Identification of a Position and Time Dependent Heat Flux Using the Kalman Filter and Improved Lumped Analysis in Heat Conduction

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Abstract

This paper aims to estimate a position and time dependent heat flux with high magnitude in a heat conduction problem. The heat flux is applied on one side of a flat plate, while the inverse problem is solved by using temperature measurements taken on the opposite side. The proposed forward problem is a surrogate model, derived from the simplification of a complete model. The inverse problem is then solved with the Kalman Filter. The temperature at the surface of the plate is approximated by using the improved lumped analysis, where the temperature gradients across the thickness of the plate are accounted for in an approximate manner. The measurements are simulated with the complete model, while the inverse problem is solved with the surrogate model. The temperature estimates show a good agreement with reference values.

Keywords: Inverse Problems, Heat Conduction, Kalman Filter, Improved Lumped Analysis

Introduction

Despite the modern and reliable available techniques for measuring temperature and heat flux, some particular scenarios are still challenging. Situations involving complex geometries or hazardous environments might make direct measurements of these quantities impractical [Dennis and Dulikravich, 2001]. Thus, estimation of these unknowns by using inverse analysis with temperature and/or heat flux measurements taken at other regions of the body of interest should be considered as a possible solution. Situations of this type become increasingly common, for example, due to the recent development of powerful microprocessors, which dissipates high amounts of heat. Techniques for dealing with such thermal loads are available, but new methodologies for proper quantification and more efficient cooling of these thermal loads are desired. Some results on the estimation of a high magnitude boundary heat flux in a heat conduction problem can be found in the literature [Dennis and Dulikravich, 2001; Feng et al., 2011; Dennis and Dulikravich, 2012; Afrin et al., 2013; Orlande et al., 2013]. All of these works emphasize the difficulties of solving the inverse problem with an accurate mathematical model, which would be a three-dimensional nonlinear heat conduction problem, thus resulting in high computational times [Dennis and Dulikravich, 2012]. In this work, the proposed forward problem is obtained by simplification of a more general heat conduction problem which, together with the modeling of uncertainties of observations and unknowns as Gaussians, allows one to use the Kalman filter [Kalman, 1960; Chen, 2003; Kaipio and Somersalo, 2004; Grewal and Andrews, 2008]. The physical problem considered in this paper involves heat conduction in a flat plate, where the temperatures at both surfaces are approximated by the Improved Lumped System Analysis [Cotta and Mikhailov, 1997]. In this formulation, the temperature gradient across the plate is approximated by Hermite's formulae. The use of the Kalman filter requires much less computational effort in comparison with techniques such as particle filters and is more readily adaptable to parallel processing.

Forward Problem

The physical problem considered in this paper involves a high magnitude heat flux applied to the top surface of a flat plate, while temperature measurements are taken at the opposite side, as shown in Fig. 1. The dimensions of the flat plate are given by Tab. 1.

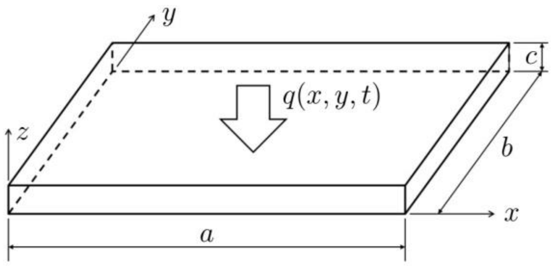


Figure 1: Geometry of the physical problem

Table 1: Dimensions of the flat plate

Dimension	Value [mm]
a	120
b	120
c	3

All other boundaries are thermally insulated. The heat flux is position-and-time dependent and the initial temperature distribution is considered to be uniform. Based on these assumptions, the resulting mathematical model [Ozisik, 1993], named "Complete Model", is given by Eqs. (1.a)-(1.f).

$$C(T_c)\frac{\partial T_c}{\partial t}=\frac{\partial}{\partial x}\left[k_T(T_c)\frac{\partial T_c}{\partial x}\right]+\frac{\partial}{\partial y}\left[k_T(T_c)\frac{\partial T_c}{\partial y}\right]+\frac{\partial}{\partial z}\left[k_T(T_c)\frac{\partial T_c}{\partial z}\right] \text{ in } \begin{matrix} 0\le x\le a, & 0\le y\le b \\ 0\le z\le c, & t\ge 0\end{matrix} \tag{1.a}$$

$$\frac{\partial T_c}{\partial x}=0 \quad \text{at} \quad x=0 \quad \text{and} \quad x=a \tag{1.b}$$

$$\frac{\partial T_c}{\partial y}=0 \quad \text{at} \quad y=0 \quad \text{and} \quad y=b \tag{1.c}$$

$$\frac{\partial T_c}{\partial z}=0 \quad \text{at} \quad z=0 \tag{1.d}$$

$$k_T(T_c)\frac{\partial T_c}{\partial z}=q(x,y,t) \quad \text{at} \quad z=c \tag{1.e}$$

$$T_c=T_0 \quad \text{at} \quad t=0 \tag{1.f}$$

Since high temperature variations are expected, the volumetric heat capacity and the thermal conductivity are supposed to vary with respect to the temperature according to [Orlande et al., 2013]:

$$C(T)=1324.75T+3557900 \ \ [\text{J/m}^3] \tag{2.a}$$

$$k_T(T)=12.45+0.014T+2.517\times 10^{-6}\text{T}^2 \ \ [\text{W/mK}] \tag{2.b}$$

The Kalman filter cannot be used to solve the inverse problem related to the estimation of the applied heat flux using this mathematical model, since it is non-linear. Regarding other techniques, a similar inverse problem solved with the complete model, using the Metropolis-Hastings algorithm in a time range of 2.0 seconds with $\Delta x=\Delta y=5\text{mm}$, $\Delta z=0.5\text{mm}$, $\Delta t=0.01\text{s}$ and 10^5 states of the Markov Chain, led to 8 days of computational time [Orlande et al., 2013]. In order to reduce this extremely high computational cost, a surrogate model is proposed in this paper as described below.

Surrogate Model

The first step to obtain the surrogate model is to linearize the thermal properties, evaluating Eqs. (2.a) and (2.b) at a reference temperature $T^* = 600K$. This gives rise to the constant thermal properties presented on Eq. (3).

$$C^* = C(T^*) \quad \text{and} \quad k_T^* = k_T(T^*) \tag{3}$$

The next step aims to reduce the number of dimensions of the model. This is achieved by calculating the mean temperature in the z direction, using the operator described in Eq. (4).

$$\overline{T}(x,y,t) = \frac{1}{c}\int_0^c T(x,y,z,t)dz \tag{4}$$

Application of this operator in Eq. (1.a) is straightforward, except for the diffusion term in the z-direction, where the result is the heat flux at z = 0 and z = c surfaces of the plate. This result can be combined with the linearized versions of Eq. (1.d) and (1.e), as shown in Eq. (5).

$$\frac{1}{c}\int_0^c \frac{\partial}{\partial z}\left[k_T^* \frac{\partial T}{\partial z}\right]dz = \frac{1}{c}k_T^* \frac{\partial T}{\partial z}\bigg|_0^c = \frac{q(x,y,t)}{c} \tag{5}$$

Operation of the linearized versions of Eqs. (1.b), (1.c) and (1.f), is also straightforward. The final result is the following surrogate model, which is a linear two dimensional problem:

$$C^* \frac{\partial \overline{T}}{\partial t} = k_T^* \frac{\partial^2 \overline{T}}{\partial x^2} + k_T^* \frac{\partial^2 \overline{T}}{\partial y^2} + \frac{q(x,y,t)}{c} \quad \text{in} \quad \begin{matrix} 0 \le x \le a, \quad t \ge 0 \\ 0 \le y \le b \end{matrix} \tag{6.a}$$

$$\frac{\partial \overline{T}}{\partial x} = 0 \quad \text{in} \quad x = 0 \text{ and } x = a \tag{6.b}$$

$$\frac{\partial \overline{T}}{\partial y} = 0 \quad \text{in} \quad y = 0 \text{ and } y = b \tag{6.c}$$

$$\overline{T} = T_0 \quad \text{in} \quad t = 0 \tag{6.d}$$

This model is much simpler to solve than the complete model. However, its solution leads to the mean temperature in the z-direction, but the desired quantity is the temperature at the z = 0 surface. The Improved Lumped Analysis [Cotta and Mikhailov, 1997] allows one to approximate this quantity by using the Hermite's formulas for integrals given by:

$$\int_0^h y(x)dx = \frac{h}{2}\left[y(0) + y(h)\right] + O(h^3) \tag{7.a}$$

$$\int_0^h y(x)dx = \frac{h}{2}\left[y(0) + y(h)\right] + \frac{h^2}{12}\left[\frac{dy}{dx}\bigg|_{x=0} - \frac{dy}{dx}\bigg|_{x=h}\right] + O(h^5) \tag{7.b}$$

These formulas are used to approximate the mean temperature in the z-direction and the integral of the temperature gradient in the z-direction, that is,

$$\overline{T}(x,y,t) \approx \frac{1}{2}\left[T(x,y,0,t) + T(x,y,c,t)\right] + \frac{c}{12}\left[\frac{\partial T}{\partial z}\bigg|_{z=0} - \frac{\partial T}{\partial z}\bigg|_{z=c}\right] \tag{8.a}$$

$$\int_0^c \frac{\partial T(x,y,z,t)}{\partial z} dz = T(x,y,c,t) - T(x,y,0,t) \approx \frac{c}{2}\left[\left.\frac{\partial T}{\partial z}\right|_{z=0} + \left.\frac{\partial T}{\partial z}\right|_{z=c} \right] \tag{8.b}$$

The final result is an approximation of the temperature at the $z = 0$ surface given by:

$$T(x,y,0,t) \approx \overline{T}(x,y,t) - \frac{c}{6k_T^*} q(x,y,t) \tag{9}$$

Inverse Problem

The inverse problem related to the estimation of the applied high intensity heat flux is solved in this paper by using a Bayesian approach. A probability density function (pdf) of the unknown state variables $\mathbf{x}_n$ given the set of observations $\mathbf{y}_{0:n}$ is built with Bayes' Theorem. Statistical inference techniques can be applied to this pdf, called "posterior", to extract information about the unknowns [Chen, 2003]. This work uses the Kalman filter, which requires the forward problem to be cast in the form of the Evolution-Observed Model given by Eqs. (10.a) and (10.b), where $\mathbf{w}_n$ and $\mathbf{v}_n$ are zero mean Gaussian noise vectors, with covariance matrices $\mathbf{Q}_n$ and $\mathbf{R}_n$, respectively.

$$\mathbf{x}_n = \mathbf{F}_n \mathbf{x}_{n-1} + \mathbf{w}_{n-1} \tag{10.a}$$

$$\mathbf{y}_n = \mathbf{H}_n \mathbf{x}_n + \mathbf{v}_n \tag{10.b}$$

The state vector for this problem, given by Eq. (11), is composed by the mean temperature and heat flux values, represented by the vectors $\overline{\mathbf{T}}_n$ and $\mathbf{q}_n$, at each control volume of the discretization grid. Thus, considering a grid with I volumes in the x direction and J volumes in the y direction, the number of unknowns is $2IJ$.

$$\mathbf{x}_n = \begin{bmatrix} \overline{\mathbf{T}}_n \\ \mathbf{q}_n \end{bmatrix} \tag{11}$$

The $\mathbf{F}_n$ matrix for the evolution model, with size $2IJ \times 2IJ$, is built with four matrices of size $IJ \times IJ$, as:

$$\mathbf{F}_n = \begin{bmatrix} \mathbf{A}_n & \mathbf{B}_n \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{12}$$

The $\mathbf{A}_n$ and $\mathbf{B}_n$ matrices result from the discretization of the forward surrogate problem with the explicit scheme of the finite volume method [Patankar, 1980; Versteeg and Malalasekera, 1995; Ferziger and Peric, 2002]. $\mathbf{A}_n$ accounts for the heat diffusion in the domain, and $\mathbf{B}_n$ is given by Eq. (13).

$$\mathbf{B}_n = \frac{\Delta t}{cC^*} \mathbf{I} \tag{13}$$

The $\mathbf{0}$ and $\mathbf{I}$ terms are the zero and identity matrices, where a *random walk model* was used for the evolution of the unknown local heat fluxes. Considering the state noise as uncorrelated and with a standard deviation σ_q, this model can be described by Eq. (14), where $\boldsymbol{\omega}$ is a standard Gaussian vector.

$$\mathbf{q}_n = \mathbf{q}_{n-1} + \sigma_q \boldsymbol{\omega} \tag{14}$$

For the observation model, the matrix $\mathbf{H}_n$ is described by Eq. (15), where the diagonal matrix results from the Improved Lumped formulation:

$$\mathbf{H}_n = \begin{bmatrix} \mathbf{0} & -\dfrac{c}{6k_T^*}\mathbf{I} \end{bmatrix} \tag{15}$$

The Kalman filter is applied to the solution of the present state estimation problem, (see Eqs. (16.a)-(16.e)). If the hypotheses of linear problem and Gaussian noise are respected, this set of equations produces an unbiased and minimal variance recursive estimator [Chen, 2003; Grewal and Andrews, 2008; Orlande et al., 2012]. Also, the covariance matrix of the estimates error $\mathbf{P}_n$ allows the construction of confidence interval for better analysis of the obtained results.

$$\mathbf{x}_{n|n-1} = \mathbf{F}_n \mathbf{x}_{n-1} \tag{16.a}$$

$$\mathbf{P}_{n|n-1} = \mathbf{F}_n \mathbf{P}_{n-1} \mathbf{F}_n^T + \mathbf{Q}_n \tag{16.b}$$

$$\mathbf{K}_n = \mathbf{P}_{n|n-1} \mathbf{H}_n^T (\mathbf{H}_n \mathbf{P}_{n|n-1} \mathbf{H}_n^T + \mathbf{R}_n)^{-1} \tag{16.c}$$

$$\mathbf{x}_n = \mathbf{x}_{n|n-1} + \mathbf{K}_n (\mathbf{y}_n - \mathbf{H}_n \mathbf{x}_{n|n-1}) \tag{16.d}$$

$$\mathbf{P}_n = (\mathbf{I} - \mathbf{K}_n \mathbf{H}_n) \mathbf{P}_{n|n-1} \tag{16.e}$$

Results

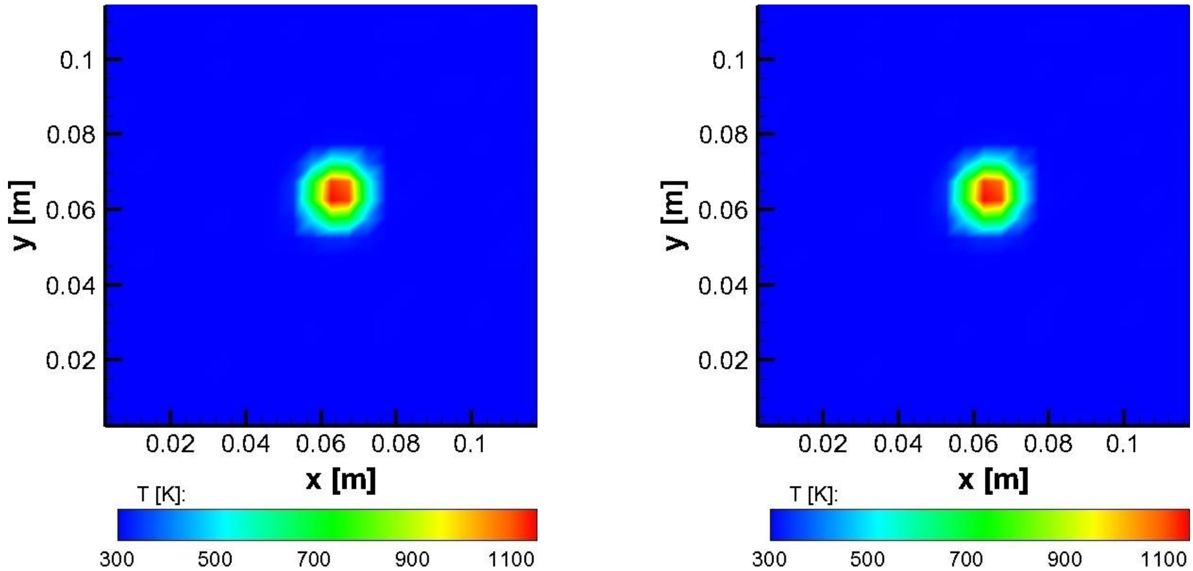
In this work, the experimental measurements were simulated with the complete model, using a fine grid with $768 \times 768 \times 64$ volumes and time step $\Delta t = 10^{-4} s$ to ensure numerical convergence. The inverse problem was solved with 24×24 volumes and a time step $\Delta t = 0.01 s$. This was done so that the simulated measurements are free of inverse crime [Kaipio and Somersalo, 2004]. The initial temperature was considered as 300K. The observation noise was assumed as Gaussian, uncorrelated, with zero mean and constant standard deviation, σ_y. In a real situation, these measurements could be obtained with modern infrared cameras, which presents standard deviations of the order of $0.01°C$ [Orlande et al., 2013]. For testing the performance of the Kalman filter, a relatively high value ($\sigma_y = 1°C$) was selected. The proposed heat flux is described by Eq. (17) and Tab. 2. The size of the region of application of the heat flux is selected so that it does not necessarily coincide with the control volume size.

$$q(x,y,t) = \begin{cases} q_0 & \text{if} \quad x_1 \le x \le x_2, \quad y_1 \le y \le y_2 \quad \text{and} \quad t \ge t_1 \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

Table 2: Parameters of the proposed heat flux

Quantity	Value	Quantity	Value	Quantity	Value
x_1	60mm	y_1	60mm	q_0	10^7 W/m²
x_2	72mm	y_2	72mm	t_1	0.4s

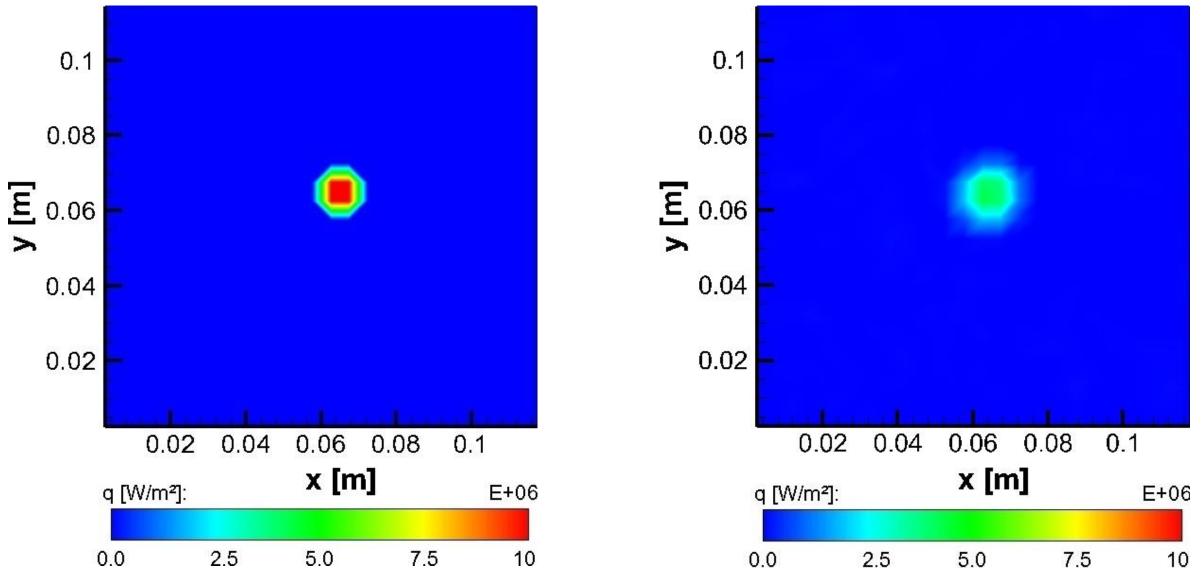
The comparison between the projection of the exact temperature field on the coarse grid and the estimated values at time $t = 2.0\text{s}$ is presented in Figs. 2.a-2.b. The agreement between these values is excellent, once the region where the heating occurs is adequately identified and the estimated temperatures are very close to the exact values. No signs of correlation were detected in the obtained residuals. The largest residual was in the heated region and its vicinity, but its value was approximately 0.45°C. Thus, both the largest residual and the standard deviation of the experimental measurements have the same order of magnitude ($1°C$), and the temperature estimates can be considered as good [Ozisik and Orlande, 2000].



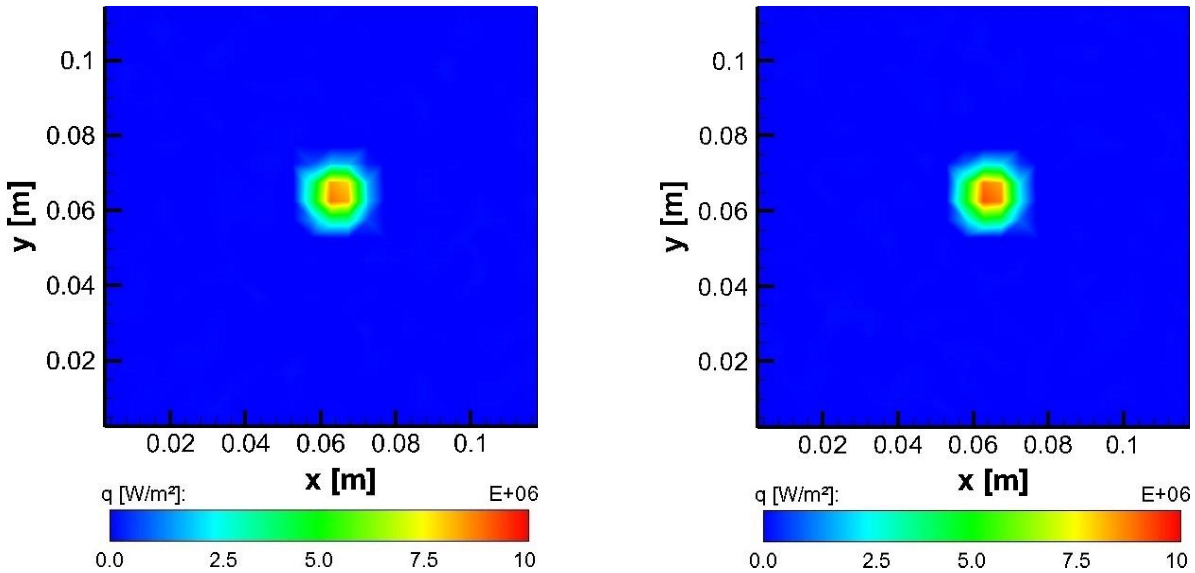
(a) Exact (b) Estimated

Figure 2: Comparison of the exact and estimated temperature field at time t = 2.0s.

The same comparison made for the temperature is presented for the heat flux in Figs. 4.a-4.d, where the projections of the exact heat flux values in the coarse grid is presented in Fig. 4.a, while the estimated values at times t = 1.0s, 1.5s and 2.0s are presented in Figs. 4.b, 4.c and 4.d, respectively.



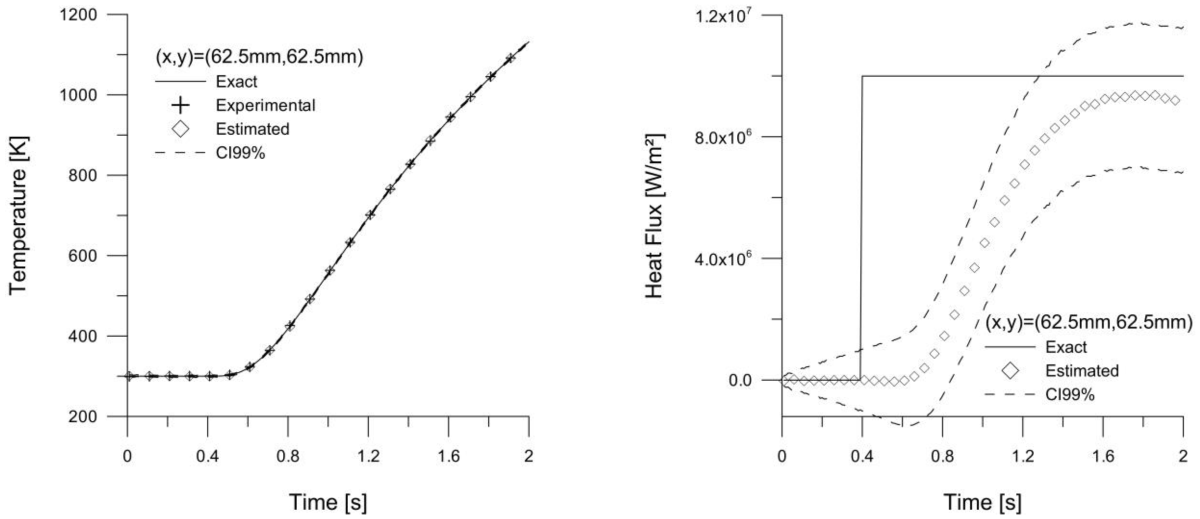
(a) Exact (b) t=1.0s.



(c) t=1.5s **(d) t=2.0s**

Figure 3: Comparison between the exact heat flux and the estimates at different times.

The results show that the region where the heating occurs is very well identified. However, some quantitative differences can be observed between the reference values and the estimates. For better understanding of these differences, the evolutions in time of the exact and estimated values are presented in Figs. 5.a-5.b for the point $(x, y) = (62.5; 62.5)\text{mm}$, located inside the heated region.



(a) Temperature **(b) Heat Flux**

Figure 4: Evolution of the reference and estimated values with time at $(\mathbf{x}, \mathbf{y}) = (62.5; 62.5)\text{mm}$.

For these results, the 99% confidence intervals show again the good quality of the temperature estimates. However, since the heating occurs on the opposite side from where the measurements are obtained, a time of approximately 0.2s is elapsed before the filter shows any change in the heat flux estimates resulting from the applied heat flux. This is due to the time required for the diffusion of heat through the thickness of the plate. Also, in the vicinity of $t = 2.0\text{s}$, the estimates show a decreasing behavior as a result of the modeling errors of the surrogate model at high temperatures.

On the other hand, it is possible to extract information about the order of magnitude of the heat flux and its region of application.

Conclusions

The proposed inverse problem, for which nonlinear and three dimensional models would be needed, could be reasonably well solved with simplified models, allowing for the use of fast and computationally efficient algorithms, such as the Kalman Filter. The temperature estimates present very good agreement with reference values. For the estimation of the heat flux, although the effect of the modeling errors of the surrogate model is noticeable, the heating location is adequately identified and the obtained estimates have the same order of magnitude as the exact values. Improvement of these results relies in accounting for modeling errors in the solution of the inverse problem.

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