Finite elements containing any nodes having non-zero q -function values are included in the domain of integral. As the result, the domain of integral has angular outer shape, as depicted in Fig. 2 (b). For the full details, the readers are referred to Okada and Ohata (2013).



(a) Trapezoid in $r - x_3$ plane (b) A typical outer shape of domain of integral

Figure 3. Domain of integral for the evaluation of J-integral

Description of interaction integral method

The interaction integral method is a technique to evaluate the mixed mode stress intensity factors based on the J-Integral [Yau et al. (1980)].

The J-integral expresses the energy release rate of a crack in linearly elastic material. The energy release rate G in an elastic body can also be expressed by the stress intensity factors (K_I, K_{II}, K_{III}) . Therefore, we can write:

$$G_{total} = J = \frac{1}{E'} (K_I^2 + K_{II}^2) + \frac{1}{2\mu} K_{III}^2 = -\frac{1}{\delta A} \int_{\Omega} \left\{ W \delta_{li} - \sigma_{ij} \frac{\partial u_j}{\partial x_1} \right\} \frac{\partial q}{\partial x_i} d\Omega \quad (4)$$

Here, E' equals E for the plane stress and E' is $E/(1-\nu^2)$ for the plane strain. μ is the shear modulus and is expressed by $E/2(1+\nu)$. E is the Young's modulus and ν is the Poisson's ratio.

The superposition of two solutions which satisfy the equilibrium can generate another equilibrium state. The independent equilibrium states are designated by superscripts (1) and (2). We can write:

$$u_j = u_j^{(1)} + u_j^{(2)}, \quad \sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}, \quad \varepsilon_{ij} = \varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)} \quad (5)$$

$$W = \frac{1}{2} (\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}) (\varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)}) = W^{(1)} + W^{(2)} + \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} \quad (6)$$

$$K_I = K_I^{(1)} + K_I^{(2)}, \quad K_{II} = K_{II}^{(1)} + K_{II}^{(2)}, \quad K_{III} = K_{III}^{(1)} + K_{III}^{(2)} \quad (7)$$

We can derive the following equation by substituting Eqs. (5), (6), (7) in Eq. (4).

$$\frac{2}{E'} (K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)}) + \frac{1}{\mu} K_{III}^{(1)} K_{III}^{(2)} = - \int_{\Omega} \left\{ \sigma_{kl}^{(1)} \varepsilon_{kl}^{(2)} \delta_{li} - \sigma_{ij}^{(1)} \frac{\partial u_j^{(2)}}{\partial x_1} - \sigma_{ij}^{(2)} \frac{\partial u_j^{(1)}}{\partial x_1} \right\} \frac{\partial q}{\partial x_i} d\Omega \quad (8)$$

We let the equilibrium state (1) be the solution of the boundary value problem which is solved by the finite element method and the equilibrium state (2) be the asymptotic solution at the vicinity of the crack. We can set the displacements of the equilibrium (2) by using asymptotic solution of displacement near the crack front as described in text book of Broek (1986).

By setting the mode I, II or III stress intensity factor of the equilibrium state (2) be unity, we extract the respective stress intensity factor. For example, if we set $K_I^{(2)} = 1$ and $K_{II}^{(2)} = K_{III}^{(2)} = 0$, we extract $K_I^{(1)}$ of the equilibrium state (1).

The asymptotic solution of the crack with respect to the stress intensity factors can be expressed by using the local Cartesian coordinates (x'_1, x'_2, x'_3) and cylindrical coordinates (r, θ, x'_3) that are defined at the crack front as depicted in Fig. 4. The displacements can be written by:

$$u_i(r, \theta) = \sqrt{\frac{r}{2\pi}} \left\{ K_I f_i^I(\theta) + K_{II} f_i^{II}(\theta) + K_{III} f_i^{III}(\theta) \right\} \quad (9)$$

If a node is above the crack face, a positive angle θ_u as shown in Fig. 4 is used in Eq. (9). If a node is below the crack face, a positive angle θ_d is used.

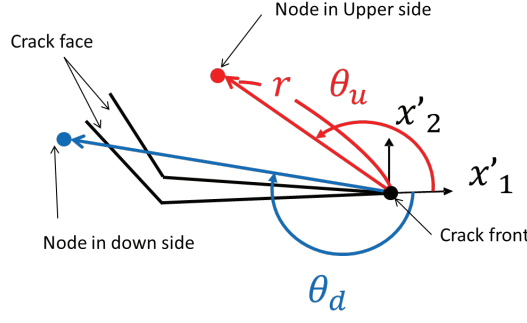


Figure 4. The local Cartesian and polar coordinate systems around the crack front

Correction terms to the asymptotic solutions (auxiliary solutions) adopted in the interaction integral method

The auxiliary solutions defined by the asymptotic solutions of crack do not satisfy the equilibrium in terms of the finite element. Therefore, a method to recover the equilibrium by adding correction terms to the asymptotic solution was proposed by Daimon et al. (2014). We write the new auxiliary solutions by the following equations.

$$\hat{u}_j^{(2)} = u_j^{(2)} + \hat{u}_j^{(2)}, \quad \hat{\sigma}_j^{(2)} = \sigma_j^{(2)} + \hat{\sigma}_j^{(2)} \quad (10)$$

Here, $\hat{u}_j^{(2)}$ and $\hat{\sigma}_j^{(2)}$ are the correction terms for the displacements and the stresses.

We write the weak form of equilibrium equations in terms of $\hat{\sigma}_j^{(2)}$, as:

$$\int w_j \left(\frac{\partial \sigma_{ij}^{(2)}}{\partial x_i} + \frac{\partial \hat{u}_{ij}^{(2)}}{\partial x_i} \right) dV = 0 \quad (11)$$

Here, V is the domain of integral in the domain integral method, and w_j are the weight functions. We can derive a symmetric weak form by applying the Gauss divergence theorem and the generalized Hooke's law $\left(\hat{\sigma}_{ij}^{(2)} = E_{ijkl} \frac{\partial \hat{u}_k^{(2)}}{\partial x_l} \right)$, as:

$$\int \frac{\partial w_j}{\partial x_i} E_{ijkl} \frac{\partial \hat{u}_k^{(2)}}{\partial x_l} dV = - \int \frac{\partial w_j}{\partial x_i} \sigma_{ij}^{(2)} dV \quad (12)$$

where E_{ijkl} are the components of fourth order tensor representing the generalized Hook's law.

We can establish a finite element formulation based on the symmetric weak form and determine the correction terms.

For cracks with kinks

If a crack involve kinks, we must account for the surface integral terms of Eq. (2). Because the crack surface and the crack propagation direction are not parallel to each other, $n_k q_k$ does not equal 0. [Okada et al. (2013)] proposed a method to vanish the surface integral terms in the domain

integral method by setting the value of q -function on the crack surface that is not parallel to crack propagation direction, as shown in Fig.5.

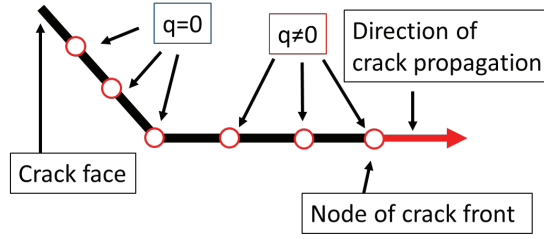


Figure 5. How to set the value of q

Numerical examples

In this section, some numerical results on the evaluation of the stress intensity factors of cracks involving kinks are presented. The stress intensity factors were also computed by VCCM proposed by Okada et al. (2008) and they are compared with those by present interaction integral method. The stress intensity factor evaluations by the interaction integral method were performed in two ways. One is to set the q -function value to be zero on the part of crack surfaces whose normal vector is not perpendicular to the direction of virtual crack extension. The other does not set q -function value to be zero on the crack surfaces.

The load conditions and the sizes of the tensile plate and the crack are as presented in Fig. 6 and Table 1. Uniformly distributed tractions are applied to the upper and lower surfaces of the plate. The crack has a kink as shown in Fig. 6. The kink angle θ is set to be 30 or 45 degrees.

The finite element analysis model is shown in Fig. 7 for the case of kink angle θ being 45 degrees. The total numbers of nodes and elements are 119152 and 82815, respectively. The size parameters of domain of integral are set to be $r_1/d_c = 64.0$, $h_0/d_c = 10.0$ and $h_1/d_c = 14.0$. The size d_c of finite element along the crack front is 0.0625 mm. The domain of integration includes the kink.

The distributions of the stress intensity factors along the crack front are presented in Figs. 8 and 9. In Figs. 8(a) and 9(a), the results of the conventional way, that does not set the q -function value to be zero at all, are presented. The results of proposed way are presented in Figs. 8(b) and 9(b). All the stress intensity factors are normalized by that of two dimensional straight edge crack subject to a remote tension (see, for example, Murakami et al. (1987)). The formula is written by:

$$K_{ref} = \sigma\sqrt{\pi a} \left\{ 1.12 - 0.231 \times (a/w) + 10.55 \times (a/w)^2 - 21.72 \times (a/w)^3 + 30.39 \times (a/w)^4 \right\} \quad (13)$$

Here, σ , w and a in present problem are 1.0 (MPa), 20.0 (mm) and 10.0 (mm), respectively.

It is seen in Figs. 8 (a) and 9 (a) that the distributions of the stress intensity factors exhibit some oscillations. They are notable in the mode II and III components. The oscillations somewhat disappeared when the proposed method is applied to compute the stress intensity factors, as seen in Figs. 8 (b) and 9 (b).

Table 1. Sizes of the plate and crack

h_1	h_2	w	t
20.0[mm]	20.0[mm]	20.0[mm]	10.0[mm]
a_1	a_2	θ	σ
9.0[mm]	1.0[mm]	30,45°	1.0[MPa]

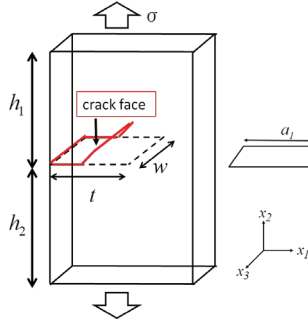


Figure 6. Plate with a through crack with a kink

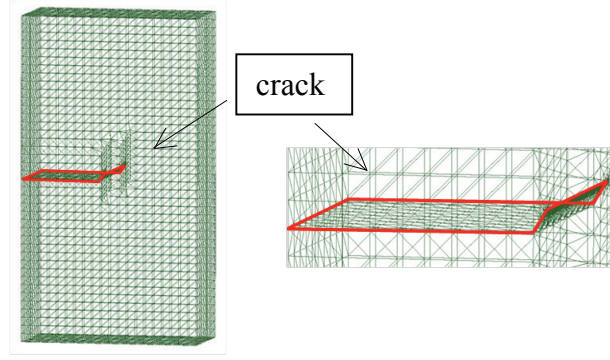
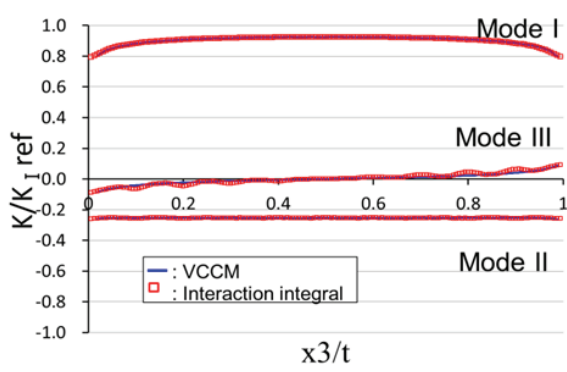
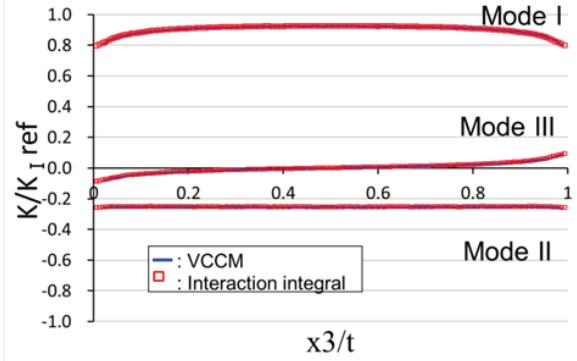


Figure 7. Finite element analysis model

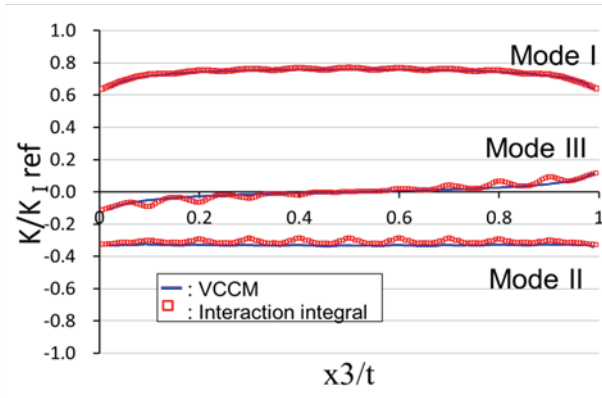


(a) Conventional way

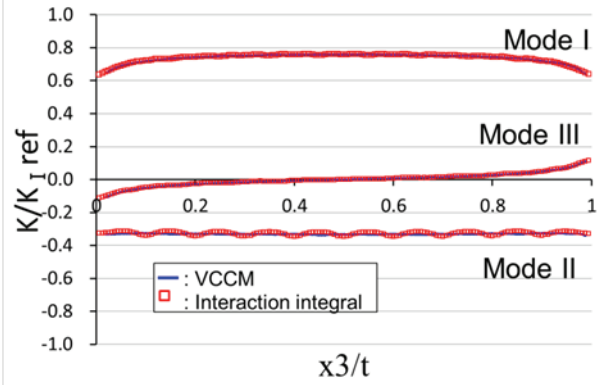


(b) Proposed way

Figure 8. Distributions of the stress intensity factors for the case of kink angle (30°)



(a) Conventional way



(b) Proposed way

Figure 9. Distributions of the stress intensity factors for the case of kink angle (45°)

Conclusions

The interaction integral method for tetrahedral finite element to accurately compute the stress intensity factors of mixed mode for cracks involving kinks is proposed in this paper. Comparing results of proposed method with those of conventional way, proposed method is found to serve more accurate and stable solutions than the conventional way.

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