Hyperelastic Fourth-Order Tensor Functions for Orthotropic Continua

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Abstract

We have developed a general hyperelastic strain energy function for the modelling of orthotropic continua that is able to maintain the same logical properties of advanced isotopic hyperelastic constitutive laws. The isotropic model of Simo and Pister, due to its exceptional fit with experimental data and superior mathematical and logical features, is replicated in orthotropic form. Through the use of the proposed Intrinsic-Field Tensors yielding asymmetric strain tensors with additional degrees of freedom, orthotropic Lamé parameters where scalars are replaced by fourth-order tensors and advanced fourth-order tensor operators, the desired general form of orthotropic hyperelastic strain energy function is achieved.

Keywords: Orthotropic hyperelasticity, composites, additional degrees of freedom, intrinsic-field tensors, fourth-order tensors, Simo and Pister, strain energy function.

Introduction

Hyperelastic materials are a class of solids that can be modelled as continua with rate-independent strain energy defined purely as a function of deformation and the material parameters. The Simo and Pister model[1], like most hyperelastic Strain Energy Functions (SEFs), is restricted to isotropic materials; one of its particular benefits is that pure distortional deformation is independent of the volumetric modulus for finite strain, and that the volumetric strain is a logarithmic function of deformation. From a mathematical standpoint, the scalar strain energy function is expressed as the product of scalar deformation invariants and scalar coefficients.

In this paper we posit that there is a generalised form of the SEF that a large class of hyperelastic functions should be able to be written within, and that those that cannot *can* either be closely approximated by the general form or do not satisfy certain expected boundary conditions of finite strain hyperelasticity. This general form is an abstraction one level up of the classical SEF and is mathematically encompassing; we called it the Generalised Strain Energy (GSE).

After first demonstrating that there is an exact representation for the Simo and Pister strain energy within the GSE, we further revisit the class of orthotropic tensors that are asymmetric and of the form of Intrinsic-Field Tensors (IFTs)[2]. We also propose a natural separation of the extended form of the Hookean material tensor for stiffness, which is naturally extended for IFTs such that it utilises all free terms within a fourth order tensor having major symmetry. These are the orthotropic Lamé material tensors for stiffness and compliance.

These tools allow a new model for *orthotropic Simo and Pister hyperelasticity* that we purport to be the first of its kind and the only such model to inherit and maintain so may logical properties of isotropic hyperelasticity, structural tensors[3] and orthotropic material models simultaneously. Since the proposed model achieves these features by derivation and as pure theoretical development, the properties are ensured. Hence we do not, in this short paper, provide numerical examples or experimental correlations. The following section begins by analysing an alternate representation for the Simo and Pister model that is conducive to our subsequent transformations.

Isotropic Simo And Pister Model

Classical form of the isotropic Simo and Pister strain energy function

The isotropic hyperelastic model of Simo and Pister[1] is desirable due to various logical properties. Shown as follows,

$$\underset{\text{S\&P}}{W} = \tfrac{1}{2}\lambda\left(\ln J\right)^2 - G\ln J + \tfrac{1}{2}G\left(\operatorname{tr}\mathbf{b} - 3\right) = \tfrac{1}{2}\left(\lambda\left(\ln J\right)^2 - \mu\ln J + \mu\operatorname{tr}\underset{2}{\mathbf{E}}\right), \quad \text{where} \tag{1}$$

$$\mu = 2G \quad \text{and} \quad \operatorname{tr}\underset{2}{\mathbf{E}} = \tfrac{1}{2}\left(\operatorname{tr}\mathbf{C} - 3\right) = \tfrac{1}{2}\left(\operatorname{tr}\mathbf{b} - 3\right),$$

the derivative of this gives the Kirchhoff stress $\boldsymbol{\tau}$:

$$\boldsymbol{\tau} = \lambda\ln J\mathbf{I} + \mu\left(\mathbf{b} - \mathbf{I}\right). \tag{2}$$

The strain energy function *W* uses the Lamé parameters λ and μ in a scalar product with invariant components of the deformation/strain tensor. Here, ln*J* is the natural logarithm of *J*, the determinant of the stretch tensor **U** or similarly of the deformation gradient **F**. Additionally, $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ is the left Cauchy–Green tensor, noting that tr**b** is the trace function of **b**, which is equal to tr**C**, where $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$. Two particularly valued properties of the Simo and Pister model are:

a) The deviatoric component of stress is only a function of μ
b) The strain energy goes to infinity as either the volume goes to infinity or to zero (singularity)

Surprisingly few models meet criteria a) and b), which can easily be demonstrated. First, the deviatoric part of the stress measure **S** is

$$\boldsymbol{\tau}_{dev} = \boldsymbol{\tau} - \boldsymbol{\tau}_{vol}, \quad \text{where} \quad \boldsymbol{\tau}_{vol} = \tfrac{1}{3}\operatorname{tr}\left(\boldsymbol{\tau}\right)\mathbf{I} \tag{3}$$

Substituting Eq. (2), the volumetric part becomes

$$\boldsymbol{\tau}_{vol} = \lambda\ln J\mathbf{I} + \tfrac{1}{3}\mu\left(\operatorname{tr}\mathbf{b} - 3\right)\mathbf{I} \tag{4}$$

and, where **e** is the Almansi–Euler strain, the deviatoric part in Eq. (3) becomes

$$\boldsymbol{\tau}_{dev} = \lambda\left(\ln J\mathbf{I} - \ln J\mathbf{I}\right) + \mu\left(\mathbf{e} - \tfrac{1}{3}\operatorname{tr}\mathbf{e}\mathbf{I}\right) = \mu\left(\mathbf{e} - \tfrac{1}{3}\operatorname{tr}\mathbf{e}\mathbf{I}\right) \tag{5}$$

noting that this is independent of the parameter λ. The next property, that of infinite strain energy at zero volume, can simply be seen to follow the logarithm of zero, $\ln 0 = \infty$.

In this paper, we shall propose an orthotropic expansion of Simo and Pister's model that preserves these properties while also remaining a valid orthotropic continuum model that collapses down to the isotropic model by nothing more than material parameters becoming isotropic.

Transformation into standard scalar form using series strain

In order to elevate the form of the strain energy function in Eq. (1) we need to first turn the strain energy in to a standard form that is similar to the St Venant Kirchhoff model.
The first component of the function is simply transformed through the identity

$$\ln J = \ln\left(\det\mathbf{U}\right) = \operatorname{tr}\left(\ln\mathbf{U}\right) = \operatorname{tr}\underset{0}{\mathbf{E}}, \tag{6}$$

where $\underset{0}{\mathbf{E}}$ is the logarithmic strain following the Seth-Hill[4], [5] form of general strain:

$$\underset{n}{\mathbf{E}} = \tfrac{1}{n}\left(\mathbf{U}^n - \mathbf{I}\right) \quad , \quad \begin{cases} n > 0: & \underset{\text{n}}{\mathbf{E}} = \dfrac{1}{n}\left(\mathbf{U}^n - \mathbf{I}\right) \\ n = 0: & \underset{0}{\mathbf{E}} = \ln\left(\mathbf{U}\right) \end{cases} \tag{7}$$

This can be used to develop and interesting equality to replace $\underset{2}{\mathbf{E}}$ in Eq. (1). Initially we note

$$\underset{2}{\mathbf{E}}=\tfrac{1}{2}\left(\mathbf{U}^2-\mathbf{I}\right),\quad \underset{1}{\mathbf{E}}=\mathbf{U}-\mathbf{I}\quad\rightarrow\quad 2\underset{2}{\mathbf{E}}+\mathbf{I}=\left(\underset{1}{\mathbf{E}}+\mathbf{I}\right)^2\quad\rightarrow\quad \underset{2}{\mathbf{E}}=\tfrac{1}{2}\underset{1}{\mathbf{E}}^2+\underset{1}{\mathbf{E}}, \tag{8}$$

which represents $\underset{2}{\mathbf{E}}$ in terms of $\underset{1}{\mathbf{E}}$. This process is repeated to the limit as n → ∞, yielding

$$\begin{aligned}\underset{2}{\mathbf{E}}&=\tfrac{1}{2}\underset{1}{\mathbf{E}}^2+\tfrac{1}{4}\underset{1/2}{\mathbf{E}}^2+\tfrac{1}{8}\underset{1/4}{\mathbf{E}}^2+\tfrac{1}{16}\underset{1/8}{\mathbf{E}}^2+\ldots+\tfrac{1}{\infty}\underset{1/\infty}{\mathbf{E}}^2+\underset{1/\infty}{\mathbf{E}}=\tfrac{1}{2^1}\underset{2^{-0}}{\mathbf{E}}^2+\tfrac{1}{2^2}\underset{2^{-1}}{\mathbf{E}}^2+\tfrac{1}{2^3}\underset{2^{-2}}{\mathbf{E}}^2+\ldots+\tfrac{1}{\infty}\underset{0}{\mathbf{E}}^2+\underset{0}{\mathbf{E}}\\&=\sum_{n=0}^{\infty}\left(\tfrac{1}{2^{n+1}}\underset{2^{-n}}{\mathbf{E}}^2\right)+\underset{0}{\mathbf{E}}\end{aligned}. \tag{9}$$

Now, suppose we define a strain measure called Series Strain $\underset{\Sigma}{\mathbf{E}}$, defined by

$$\underset{\Sigma}{\mathbf{E}}^2=\sum_{n=0}^{\infty}\left(\tfrac{1}{2^{n+1}}\underset{2^{-n}}{\mathbf{E}}^2\right), \tag{10}$$

then we now have the identity

$$\underset{2}{\mathbf{E}}=\underset{\Sigma}{\mathbf{E}}^2+\underset{0}{\mathbf{E}}. \tag{11}$$

Substitution of Eq. (6) and (11) into Eq. (1) gives

$$\underset{\text{S\&P}}{W}=\tfrac{1}{2}\left[\lambda\left(\operatorname{tr}\underset{0}{\mathbf{E}}\right)^2-\mu\operatorname{tr}\underset{0}{\mathbf{E}}+\mu\operatorname{tr}\left(\underset{\Sigma}{\mathbf{E}}^2+\underset{0}{\mathbf{E}}\right)\right]=\tfrac{1}{2}\left[\lambda\left(\operatorname{tr}\underset{0}{\mathbf{E}}\right)^2+\mu\operatorname{tr}\underset{\Sigma}{\mathbf{E}}^2\right], \tag{12}$$

which is remarkably similar to the St Venant Kirchhoff (SVK) model:

$$\underset{\text{SVK}}{W}=\tfrac{1}{2}\left[\lambda\left(\operatorname{tr}\underset{2}{\mathbf{E}}\right)^2+\mu\operatorname{tr}\underset{2}{\mathbf{E}}^2\right] \tag{13}$$

This form provides the basis for representation by the generalised strain energy function present in the section that follows.

Generalised Strain Energy (Gse)

GSE formulation

Given a fourth order tensor possessing major symmetry $\mathbb{B}=\mathbb{B}^{\mathrm{T}}$, we note the identity

$$\mathbf{A}:\mathbb{B}:\mathbf{C}=\mathbb{B}::\left(\mathbf{A}\otimes\mathbf{C}\right), \tag{14}$$

where the operator $\otimes$ is the tensor product used by Itskov[6], $\left(\mathbf{A}\otimes\mathbf{B}\right)_{ijkl}=A_{ij}B_{kl}$. Using Eq. (14), we can represent the classical linear Hooke's Law

$$\tilde{W}=\tfrac{1}{2}\boldsymbol{\varepsilon}:\bar{\mathbb{C}}:\boldsymbol{\varepsilon}\quad\text{as}\quad\tilde{W}=\tfrac{1}{2}\bar{\mathbb{C}}::\left(\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}\right). \tag{15}$$

For a general representation of the model, we allow any order n of strain as per the Seth–Hill formula in Eq. (7), and so define a fourth-order tensor

$$\underset{n}{\mathbb{E}}=\underset{n}{\mathbf{E}}\otimes\underset{n}{\mathbf{E}}. \tag{16}$$

This yields a general model than can encompass the sum of any number of strain orders in consideration of the repeated indices on one side of the equation (Einstein summation convention):

$$W=\tfrac{1}{2}\underset{n}{\mathbb{C}}::\underset{n}{\mathbb{E}}. \tag{17}$$

The capacity of this to represent various models will become apparent in the coming sections. Further to this, we can define a fourth-order *Strain Energy Tensor* (SET), which maintains identity to the components of strain energy:

$$\mathbb{W} = \tfrac{1}{2} \underset{n}{\mathbb{C}} \circ \underset{n}{\mathbb{E}} \tag{18}$$

This can be simply reduced back to the scalar value by summation of all elements of the tensor, as

$$W = \sum_{i,j,k,l} W_{ijkl} \,. \tag{19}$$

The form of Eq. (17) is not limited to a typical Hookean stiffness tensor– as mentioned, the summation index n refers to the order identifier of the general strain, but it also has a corresponding component of $\mathbb{C}$ such that $\mathbb{C} = \sum_n \underset{n}{\mathbb{C}}$, where in the isotropic form it is split up into two fourth-order tensors separating the Lamé parameters, i.e.

$$\boxed{W = \tfrac{1}{2}\left(\underset{m}{\mathbb{L}} :: \underset{m}{\mathbb{E}} + \underset{n}{\mathbb{G}} :: \underset{n}{\mathbb{E}} \right)} \,, \tag{20}$$

where

$$\mathbb{C} = \mathbb{L} + \mathbb{G} \,, \quad \underset{\text{iso}}{\mathbb{L}} = \lambda.\mathbf{I} \otimes \mathbf{I} \,, \quad \underset{\text{iso}}{\mathbb{G}} = \mu.\mathbf{I} \odot \mathbf{I} \tag{21}$$

Eq. (20) has the capacity to encompass a wide range of existing strain energy functions with no approximation. Essentially, it is the transformation of the function of strain energy from the scalar product of *scalar parameters* and *invariants of strain* into the quadruple contractions of *fourth-order material tensors* and *fourth-order strains*.

Simo and Pister in GSE form

With the development on the GSE in Eqs. (17) and (20) we can now easily transform the strain energy function of Simo and Pister, represented in a general scalar form in Eq. (12), simply by specifying the order of the strains:

$$W = \tfrac{1}{2}\left(\mathbb{L} :: \underset{0}{\mathbb{E}} + \mathbb{G} :: \underset{\Sigma}{\mathbb{E}} \right) \tag{22}$$

In the case of the series strain, the corresponding fourth-order tensor is defined as

$$\underset{\Sigma}{\mathbb{E}} = \sum_{n=0}^{\infty} \left(\tfrac{1}{2^{n+1}} \underset{2^{-n}}{\mathbf{E}} \otimes \underset{2^{-n}}{\mathbf{E}} \right). \tag{23}$$

Eq. (22) is an exact representation of Simo and Pister's isotropic hyperelastic model, though it is now in a form that is conducive to the introduction of direction dependence.

Intrinsic-Field Tensors: Strain

Deformation IFTs

Earlier in this we referred to the classical stretch tensor **U**, we must now differentiate the stretch based on the material property-based domain. We shall introduce E to represent the domain of isotropic materials and the domain Œ to represent orthotropic materials. The property of symmetry is herein only afforded to the stretch tensor existing within the domain of isotropy, hence $\mathbf{U}^{\mathrm{E}}$.

The well-known polar decomposition of the deformation gradient **F** into stretch and rotation **R** can be represented in isotropic parts

$$\mathbf{F} = \mathbf{R}^{\mathrm{E}} \cdot \mathbf{U}^{\mathrm{E}} \quad , \quad \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} = \left(\mathbf{U}^{\mathrm{E}} \right)^{\mathrm{T}} \cdot \mathbf{U}^{\mathrm{E}} \,, \tag{24}$$

where the former equation indicates the multiplicative decomposition, and the latter implies the unitary and orthogonal nature of **R**. These equations are indeterminate – they have infinite solutions – and so in mechanics we impose a symmetry condition onto the stretch tensor as follows:

$$\left(\mathbf{U}^{\mathrm{E}}\right)^{\mathrm{T}}=\mathbf{U}^{\mathrm{E}} \quad \therefore \quad \mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}=\left(\mathbf{U}^{\mathrm{E}}\right)^{2} \tag{25}$$

In the present method, we require a stretch whereby the condition of symmetry is removed, which it turns out is only necessary for orthotropic and anisotropic domains. Thus the stretch tensor $\mathbf{U}^{\text{Œ}}$ is potentially asymmetric. This has been published in the thesis by Kellermann[2]. Hence Eq. (24) remains similar

$$\mathbf{F}=\mathbf{R}^{\text{Œ}}\cdot\mathbf{U}^{\text{Œ}} \quad , \quad \mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}=\left(\mathbf{U}^{\text{Œ}}\right)^{\mathrm{T}}\cdot\mathbf{U}^{\text{Œ}}, \tag{26}$$

while the enforced symmetry is replaced by dependence on the $\mathrm{R}^{\text{Œ}}$, the IFT rotation as a function of the Rodrigues Rotation Vector **Ω**, hence

$$\mathbf{U}^{\text{Œ}}=\left[\mathbf{R}^{\text{Œ}}(\mathbf{\Omega})\right]^{\mathrm{T}}\cdot\mathbf{F}. \tag{27}$$

We do not go into detail of the physical implications of IFTs here, though it should be noted that Eq. (27)is solved simply by minimisation of the strain energy function, the variables being the components of the Rodrigues vector. The result is an asymmetric stretch tensor such that $U^{\text{Œ}}{}_{ij}\neq U^{\text{Œ}}{}_{ji}$.

Generalised strain as and IFT

It follows from the Seth–Hill strain in Eq. (7) and the redefinition of stretch in Eq. (27) that we can define a new IFT form of generalised strain:

$$\underset{\mathrm{n}}{\mathbf{Œ}}=\tfrac{1}{n}\left[\left(\mathbf{U}^{\text{Œ}}\right)^{n}-\mathbf{I}\right] \tag{28}$$

This measure is for the domain of orthotropic continua, and is not limited to positive integers, indeed negative values yield Eulerian measures; and, fractions to the limit of zero (the logarithmic strain as an IFT) are similarly useful.

Material Tensors For IFTs

Orthotropic Hookean tensors for IFTs

Since IFT theory differentiates between in-plane shear components[7], we require additional shear parameters in the sense that *xy* and *yx* properties become unique. This is quite a natural extension, as it simply means using the 9×9 stiffness matrix that follows from a 3×3×3×3 material tensor. The most compact form of such properties uses indicial notation, where the compliance material tensor $\mathbb{S}$ in the orthotropic orientation denoted by ${}^{\{\mathcal{M}\}}(\cdot)$ is expressed as

$${}^{\{\mathcal{M}\}}\mathbb{S}_{ijkl}=\left(\delta_{ik}\delta_{jl}\left(1+\nu_{ji}\right)-\delta_{ij}\delta_{kl}\nu_{lj}\right)\Big/\underline{\underline{E}}_{l}, \tag{29}$$

where δ_{ij} is the Kronecker delta, $\underline{\underline{E}}_{i}$ are the components of the Young's Modulus vector and ν_{ij} are the components of the Poisson Ratio matrix (see Reference [7]).
Representing the compliance tensor in flattened matrix form $[\mathbb{S}]$, it then is inverted as shown

$$[\mathbb{C}]=[\mathbb{S}]^{-1} \tag{30}$$

to yield the orthotropic Hookean material tensor $\underset{\mathrm{orth}}{\mathbb{C}}$ for use with IFTs.

Orthotropic Lamé tensors for IFTs

Various previous efforts have proposed a set of "orthotropic Lamé parameters", though none meet a very simple requirement set out here:

a) Reduces to the two isotropic Lamé material tensors in Eq. (21) when properties are isotropic
b) Addition of each yields the Hookean orthotropic material tensor of Eqs. (29) and (30), ensuring consistent tangent stiffness

The resulting proposed orthotropic Lamé tensors are $\underset{\text{orth}}{\mathbb{L}}$ and $\underset{\text{orth}}{\mathbb{G}}$ corresponding to λ and μ in isotropy. These are given for both compliance and stiffness in two-dimension as follows.

$$\left[\underset{\text{orth}}{\mathbb{L}}\right]=\begin{bmatrix} \frac{1}{2}\left(\nu_{23}+\nu_{32}+\nu_{23}{}^{2}+\nu_{32}{}^{2}\right)\tilde{E}_1/\bar{\nu} & \left(\nu_{21}+\nu_{23}\nu_{31}\right)\tilde{E}_1/\bar{\nu} & 0 & 0 \\ \left(\nu_{12}+\nu_{13}\nu_{32}\right)\tilde{E}_2/\bar{\nu} & \frac{1}{2}\left(\nu_{31}+\nu_{13}+\nu_{31}{}^{2}+\nu_{13}{}^{2}\right)\tilde{E}_2/\bar{\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[\underset{\text{orth}}{\mathbb{G}}\right]=\begin{bmatrix} \left(2+\nu_{23}+\nu_{32}\right)\left(1-\nu_{23}-\nu_{32}\right)\tilde{E}_1/2\bar{\nu} & 0 & 0 & 0 \\ 0 & \left(2+\nu_{31}+\nu_{13}\right)\left(1-\nu_{31}-\nu_{13}\right)\tilde{E}_2/2\bar{\nu} & 0 & 0 \\ 0 & 0 & \tilde{E}_2/\left(1+\nu_{21}\right) & 0 \\ 0 & 0 & 0 & \tilde{E}_1/\left(1+\nu_{12}\right) \end{bmatrix} \tag{31}$$

$$\left[\underset{\text{orth}}{\mathbb{C}}\right]=\left[\underset{\text{orth}}{\mathbb{L}}\right]+\left[\underset{\text{orth}}{\mathbb{G}}\right]=\begin{bmatrix} \left(1-\nu_{23}\nu_{32}\right)\tilde{E}_1/\bar{\nu} & \left(\nu_{21}+\nu_{23}\nu_{31}\right)\tilde{E}_1/\bar{\nu} & 0 & 0 \\ \left(\nu_{12}+\nu_{13}\nu_{32}\right)\tilde{E}_2/\bar{\nu} & \left(1-\nu_{31}\nu_{13}\right)\tilde{E}_2/\bar{\nu} & 0 & 0 \\ 0 & 0 & \tilde{E}_2/\left(1+\nu_{21}\right) & 0 \\ 0 & 0 & 0 & \tilde{E}_1/\left(1+\nu_{12}\right) \end{bmatrix}$$

where $\bar{\nu}=1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{21}\nu_{32}\nu_{13}$.

Orthotropic Simo And Pister Model

Orthotropic Simo and Pister using GSE

Having shown the Simo and Pister model in the form of the GSE in Eq. (22), having presented the equivalent intrinsic-field tensors for the strains in orthotropy in Eq. (28) and having given the orthotropic equivalent of the fourth-order Lamé tensors in Box 1 we are able to convert Simo and Pister isotropic hyperelasticity into a fully logically-compliant hyperelastic model. This is achieved through the trivial step of replacing $\underset{\text{iso}}{\mathbb{L}}$ with $\underset{\text{orth}}{\mathbb{L}}$, $\underset{\text{iso}}{\mathbb{G}}$ with $\underset{\text{orth}}{\mathbb{G}}$ and $\underset{n}{\mathbb{E}}=f\left(\underset{n}{\mathbf{E}}\right)$ with $\underset{n}{\mathbb{E}}=f\left(\underset{n}{\boldsymbol{\OE}}\right)$.

The resulting formula, expressed entirely as fourth-order tensors, is

$$\underset{\text{orth}}{W}=\tfrac{1}{2}\left(\mathbb{L}::\underset{0}{\mathbb{E}}+\mathbb{G}::\underset{\Sigma}{\mathbb{E}}\right) \quad , \quad \underset{n}{\mathbb{E}}=\underset{n}{\boldsymbol{\OE}}\otimes\underset{n}{\boldsymbol{\OE}} \tag{32}$$

with the more familiar form as follows. Orthotropic Simo and Pister Hyperelasticity:

$$\boxed{\underset{\text{orth}}{W}=\tfrac{1}{2}\left(\underset{0}{\boldsymbol{\OE}}:\mathbb{L}:\underset{0}{\boldsymbol{\OE}}+\underset{\Sigma}{\boldsymbol{\OE}}:\mathbb{G}:\underset{\Sigma}{\boldsymbol{\OE}}\right)} \tag{33}$$

In the next section we will complete the development of the equation by demonstrating that it has the correct tangent stiffness.

Linearisation back to Hooke's law

Finally we can demonstrate that the proposed model reduces back to orthotropic Hooke's law for IFTs, which has been shown to have identical strain energy to classical orthotropic Hooke's law. Physically, this also shows that the tangent stiffness of the proposed orthotropic hyperelastic model is consistent with classical elasticity. As deformation gradient gets very close to the identity tensor, all strain measures linearise to the infinitesimal strain measure of Cauchy, though in the case of IFTs it is asymmetric:

$$\text{as } \mathbf{F} \to \mathbf{I}\,, \quad \underset{n}{\mathbf{E}} \to \underset{n}{\tilde{\mathbf{E}}} = \tilde{\mathbf{E}} = \boldsymbol{\varepsilon}\,, \quad \underset{n}{\mathbf{Œ}} \to \underset{n}{\tilde{\mathbf{Œ}}} = \tilde{\mathbf{Œ}} \tag{34}$$

Thus equation (33) can be factored by the identical linear strain measures as

$$\underset{\text{orth}}{\tilde{W}} = \tfrac{1}{2}\tilde{\mathbf{Œ}} : \left(\mathbb{L} + \mathbb{G}\right) : \tilde{\mathbf{Œ}} \tag{35}$$

and from Box 1 we know that the two orthotropic Lamé tensor combine to give the extended orthotropic Hookean material tensor $\underset{\text{orth}}{\mathbb{C}} = \underset{\text{orth}}{\mathbb{L}} + \underset{\text{orth}}{\mathbb{G}}$. Hence Eq. (35) returns to the familiar form

$$\begin{aligned} \underset{\text{orth}}{\tilde{W}} &= \tfrac{1}{2}\tilde{\mathbf{Œ}} : \underset{\text{orth}}{\mathbb{C}} : \tilde{\mathbf{Œ}} \\ &= \tfrac{1}{2}\boldsymbol{\varepsilon} : \underset{\text{orth}}{\bar{\mathbb{C}}} : \boldsymbol{\varepsilon} \quad \text{Orthotropic Hooke's Law} \end{aligned}, \tag{36}$$

where $\underset{\text{orth}}{\bar{\mathbb{C}}}$ is the classical orthotropic Hookean material tensor for stiffness. The proof of the equality between lines in Eq. (36) is obtained by using a mixing equation to generate the classical 'combined' in-plane shear moduli and then finding that the energies are always identical since

$$\tfrac{1}{2}\left(\tilde{Œ}_{ij} + \tilde{Œ}_{ji}\right) = \varepsilon_{ij}\,. \tag{37}$$

Thus classical tangent stiffness is guaranteed in the proposed model, and for that matter, any orthotropic hyperelastic model of the form of Eq. (20).

Conclusion

In this paper a new class of hyperelastic, orthotropic strain energy functions is introduced by way of demonstrating the conversion of the well-known Simo and Pister model. This is done by first elevating the model from being *isotropic & hyperelastic* to being *orthotropic & hyperelastic*, and then reducing the *orthotropic & hyperelastic* model to being *orthotropic & infinitesimal*. Both the start point (Simo and Pister's model) and the end points (Hookean infinitesimal orthotropy) are widely accepted models, and no approximations are made from the transition from one to the other. The resulting midpoint, the hyperelastic, orthotropic Simo and Pister model, maintains all the desirable qualities of its isotropic counterpart and of Hookean orthotropy. This alone should serve as a compelling argument for the introduction of intrinsic-field tensors and the greater proposed theory of *Orthotropic Continuum Mechanics* into the domain of contemporary continuum mechanics at large. This is by no means a specialised theory – its ability to encompass and adapt to a wide range of applications should be evident through the mathematics alone.

References

[1] J. C. Simo and K. S. Pister, "Remarks on rate constitutive equations for finite deformation problems: computational implications," *Computer Methods in Applied Mechanics and Engineering*, vol. 46, no. 2, pp. 201–215, 1984.

[2] D. C. Kellermann, "A theory of Strongly orthotropic continuum mechanics," The University of New South Wales, 2008.

[3] S. Klinkel, C. Sansour, and W. Wagner, "An anisotropic fibre-matrix material model at finite elastic-plastic strains," *Computational Mechanics*, vol. 35, no. 6, pp. 409–417, May 2005.

[4] R. Hill, "On constitutive inequalities for simple materials--II," *Journal of the Mechanics and Physics of Solids*, vol. 16, no. 5, pp. 315–322, Aug. 1968.

[5] B. R. Seth, "Generalized strain measure with applications to physical problems," *Second-order effects in elasticity, plasticity and fluid dynamics*, pp. 162–172, 1964.

[6] M. Itskov and A. E. Ehret, "A Universal Model for the Elastic, Inelastic and Active Behaviour of Soft Biological Tissues," *GAMM-Mitteilungen*, vol. 32, no. 2, pp. 221–236, 2009.

[7] D. C. Kellermann, T. Furukawa, and D. W. Kelly, "Strongly orthotropic continuum mechanics and finite element treatment," *International Journal for Numerical Methods in Engineering*, vol. 76, no. 12, pp. 1840–1868, 2008.