Finite integration method for solving multi-dimensional partial differential equations

***P. H. Wen[1,2], C.S. Chen[2,3], Y. C. Hon[2,4], M. Li[2]**

[1] School of Engineering and Materials Science, Queen Mary, University of London, London E1 4NS, UK
[2]College of Mathematics, Taiyuan University of Technology, Taiyuan, China
[3] Department of Mathematics, University of Southern Mississippi, Hattiesburg, MS 39406, USA
[4]Department of Mathematics, City University of Hong Kong, Hong Kong SAR, China

*Corresponding author: p.h.wen@qmul.ac.uk

Abstract

Based on the recently developed Finite Integration Method (FIM) for solving one-dimensional ordinary and partial differential equations, this paper extends the technique to higher dimensional partial differential equations. The main idea is to extend the first order finite integration matrices constructed by using either Ordinary Linear Approach (OLA) (uniform distribution of nodes) or Radial Basis Function (RBF) interpolation (uniform/random distributions of nodes) to higher order integration matrices. Illustrative two-dimensional numerical examples are given in two-dimension to compare the FIM (FIM-OLA and FIM-RBF) with the Finite Difference Method and Point Collocation Method to demonstrate its superior accuracy and efficiency.

Keywords: finite integration method, radial basis functions, partial differential equation.

Introduction

Mathematical models in terms of partial differential equations (PDEs) have commonly been used to describe a wide variety of physical phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow, and elasticity. Under various boundary conditions, it is very rare that these models can be solved in closed form solutions. Numerical methods are unavoidable for seeking approximate solutions to simulate the dynamic and characteristics of the models. Due to the advance of computational methods, these kinds of numerical approximation can usually be achieved inexpensively to high accuracy together with a reliable bound on the error between the analytical solution and its numerical approximation. There are many numerical techniques available for solving differential equations [Lambert (1991), Hairer (1993)] including the Finite Element Method (FEM) and Boundary Element Method (BEM). In the last decade, the development of the Radial Basis Functions (RBFs) as a truly meshless method has drawn attention from many researchers. In particular, the use of multiquadric radial basis function (MQ-RBF) [Hardy (1971), Goldberg and Chen (1997), Hon and Mao (1997), Atluri (2002), Liu (2003),] has shown the superior convergence of the method in comparing with FEM and BEM. Numerical results indicated that these meshless methods provide a similar optimal accuracy for solving both elliptic and parabolic equations in 2D. Recently, Wen et al (2013) and Li et al (2013) developed a Finite Integration Method (FIM) for solving differential equation in 1D and demonstrated its applications to nonlocal elasticity problems. It has been shown that the FIM gives higher degree of accuracy than the Finite Difference Method (FDM) and Point Collocation Method (PCM). In this paper, the FIM is further extended to solve multi-dimensional partial differential equations. Two-dimensional partial differential equations are given in illustrative examples. Similar to the FDM and the PCM, a finite number of points, known as field points, are distributed in the computational domain. The field points are generated either uniformly (grid) along the independent coordinate or randomly in the domain. The integration matrix of the first order is obtained by the direct integration with either OLA approximation. Based on these first order integration matrices, any finite integration matrix

with multi-layer integration can easily be obtained. To compare with other numerical methods, the PCM and analytical solution are used.

FIM for one-dimensional problems

Numerical quadrature rule based on Ordinary Linear Approach (OLA) is the simplest computational scheme for integration [see Wen et al (2013)]. Starting from one-dimension problem, an integral of a given function $u(x)$ can be written as

$$U(x)=\int_0^x u(\xi)d\xi \tag{1}$$

Applying the linear interpolation technique to Eq. (1), we have

$$U(x_k)=\int_0^{x_k} u(\xi)d\xi=\sum_{i=1}^{k} a_{ki}u(x_i) \tag{2}$$

where, using trapezoidal rule,

$$a_{1i}=0 \tag{3}$$

$$a_{ki}=\begin{cases}\Delta, & i=1,\\ 0.5\Delta, & i=2,3,...,k-1,\\ 0.5\Delta, & i=k,\\ 0, & i>k,\end{cases} \tag{4}$$

and $x_i=\Delta*(i-1)$, $\Delta=b/(N-1)$, $i=1,2,..,N$ are nodal points in [0, b], and $x_1=0, x_N=b$. Note that Eq. (2) can be written in a matrix form as

$$\mathbf{U}=\mathbf{A}\mathbf{u} \tag{5}$$

where $\mathbf{U}=\left[U_1,U_2,...,U_N\right]^T$ $\mathbf{u}=\left[u_1,u_2,...,u_N\right]^T$, the first order integration matrix

$$\mathbf{A}=(a_{ki})=\Delta\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 1/2 & 1/2 & 0 & 0 & 0 & 0\\ 1/2 & 1 & 1/2 & 0 & 0 & 0\\ 1/2 & 1 & 1 & 1/2 & 0 & 0\\ ... & ... & ... & ... & ... & ...\\ 1/2 & 1 & 1 & 1 & 1 & 1/2\end{pmatrix}_{N\times N}$$

.

and $U_i=U(x_i)$, $u_i=u(x_i)$ are the values of integration and the integral function respectively at each nodes. Thereafter, consider a multi-integral for one-dimensional problem

$$U^{(2)}(x)=\int_0^x\int_0^\zeta u(\xi)d\xi d\zeta, \qquad x\in[0,b]. \tag{6}$$

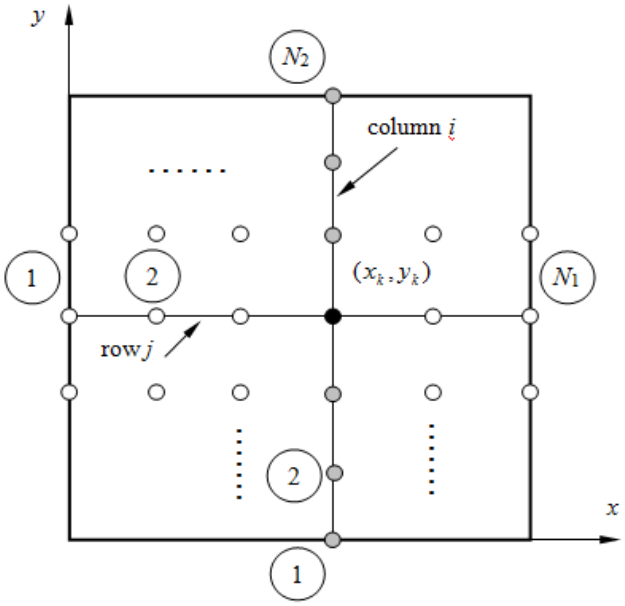

Figure 1. Uniform distribution of collocation points.

$$U^{(2)}(x_k)=\int_0^{x_k}\int_0^\zeta u(\xi)d\xi d\zeta=\sum_{i=0}^{k}\sum_{j=0}^{i}a_{ki}a_{ij}u(x_i)=\sum_{i=0}^{k}a_{ki}^{(2)}u(x_i) \tag{7}$$

.

The above multi-integral can also be written in a matrix form as

$$\mathbf{U}^{(2)}=\mathbf{A}^{(2)}\mathbf{u}=\mathbf{A}^2\mathbf{u} \tag{8}$$

where

$$\mathbf{A}^{(2)} = \mathbf{AA} = \Delta^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 3/4 & 1 & 1/4 & 0 & 0 & 0 \\ 5/4 & 2 & 1 & 1/4 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ [1+2(N-1)]/4 & N-1 & N-2 & \dots & 1 & 1/4 \end{pmatrix}_{N\times N} \tag{9}$$

and the elements of matrix $\mathbf{A}^{(2)}$ are

$$a_{1i}^{(2)} = 0 \tag{10}$$

$$a_{ki}^{(2)} = \begin{cases} [1+2(k-2)]\Delta^2/4, & i=1, \\ (k-i)\Delta^2, & i=2,3,\dots,k-1, \\ \Delta^2/4, & i=k, \\ 0, & i>k. \end{cases}$$

For two-dimensional problems, let us consider a uniform distribution of collocation points as shown in Fig. 1. Similar to Eq. (1), we define

$$U_x(x,y) = \int_0^x u(\xi,y)d\xi, U_x(x_k,y_k) = \int_0^{x_k} u(\xi,y_k)d\xi \tag{11}$$

and the total number of point is $k = N_1(j-1)+i$, where i and j denote the number of column and the number of row respectively. This numbering system is called the global number system. We can also express each nodal value of integration in Eq. (11) in a matrix form as

$$\mathbf{U}_x = \mathbf{A}_x\mathbf{u} \tag{12}$$

where integral nodal value $U_x = [U_{x1}, U_{x2}, \dots, U_{xM}]^T$, nodal value $\mathbf{u} = [u_1, u_2, \dots, u_M]^T$ and M is the total number of collocation points ($M = N_1 \times N_2$ for grid shown in Fig. 1). For a rectangular domain, the first order integration matrix

$$\mathbf{A}_x = \underbrace{\begin{pmatrix} \mathbf{A} & 0 & \dots & 0 \\ 0 & \mathbf{A} & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \mathbf{A} \end{pmatrix}}_{N_2} \tag{13}$$

in which, $\mathbf{A}$ is integration matrix for one-dimension given in Eq. (5) with dimension $N_1 \times N_1$. Similarly, the integration along y axis is

$$U_y(x,y) = \int_0^y u(x,\eta)d\eta, \quad U_y(x_k,y_k) = \int_0^{y_k} u(x_k,y)dy \tag{14}$$

which can be written in the matrix form as

$$\mathbf{U}_y = \mathbf{A}'\mathbf{u} \tag{15}$$

in the local system for the collocation points, where $k = N_2(i-1)+j$. The first order integration matrix in the local system is

$$\mathbf{A}' = \underbrace{\begin{pmatrix} \mathbf{A} & 0 & \dots & 0 \\ 0 & \mathbf{A} & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \mathbf{A} \end{pmatrix}}_{N_2} \tag{16}$$

in which $\mathbf{A}$ is the integration matrix for one-dimension integral given in Eq. (5) with dimension $N_2 \times N_2$. By a simple re-arrangement of the number of the nodes, Eq. (15) can be rewritten, in the global system, as

$$\mathbf{U}_y = \mathbf{A}_y\mathbf{u}. \tag{17}$$

For the multi-integration in two-dimensional problem in a rectangular domain, we consider the following integral with respect to coordinate x

$$U_x^{(2)}(x_i,y) = \int_0^{x_i}\int_0^{\zeta} u(\xi,y)d\xi d\zeta, \qquad x_i \in [0,b_1],\ y \in [0,b_2] \tag{18}$$

and use the same procedure for one-dimension, one has

$$U^{(2)}(x_k,y_k) = \int_0^{x_k}\int_0^{\zeta} u(\xi,y_k)d\xi d\zeta = \sum_{i=0}^{k}\sum_{j=0}^{i}(a_{kj})_x(a_{ji})_x u_i \tag{19}$$

or in a matrix form

$$\mathbf{U}_x^{(2)} = \mathbf{A}_x^2\mathbf{u} \tag{20}$$

where

$$\mathbf{A}_x^2 = \mathbf{A}_x \mathbf{A}_x = \underbrace{\begin{pmatrix} \mathbf{A}^2 & 0 & \dots & 0 \\ 0 & \mathbf{A}^2 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \mathbf{A}^2 \end{pmatrix}}_{N_2} \tag{21}$$

Similarly, one has multi-integration $U_y^{(2)}(x, y)$ with respect to coordinate *y*

$$U_y^{(2)}(x_k, y_k) = \int_0^{y_k} \int_0^{\tau} u(x_k, \eta) d\eta d\tau = \sum_{i=0}^{k} \sum_{j=0}^{i} (a_{kj})_y (a_{ji})_y u_i \tag{22}$$

and

$$\mathbf{U}_y^{(2)} = \mathbf{A}_y^2 \mathbf{u}. \tag{23}$$

This method can be extended to the higher order integrations, i.e.

$$U_x^{(m)}(x_k, y_k) = \underbrace{\int_0^{x_k} \dots \int_0^{\dots} u(\xi_1, y_k) d\xi_1 \dots d\xi_m}_{m \text{ layers}}, \; U_y^{(m)}(x_k, y_k) = \underbrace{\int_0^{y_k} \dots \int_0^{\dots} u(x_k, \eta_1) d\eta_1 \dots d\eta_m}_{m \text{ layers}} \tag{24}$$

$$x_k \in [0, b_1], y_k \in [0, b_2]$$

Applying ordinary linear interpolation technique again for integral function $U^{(m)}(x, y)$, we have

$$U_x^{(m)}(x_k, y_k) = \int_0^{x_k} \dots \int_0^{\dots} u(\xi_1, y_k) d\xi_1 \dots d\xi_1 = \sum_{i=0}^{M} \dots \sum_{j=0}^{M} (a_{kj})_x \dots (a_{ni})_x u(x_i, y_i) = \sum_{i=0}^{M} \left(a_{ki}^{(m)}\right)_x u_i \tag{25}$$

$$U_y^{(m)}(x_k, y_k) = \int_0^{y_k} \dots \int_0^{\dots} u(x_k, \eta_1) d\eta_1 \dots d\eta_m = \sum_{i=0}^{M} \dots \sum_{j=0}^{M} (a_{kj})_y \dots (a_{ni})_y u(x_i, y_i) = \sum_{i=0}^{M} \left(a_{ki}^{(m)}\right)_y u_i \tag{26}$$

Again, it can also be written, in a matrix form, as

$$\mathbf{U}_x^{(m)} = \mathbf{A}_x^m \mathbf{u}, \quad \mathbf{U}_y^{(m)} = \mathbf{A}_y^m \mathbf{u} \tag{27}$$

In addition, this method can be extended to multi-layers integration with two coordinates *x* and *y* as follow:

$$U^{(mn)}(x_k, y_k) = \int_0^{x_k} \dots \int_0^{\dots} \int_0^{y_k} \dots \int_0^{\dots} u(\xi, \eta) \underbrace{d\xi_1 \dots d\xi_n}_{n-\text{layer}} \underbrace{d\eta_1 \dots d\eta_m}_{m-\text{layer}} \qquad x_k \in [0, b_1], \; y_k \in [0, b_2] \tag{28}$$

and the nodal values of the above integration are obtained in the matrix form as

$$\mathbf{U}^{(mn)} = \mathbf{A}_x^m \mathbf{A}_y^n \mathbf{u}. \tag{29}$$

FIM with radial basis functions

For uniform distribution of nodes (grid), the multi-layer integrations at each node can be obtained quite easily in a matrix form. However, in general case, if the nodes distribution is random, the algorithm OLA discussed in the Section above is not valid. In this case, interpolation schemes have to be introduced. Recently, the radial basis functions interpolation schemes and moving least square method are very popular meshless methods. For example, the MQ-RBF was introduced by Hardy (1971) for the interpolation of topographical surfaces in the early stage of radial bases function application. Note that $u(\mathbf{x})$ in the domain Ω can be interpolated over a number of randomly distributed nodes $\mathbf{x}_i = (x_i, y_i)$, $i = 1,2,...,M$, as

$$u(\mathbf{x}) = \sum_{i=1}^{M} R_i(\mathbf{x}, \mathbf{x}_i) \alpha_i + \sum_{q=1}^{Q} P_q(\mathbf{x}) \beta_q = \mathbf{R}(\mathbf{x})\boldsymbol{\alpha} + \mathbf{P}(\mathbf{x})\boldsymbol{\beta} = \sum_{i=1}^{M} \varphi_i(\mathbf{x}) u_i \quad \mathbf{x} \in \Omega, \tag{30}$$

where $\mathbf{R}(\mathbf{x})=\left[R_1(\mathbf{x},\mathbf{x}_1),R_2(\mathbf{x},\mathbf{x}_2),...,R_M(\mathbf{x},\mathbf{x}_M)\right]$ is a set of radial basis functions centred at $\mathbf{x}=(x,y)$, $\boldsymbol{\alpha}=[\alpha_1,\alpha_2,\cdots,\alpha_M]^T$ and $\boldsymbol{\beta}=[\beta_1,\beta_2,\cdots,\beta_Q]^T$ are the coefficients to be determined, $\varphi_i(\mathbf{x})$ is shape function. Therefore, the integration matrices of the first order are

$$\mathbf{A}_x=\begin{pmatrix}\bar{\phi}_{x1}(\mathbf{x}_1) & \bar{\phi}_{x2}(\mathbf{x}_1) & ... & \bar{\phi}_{xM}(\mathbf{x}_1)\\ \bar{\phi}_{x1}(\mathbf{x}_2) & \bar{\phi}_{x2}(\mathbf{x}_2) & ... & \bar{\phi}_{xM}(\mathbf{x}_2)\\ ... & ... & ... & ...\\ \bar{\phi}_{x1}(\mathbf{x}_M) & \bar{\phi}_{x2}(\mathbf{x}_M) & ... & \bar{\phi}_{xM}(\mathbf{x}_M)\end{pmatrix}_{M\times M} \quad \text{and} \quad \mathbf{A}_y=\begin{pmatrix}\bar{\phi}_{y1}(\mathbf{x}_1) & \bar{\phi}_{y2}(\mathbf{x}_1) & ... & \bar{\phi}_{yM}(\mathbf{x}_1)\\ \bar{\phi}_{y1}(\mathbf{x}_2) & \bar{\phi}_{y2}(\mathbf{x}_2) & ... & \bar{\phi}_{yM}(\mathbf{x}_2)\\ ... & ... & ... & ...\\ \bar{\phi}_{y1}(\mathbf{x}_M) & \bar{\phi}_{y2}(\mathbf{x}_M) & ... & \bar{\phi}_{yM}(\mathbf{x}_M)\end{pmatrix}_{M\times M}.$$

where $\bar{\phi}_{xi}=\int\phi_i(\mathbf{x})dx$, $\bar{\phi}_{yi}=\int\phi_i(\mathbf{x})dy$.

The FIM for multi-dimensional problems

The FIM is readily extendable to solving higher dimensional problems. For illustration, consider the following two-dimensional partial differential equation

$$\begin{aligned}&\alpha_1(x,y)\frac{\partial^2 u}{\partial x^2}+\alpha_2(x,y)\frac{\partial^2 u}{\partial y^2}+\alpha_3(x,y)u=b(x,y), && \mathbf{x}\in\Omega,\\ &\Lambda[u(x,y)]=h(x,y), && \mathbf{x}\in\partial\Omega,\end{aligned} \tag{31}$$

where Λ is a boundary operator, $\alpha_1(x,y),\alpha_2(x,y),\alpha_3(x,y),b(x,y)$ and $h(x,y)$ are given functions. u is generally referred as potential, which represents the transversal displacement of a membrane. Ω and $\partial\Omega$ are simple connected domain and its boundary respectively. Integrating twice in Eq. (31) with respect to coordinates x and y respectively, one has

$$\begin{aligned}&\iint\iint\left[\alpha_1(x,y)\frac{\partial^2 u}{\partial x^2}+\alpha_2(x,y)\frac{\partial^2 u}{\partial y^2}+\alpha_3(x,y)u\right]dxdxdydy=\iint\iint b(x,y)dxdxdydy+\\ &xf_0(y)+f_1(y)+yg_0(x)+g_1(x)\end{aligned} \tag{32}$$

where $f_0(y)$, $f_1(y)$, $g_0(x)$ and $g_1(x)$ are unknown one-dimensional functions. Using the technique of integration by part, we have

$$\begin{aligned}&\iint\left[\alpha_1 u-2\int u\frac{\partial\alpha_1}{\partial x}dx+\iint u\frac{\partial^2\alpha_1}{\partial x^2}dxdx\right]dydy+\iint\left[\alpha_2 u-2\int u\frac{\partial\alpha_2}{\partial y}dx+\iint u\frac{\partial^2\alpha_2}{\partial y^2}dydy\right]dxdx+\\ &\iint\iint\alpha_3 udxdxdydy=\iint\iint b(x,y)dxdxdydy+xf_0(y)+f_1(y)+yg_0(x)+g_1(x).\end{aligned} \tag{33}$$

By using integration matrix mentioned in the previous sections, we have

$$\begin{aligned}&\left[\mathbf{A}_y^2\boldsymbol{\alpha}_1+\mathbf{A}_x^2\boldsymbol{\alpha}_2-2\mathbf{A}_x\mathbf{A}_y\left(\mathbf{A}_y\boldsymbol{\alpha}_{1,x}+\mathbf{A}_x\boldsymbol{\alpha}_{2,y}\right)+\mathbf{A}_x^2\mathbf{A}_y^2\left(\boldsymbol{\alpha}_{1,xx}+\boldsymbol{\alpha}_{2,yy}+\boldsymbol{\alpha}_3\right)\right]\mathbf{u}=\\ &\mathbf{A}_y^2\mathbf{A}_x^2\mathbf{b}+\mathbf{X}\boldsymbol{\Psi}_y\mathbf{f}_0+\boldsymbol{\Psi}_y\mathbf{f}_1+\mathbf{Y}\boldsymbol{\Psi}_x\mathbf{g}_0+\boldsymbol{\Psi}_x\mathbf{g}_1\end{aligned} \tag{34}$$

where $\mathbf{X}=\{x_1,x_2,...,x_M\}$, $\mathbf{Y}=\{y_1,y_2,...,y_M\}$, $\mathbf{f}_0=\left[f_0^1,f_0^2,...,f_0^r\right]^{\mathrm{T}}$, $\mathbf{f}_1=\left[f_1^1,f_1^2,...,f_1^r\right]^{\mathrm{T}}$, $\mathbf{g}_0=\left[g_0^1,g_0^2,...,g_0^p\right]^{\mathrm{T}}$, $\mathbf{g}_1=\left[g_1^1,g_1^2,...,g_1^p\right]^{\mathrm{T}}$, p and r are numbers of point to be used for interpolation of functions $f(y)$ and $g(x)$ respectively, $\boldsymbol{\Psi}_x$ and $\boldsymbol{\Psi}_y$ are matrices of one-dimensional shape functions with respect to coordinates x and y respectively, and

$$\boldsymbol{\alpha}_l=\begin{pmatrix}\alpha_l(\mathbf{x}_1) & 0 & ... & 0\\ 0 & \alpha_l(\mathbf{x}_2) & ... & 0\\ ... & ... & ... & ...\\ 0 & 0 & ... & \alpha_l(\mathbf{x}_M)\end{pmatrix} \qquad l=1,2,3 \tag{35}$$

Integral functions $f_0(y)$, $f_1(y)$, $g_0(x)$ and $g_1(x)$ can be interpolated in terms of the nodal values in the following procedure:

(1) Determine the regions of functions $f(y)$ and $g(x)$, i.e. $[\bar{y}_1, \bar{y}_r]$, $[\bar{x}_1, \bar{x}_p]$, and uniformly distributed points in these regions as shown in Fig. 2;
(2) Determine one-dimensional shape function matrices $\mathbf{\Psi}_x$ and $\mathbf{\Psi}_y$

By using linear interpolation, one has

$$\begin{aligned} f(y) &= 0 \quad \text{if} \quad \bar{y}_1 < y < \bar{y}_{m-1} \ \text{or} \ \bar{y}_m < y < \bar{y}_r \\ g(x) &= 0 \quad \text{if} \quad \bar{x}_1 < x < \bar{x}_{n-1} \ \text{or} \ \bar{x}_n < x < \bar{x}_p \end{aligned} \tag{36}$$

$$f(y) = \frac{\bar{y}_m - y}{\bar{y}_m - \bar{y}_{m-1}} f_{m-1} + \frac{y - \bar{y}_{m-1}}{\bar{y}_m - \bar{y}_{m-1}} f_m \quad \text{if} \quad \bar{y}_{m-1} < y < \bar{y}_m$$

$$g(x) = \frac{\bar{x}_n - x}{\bar{x}_n - \bar{x}_{n-1}} g_{n-1} + \frac{x - \bar{x}_{n-1}}{\bar{x}_n - \bar{x}_{n-1}} g_n \quad \text{if} \quad \bar{x}_{n-1} < x < \bar{x}_n \tag{37}$$

Therefore, the matrices of shape function are

$$\mathbf{\Psi}_x = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \dfrac{\bar{x}_n - x_i}{\bar{x}_n - \bar{x}_{n-1}} & \dfrac{x_i - \bar{x}_{n-1}}{\bar{x}_n - \bar{x}_{n-1}} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{matrix} \text{row } 1 \\ \\ i \\ \\ M \end{matrix}$$

$$\begin{matrix} \text{columm } 1 & & n\text{-}1 & n & & p \end{matrix} \tag{38}$$

and

$$\mathbf{\Psi}_y = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \dfrac{\bar{y}_m - y_i}{\bar{y}_m - \bar{y}_{m-1}} & \dfrac{y_i - \bar{y}_{m-1}}{\bar{y}_m - \bar{y}_{m-1}} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{matrix} \text{row } 1 \\ \\ i \\ \\ M \end{matrix}$$

$$\begin{matrix} \text{columm } 1 & & m\text{-}1 & m & & r \end{matrix} \tag{39}$$

in which $\psi_i(x)$ and $\psi_i(y)$ are shape functions in one dimensional case as shown in Fig. 2. In Eq. (34), we have M nodal unknowns of $\mathbf{u}$, $2q$ unknowns of $\mathbf{f}_0$, $\mathbf{f}_1$ and $2r$ unknowns of $\mathbf{g}_0$, $\mathbf{g}_1$.

For a rectangular plate with uniform distribution of nodes $(N_1 \times N_2)$, obviously one has $2p + 2r$ nodes located on the boundary. By selecting $p = N_1 - 1$ and $r = N_2 - 1$ for uniform distribution of node, there are $N_1 \times N_2 + 2(N_1 + N_2 - 2)$ linear system of equations to determine all unknowns, i.e. $\mathbf{u}$, $\mathbf{f}_0$, $\mathbf{f}_1$, $\mathbf{g}_0$, and $\mathbf{g}_1$. In fact, the number of boundary points to determine four one-dimensional integral functions is arbitrary. The number of points (L) on the boundary should be greater than or equal to $2(p+r)$. If $L = 2(p+r)$, the standard Gaussian solver can be used directly. Otherwise, the Singular Value Decomposition [Press et al (1992)] scheme should be introduced.

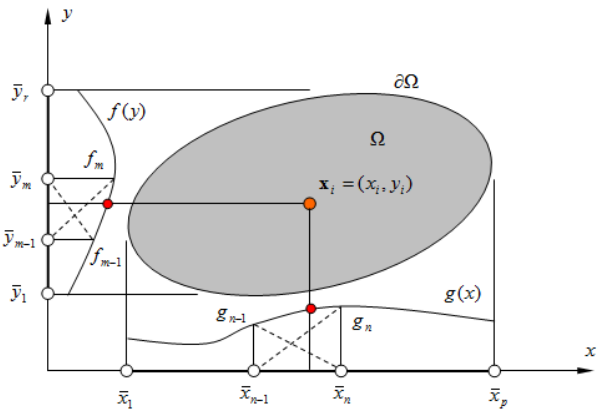

Figure 2. Interpolations for one dimensional function $f(y)$ and $g(x)$.

Numerical example

Consider the following partial differential equation

$$\begin{aligned} x(1-x)\frac{\partial^2 u}{\partial x^2} + y(1-y)\frac{\partial^2 u}{\partial y^2} &= -4xy(1-x)(1-y), \quad (x,y) \in \Omega, \\ u(x,y) &= 0, \qquad\qquad\qquad (x,y) \in \partial\Omega, \end{aligned} \tag{41}$$

where $\Omega \cup \partial\Omega = [0,1] \times [0,1]$. The analytical solution is given by $u^*(x,y) = xy(1-x)(1-y)$. The average relative error is defined as

$$\varepsilon = \frac{1}{M}\sum_{i=1}^{M} \frac{|u_i - u_i^*|}{|u_{\max}^*|}, \quad u_{\max}^* = u^*(0.5, 0.5) = \frac{1}{16} \tag{42}$$

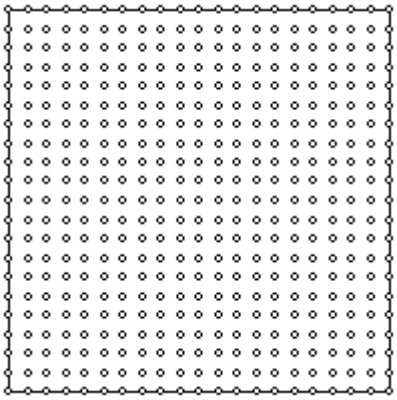

Figure 3. Distributions of nodes.

Uniformly distributed nodes is shown in Fig.3. For radial basis functions approach in Eq. (30), three radial basis functions are considered, i.e.

(1) MQ function: $R(r) = \sqrt{c^2 + r^2}$;

(2) Linear function (LF): $R(r) = r$;

(3) Thin-Plate Splines (TPS): $R(r) = r^2 \ln r$.

In this example, we chose $N_1 = N_2 = N$ and $p = r = N$. The shape parameter c of MQ is selected as $c = 1 / N$. For the RBF approach, single integration matrix in Eq. (70) is used. The average errors ε for various number of collocation point are shown in Table 1. Among these algorithms, the accuracy of OLA is the lowest and PSF of radial basis function is the highest.

Table 1. Average errors (ε).

N	OLA	MQF	LF	TPS
10	0.019110	0.013707	0.017554	0.012985
20	0.005689	0.005365	0.006462	0.004496
30	0.013820	0.003083	0.003671	0.002373

Conclusion

In this paper, the Finite Integration Method (FIM) with Ordinary Linear Approach and Radial Basis Functions interpolation was extended to solve multi-dimensional differential equations. Compared with the Point Collocation Method (PCM) and the Finite Difference Method, the proposed FIM performs much superior in accuracy and stability. For the FIM with Radial Basis Functions interpolation, the use of randomly distributed nodes in the domain allows solving problems under irregular domains.

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