Active Control and Potential Exploitation of Parametrically Excited Systems

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Abstract

This paper presents active control of parametrically excited systems. Parametrically excited systems can exhibit complex dynamic behavior such as inherent instability. Active control can be used to first stabilize and then increase the stability regions of such systems, using velocity feedback. A beam subject to an axial load is considered, representing a parametrically excited system with periodic time-varying stiffness. For amplitudes that are well below the critical buckling load and for axial load excitation at twice the first natural frequency of the beam, the system becomes parametrically unstable. It is demonstrated, how the system can be stabilized using active control. Alternatively, parametric excitation can be exploited for energy harvesting. When the system is close to the transition curves or instability regions, due to the high amplitude level of vibrations, parametrically excited systems can harvest much more energy compared to the time-invariant systems.

Keywords: Parametrically excited systems, Active control

Introduction

Parametrically excited systems include a parameter(s) in their dynamic equations, which varies periodically in time. Parametric resonance is a dynamic instability associated with such systems. It involves interaction between the parametric excitation frequency and the natural frequency of the system, leading to negatively damped modes and unstable oscillations. When the parametric excitation frequency is at about twice the natural frequency, the system exhibits instability, leading to large oscillations and potentially fatigue or failure. One example of parametrically excited systems is cable-stayed bridges, in which the tension of the cables can vary periodically due to the vibration of the deck, resulting in parametric resonance and instability [Reynolds et al. (2006)]. Other examples of such systems are flexible risers or ships, in which the wave motion can be the source of parametric excitation [Ahmed et al. (2010)].

Parametrically excited systems have been the subject of research investigation for decades [Cartmell (1990), Nayfeh and Mook (1995)]. The most simple and widely used equation of parametric excitation is the well-known Mathieu equation with linear, periodic, time varying stiffness coefficient. This rather simple single degree-of-freedom (DOF) mechanical system exhibits complex unstable behaviour but also interesting stability regions, as known from the inverted pendulum problem, depending on the amplitude and frequency parameters of the time periodic (harmonic) term. Parametrically excited systems exhibit combination resonances of summed or difference types. When subjected to an external forcing frequency, a periodically time-varying system will be resonant when the external frequency equals the combination of natural frequency and parametric excitation frequency.

Recent research emphasizes the potential for deliberately introducing parametric excitation to increase the capability of a system to suppress vibrations [Ecker (2010), Dohnal and Mace (2008)]. Parametric excitation can also be exploited for energy harvesting. Daqaq et al. [Daqaq et al. (2009)] investigated the problem of energy harvesting using a parametrically excited cantilever beam. The cantilever beam was excited vertically, perpendicular to the direction of the oscillatory displacement, at twice of its fundamental frequency. In 2011, the same parametrically excited

harvesting beam configuration was considered including higher modes and nonlinear effects of the piezoelectric patch [Abdelkefi et al. (2012)].
This paper describes both active control of parametrically excited systems and their potential exploitation for energy harvesting. In active control, the problem is to suppress large amplitude of vibration or to extend the stability boundaries using velocity feedback control. A beam under axial load is considered, representing a parametrically excited system. For certain amplitudes and frequencies of the axial load, the system can exhibit parametric instability. Active control is used to stabilise the system. In addition, an investigation is carried out to determine the amount of energy that can be harvested from the beam with and without parametric excitation. It is shown that when the system is parametrically excited, it can harvest much more energy.

Parametrically Excited Systems

A vertical cantilever steel beam subjected to an axial time-harmonic load $P(t)$ associated with the base acceleration is shown in Figure 1. A static compressive load is expected to reduce the first natural frequency. When the amplitude of the axial load reaches the critical buckling load, the beam can experience buckling instability and zero stiffness. If the axial load is harmonic, i.e. $P(t) = P\cos\Omega t$, the bending stiffness of the beam varies periodically. For a specific amplitude of parametric excitation P_0 at a frequency Ω almost twice the first bending natural frequency ω_1, the system becomes parametrically unstable.

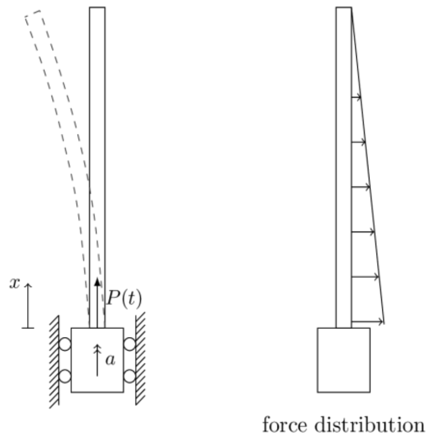


Figure 1: A parametrically excited system- a beam subject to a harmonic axial load

Vibration of an axially loaded cantilever at its first mode can be described using a single degree-of-freedom equation with the time-varying natural frequency. We add a damping term to obtain:

$$\ddot{q} + 2\varsigma\omega_0\dot{q} + \omega_0^2\left(1 - \frac{P(t)}{P_{cr}}\right)q = 0 \tag{1}$$

This is the well-known damped Mathieu Equation with a periodic time-varying stiffness.
The cantilever beam considered in this paper is mounted vertically, therefore, accounting for the effect of gravity, the forcing term takes the following form:

$$P(x,t) = P_0(x) + P_1(x)\cos\Omega t = m(x)g + m(x)a\cos\Omega t \tag{1}$$

where P_0 represents the weight of the beam and $P_1(t)$ is the force resulting from the acceleration of the base. We note that the load is not uniform over the length of the beam since the mass is distributed, thus the space average of the mass distribution is used:

$$\frac{1}{L}\int_0^L \rho Ax\,dx = \frac{\rho AL}{2} \tag{3}$$

The final form of the governing equation can be written as,

$$\ddot{q} + 2\varsigma\omega_P\dot{q} + \omega_P^2\left(1 - \frac{P_1\cos\Omega t}{2P_{cr'}}\right)q = 0 \tag{4}$$

with $P_{cr'} = P_{cr} - \frac{P_0}{2}$ and $\omega_P^2 = \omega_0^2\left(1 - \frac{P_0}{2P_{cr}}\right)$.

Stability Analysis

We approximate the response with three terms in the Fourier series:

$$q(t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\Omega}{2}t\right) + \sum_{n=1}^{\infty} B_m \sin\left(\frac{n\Omega}{2}t\right) \tag{5}$$

Taking derivatives, substituting into Eq. (1), and partitioning the $\sin\left(\frac{n\Omega t}{2}\right)$ and $\cos\left(\frac{n\Omega t}{2}\right)$ terms, leads to a set of equations in terms of the coefficients A_n and B_n. The transition curves (stability curves) are obtained from solving the determinant of the coefficient matrices.

$$\begin{vmatrix} \omega_P^2\left(1 - \frac{P_1}{4P_{cr'}}\right) - \frac{\Omega^2}{4} & \zeta_1\omega_P\Omega \\ -\zeta_1\omega_P\Omega & \omega_P^2\left(1 + \frac{P_1}{4P_{cr'}}\right) - \frac{\Omega^2}{4} \end{vmatrix} = 0 \tag{6}$$

which results in the two transition curves separating the stable from the unstable regions,

$$\left(\frac{\omega_P}{\Omega}\right)^4\left(1 - \frac{P_1^2}{16P_{cr'}^2}\right) + \left(\frac{\omega_P}{\Omega}\right)^2\left(\zeta_1^2 - \frac{1}{2}\right) + \frac{1}{16} = 0 \tag{7}$$

Transition curves were computed for a steel cantilever beam which dimensions and properties are listed in **Table 1**. The natural frequency associated with the first bending mode accounting for the weight of the beam was found to be $f_P = \omega_P / 2\pi = 3.18$ Hz, whereas the critical buckling load diminished by the average weight of the beam was calculated as $P_{cr'} = 2.91$ N.

Table 1: Beam dimensions and material properties

property	b, m	d, m	L, m	E, GPa	ρ, kg/m^3	ς_1
value	0.0105	0.00144	0.583	186	7850	0.002

The two transition curves are plotted using the analytical approximation in Figure 2. It can be seen that parametric instability occurs when the parametric excitation frequency is twice the first natural frequency ($\Omega = 2\omega_P$). The effect of damping is to move the transition curves upwards, hence increasing the stable region, when comparing a damped system (black solid line) with an undamped system (grey dashed line). Considering higher order terms in the Fourier series, other transition curves can be obtained.

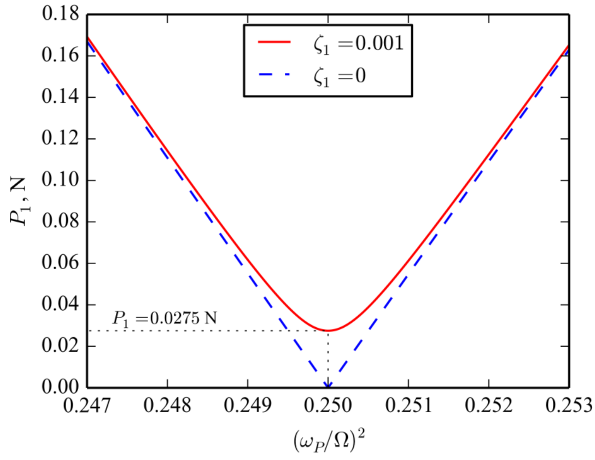
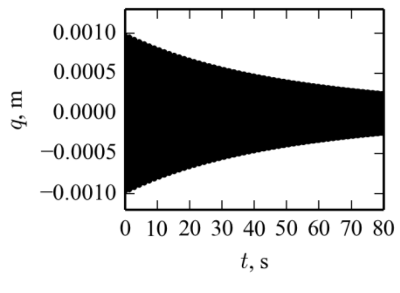
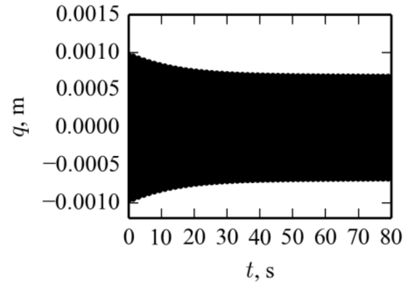


Figure 2: Transition curves for the beam subject to axial load, where the area above the curve is unstable. The transition curves are shown with no damping and for a damping ratio of 0.1%.

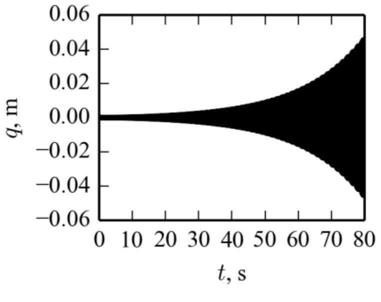
For the beam example, the parametric amplitude is found analytically to be 0.0275 N, which is well below the critical buckling load 2.91N. For lower level of damping, a small axial perturbation can make the system unstable. Numerical simulation using ode45 in Matlab also validates the analytical results. The time response for different amplitudes P_1 is shown in Figure 3 for $\varsigma_1 = 0.1\%$. The system is stable for amplitudes below 0.0275 N, Figure 3(a), and unstable above this amplitude, Figure 3(c). The response of the system exhibits a limit cycle oscillation, Figure 3 (b), when excited at 0.0275N. In this case, the system is on the transition curve.



(a) $P_1 = 0.01$ N



(b) $P_1 = 0.0275$ N



(c) $P_1 = 0.1$ N

Figure 3: Time-response of the parametrically excited beam with 0.1% damping when excited at twice its natural frequency with different parametric amplitudes.

Experiments

In order to illustrate the described phenomenon a simple physical model was build. A steel cantilever beam was vertically mounted on the shaker table in a way that emulates a clamped boundary condition. The shaker provided an axial excitation to the beam which was indirectly measured by placing an accelerometer on the shaker table. Transverse vibrations of the cantilever were recorded using a laser vibrometer.

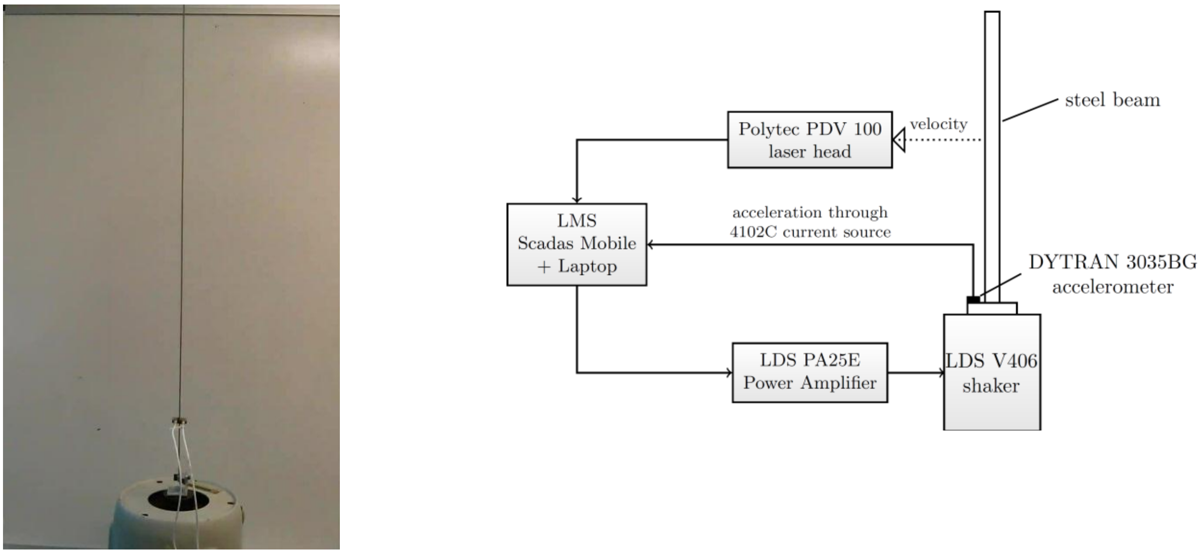


Figure 4: Schematic diagram of the experimental set-up

Firstly, the natural frequency associated with the first bending mode of the vertically mounted cantilever was determined via an impact hammer test. Based on this result, the parameters of the beam were identified (Table 1).

The transition curves were experimentally determined by sweeping through excitation frequencies close to the natural frequency associated with the first bending mode at different levels of the axial load and analysing the shape of the time histories of transverse vibrations. The system was qualified as stable if its response decayed exponentially with time and as unstable if the response was growing and experiencing a limit cycle oscillation. In order to ensure that all useful information is captured a very long time window was used (5 min) given the structure being very lightly damped.

The experimental results compared to theoretical transition curves as derived in the previous section are presented in Figure 5. Both are in a good agreement confirming the validity of the theoretical approach adopted. In Figure 6 we present chosen velocity response time histories and their Fourier transforms for the stable and the unstable case. The influence of nonlinearities is evident in the response at the unstable state since the response growth is limited.

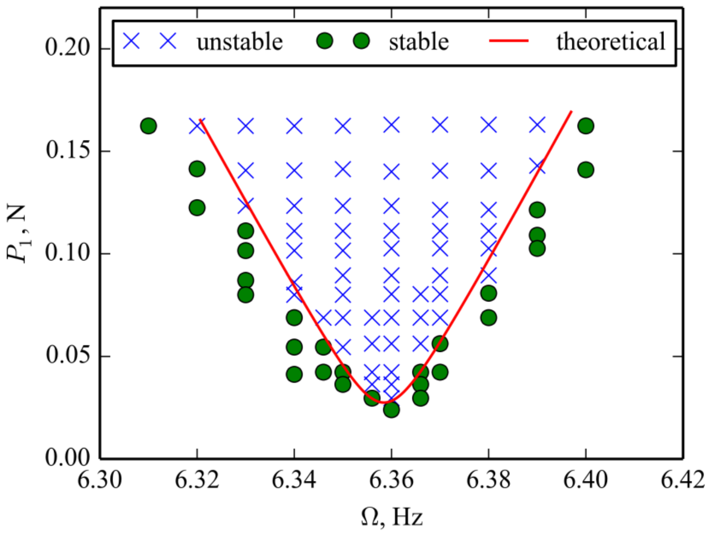


Figure 5: Transition curves – comparison between the experiment and the theory

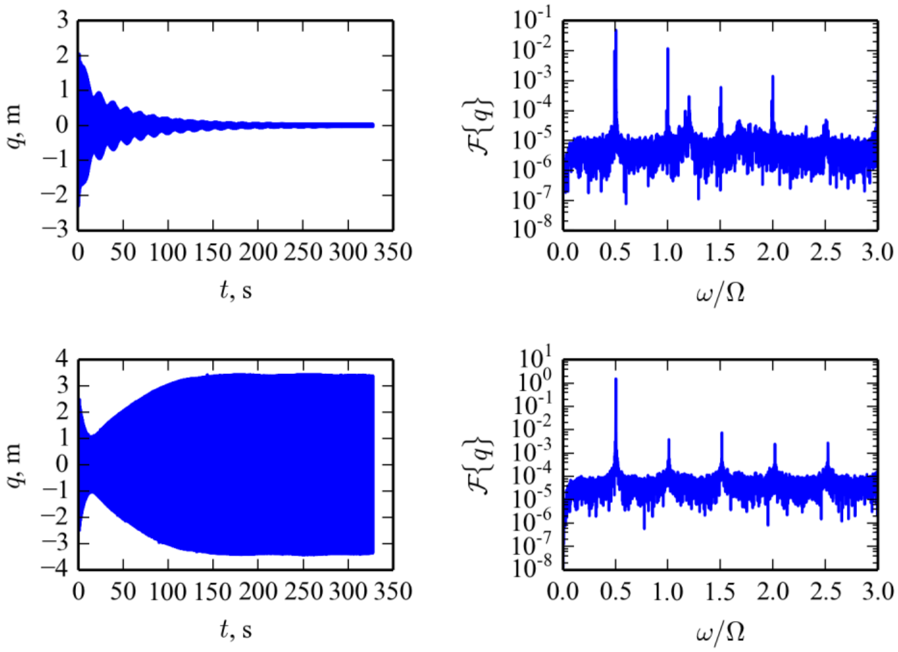


Figure 6: Response time histories and their Fourier transforms: (a) stable; (b) unstable

Active control

To stabilize the parametric instability as well as to increase the stability region, a velocity feedback and pole placement is considered. The system with control has the following form in state-space,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_P^2\left(1-\gamma\dfrac{P_1\cos\Omega t}{P_{cr'}}\right) & -2\zeta_1\omega_P \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) = \mathbf{A}(t)\mathbf{x} + \mathbf{B}u(t) \qquad (8)$$

where $u(t)$ is the control force and is a single-input.

Simulation: Velocity feedback

The amplitude of axial load is chosen to be $P_1 = 0.075\,\mathrm{N}$ and $\Omega = 2\omega_P$. The open-loop response is 7unstable as shown with the blue-dashed line in Figure 5. The control law is considered to be:

$$u(t) = -h\dot{x}(t) \tag{9}$$

where h is the velocity feedback gain. The system is stabilized using the velocity feedback gain of $h = 1$ as shown with the red solid line in Figure .

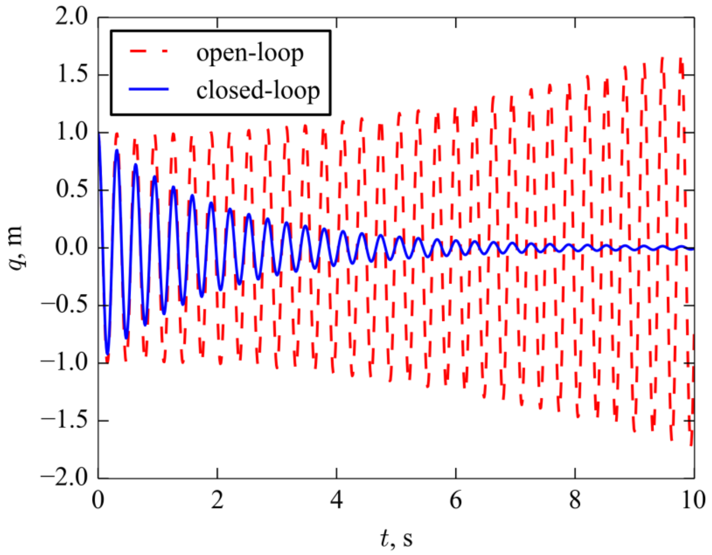


Figure 7 – Simulation: Time response - velocity feedback control; open-loop: grey dashed line, closed-loop: black solid line

Experiment: Velocity feedback

Experiment is carried out to implement the velocity feedback control and stabilize the system. The parameters of the axial load are chosen so that the open-loop system has parametric instability. A piezoelectric actuator is attached to the beam. The velocity of the beam at the other end is measured using LMS data acquisition and the velocity is fed back to the amplifier of the piezo with a feedback gain of 100.

The open-loop and closed-loop time responses are measured as shown in Figure 7. When the controller is switched on, the response decays due to the increase of the damping. This clearly demonstrates that using active control, the system is stabilized.

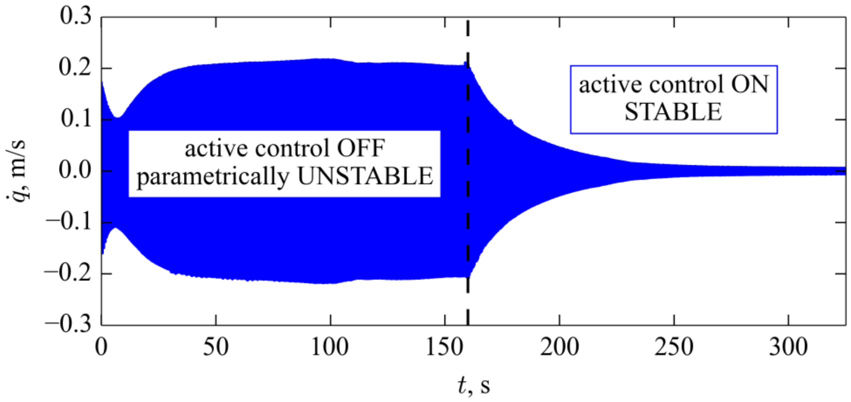


Figure 7 : Experiments: Active control of a parametrically excited beam

Exploitation: Energy Harvesting

Parametric resonance can be exploited in parametrically excited systems to harvest more vibration energy due to large amplitudes of vibrations compared to linear time-invariant systems. A piezoelectric is attached to the parametrically excited beam, as shown in Figure 8, which is shunted to a resistor for energy harvesting. The "electrical damping", used to harvest energy is assumed to be $0.1\,\mathrm{Ns/m}$.

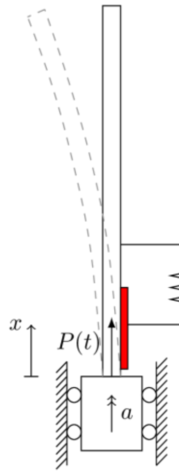
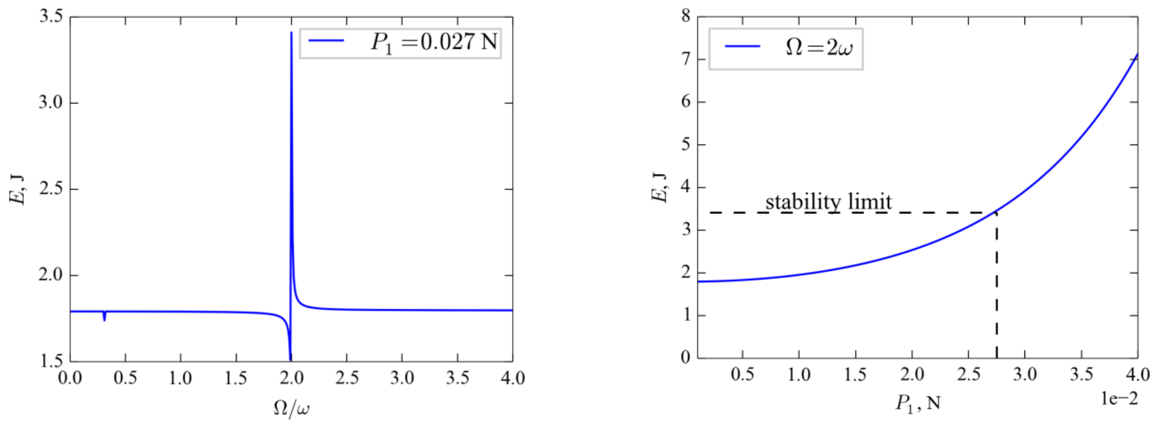


Figure 8 : Energy harvesting from a parametrically excited beam

Numerical simulation is carried out to investigate the amount of energy that can be harvested when the beam is subjected to the axial load with an amplitude, close to the parametric instability, for example $P_1 = 0.0272\,\mathrm{N}$ and $\Omega = 2\omega_1$. The harvested energy is calculated using Eq. (10) for the duration of 80 s with an initial displacement of 0.01m.

$$E = \int_0^{80} 0.1\dot{q}^2(t)dt = 3.4\,\mathrm{J} \tag{10}$$

Figure 9(a) shows that the harvested energy is maximum when the excitation frequency is almost twice the first bending frequency. In addition, the amount of energy increases when the excitation amplitude increases as shown in Figure 8(b) when $\Omega = 2\omega_1$. The amount of harvested energy from the parametrically excited beam with $P_1 = 0.0272\,\mathrm{N}$ is $E = 3.4\,\mathrm{J}$, while the harvested energy from the time-invariant system is $E = 1.78\,\mathrm{J}$. This dynamic behaviour can be exploited for the design of energy harvesters.



(a) (b)

Figure 9 : Harvested energy-(a) for different frequency ratio when $P_1 = 0.0272$ N (b) for different parametric amplitude when $\Omega = 2\omega_1$

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Conclusions

In this paper, control and exploitation of parametrically excited systems was presented. A beam subject to axial load was considered as a parametrically excited system. For an amplitude well below the critical buckling load, the beam experienced parametric resonance. Parametric instability was controlled using active vibration control. Velocity feedback control was considered to stabilize the system and assign stable poles. In addition, it was demonstrated that parametric excitation could be exploited to increase the amount of harvested energy. Practical implementation of the energy harvesting will be considered as part of future work.

References

Reynolds, P., Pavic, A. (2006) Vibration Performance of a Large Cantilever Grandstand during an International Football Match, *ASCE Journal of Performance of Constructed Facilities*, 20(3), 202-212.

Ahmed, T.M., Hudson, D.A. and Temarel, P.(2010) An investigation into parametric roll resonance in regular waves using a partly non-linear numerical model, *Ocean Engineering*, 37, 1307-1320.

Cartmell, M. (1990) *Introduction to Linear, Parametric and Nonlinear Vibrations*, Chapman and Hall, London.

Nayfeh, A. Mook, D. (1995) *Nonlinear Oscillations*, Wiley-VCH.

Ecker, H. (2010) Exploiting the use of parametric excitation, *10th International Conference RASD2010*, Southampton, July 2010.

Dohnal, F. Mace, B.R. (2008), Amplification of damping of a cantilever beam by parametric excitation, Proceedings-CD MOVIC 2008. *9th International Conference on Motion and Vibration Control (MOVIC 2008)*, September 2008, Munich, Germany, Institute of Applied Mechanics.

M.F. Daqaq, C. Stabler, Y. Qaroush, T. Seuaciuc-Osório, Investigation of power harvesting via parametric excitations, *Journal of Intelligent Material Systems and Structures*, 20, 545-557.

Abdelkefi, Nayfeh, N and Hajj,M. (2012) Global nonlinear distributed-parameter model of parametrically excited piezoelectric energy harvesters, *Nonlinear Dynamics*, 67, 1147-1160.