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Convergency of viscoelastic constraints to Nonholonomic Idealization

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Abstract

Rolling contacts are usual in various technical systems and yield usually non-holonomic constraints. A new regularization method motivated by physical considerations is investigated in the present paper. The convergence of the spring-damper regularization for the so called principal damping, which is motivated by the critical damping in the linear case, is proven. The solutions of the DAEs and the corresponding ODEs converge if a certain condition on the regularization parameters is fulfilled. A rolling disc on the flat plane and a skate on an inclined plane are analyzed as numerical examples. It is demonstrated firstly that the optimal choice of the regularization parameters corresponds to the principle damping and secondly that the sufficient convergence condition obtained in the proof is valid for the numeric simulations.

1. Introduction

In most cases the constraint equations on velocity level enforcing a rolling motion cannot be integrated, yielding nonholonomic constraint equations. Usually the nonholonomic constraints can be incorporated into the equations of motion by the method of Lagrange multipliers. This formulation leads to index-2 differential algebraic problems. In the present paper we investigate a new viscoelastic idealization of nonholonomic constraints, that is motivated by physical considerations. Pure rolling is equal to a sticking state, with a kinematically repositioned contact point. Usually sticking is modeled by introducing an elasticity in the contact as demonstrated by [3]. Here the constraint is enforced by the elastic and dissipative terms, that help to avoid numerical oscillations in the contact. In an earlier work [2] applied this kind of viscoelastic formulation to a tangential contact law, extending the classical laws of friction, like the Coulomb model, to distributed contacts, in order to circumvent the problem of indeterminacy in the sticking state. However a description of a contact law by means of viscoelastic forces is sensible only if it approximates the idealized rigid formulation in case of infinitely stiff chosen viscoelastic parameters. Thus the objective of this work is to show the convergency of the viscoelastic description to the idealized nonholonomic rigid description in a mathematical sense.

2. Statement of problem

Consider the general multibody system with m nonholonomic constraint equations, as given in definition 1.

Definition 1 (Differential algebraic initial value problem)**.** *Let $I = [t_0, t_e]$ be a closed interval. Then the equations of motion can be described by the following differential algebraic initial value problem*

$$M(q)\ddot{q} = F(q, \dot{q}, t) - G^{\mathsf{T}}(q)\Lambda, \tag{1}$$

$$0 = G(q)\dot{q} \tag{2}$$

with the consistent initial conditions $q(t_0) = q_0$, $\dot{q}(t_0) = \dot{q}_0$. Furthermore holds $M(q) \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. The functions $G(q) \in \mathbb{R}^{m\times n}$ and $F(q, \dot{q}, t) \in \mathbb{R}^n$ are sufficiently smooth, the matrix $G(q)$ is assumed to have full rank m.

Usually deformations occur in a contact area due to local deformations of asperities and the elasticity of the bodies itself. A sensible physical description of a contact should take these effects into account. Thus the constraint forces, that enforce the constraint equation, are replaced by applied forces in form of a viscoelastic force element, which leads to the viscoelastic description of the given multibody system as stated in definition 2.

Definition 2 (Viscoelastic description)**.** *Let $I = [t_0, t_e]$ be a closed interval. Then for $t \in I$ and for a fixed $\varepsilon_f \in (0, \varepsilon_0]$ the equation of motion of the viscoelastic description is given by*

$$M(q)\ddot{q} = F(q, \dot{q}, t) - G^{\mathsf{T}}(q)\Lambda, \tag{3}$$

$$\dot{z} = G(q)\dot{q} \tag{4}$$

along with the initial conditions $q(t_0) = q_0$, $\dot{q}(t_0) = \dot{q}_0$ and $z(t_0) = z_0$, where the Lagrange multiplier Λ is replaced by

$$\Lambda = \frac{c}{\varepsilon_f} z + \frac{d}{\varepsilon_f^{\kappa}} \dot{z}. \tag{5}$$

The functions $G(q) \in \mathbb{R}^{m\times n}$ and $F(q, \dot{q}, t) \in \mathbb{R}^n$ are sufficiently smooth, the matrix $G(q)$ is assumed to have full rank m.

The parameter κ is chosen as $\frac{1}{2}$, inspired by the critical damping in the linear case. The given forms of the underlying systems are not suitable, in order to obtain an estimate of the distance between the corresponding solutions. However by applying appropriate transformations, they can be transformed to a standard singular perturbation form. Transforming the systems given in definition 1 and definition 2 to autonomous systems and introducing the new variable θ according to

$$\sqrt{\varepsilon}\dot{\Lambda} = \theta + h(y,\Lambda) + S(\Lambda), \tag{6}$$

where $S(\Lambda) = \Lambda$ and $h(y,\Lambda) = c(\frac{d}{c})^{\frac{1}{1-\kappa}}(\dot{G}y_2 + GM^{-1}F - GM^{-1}G^{\mathsf{T}}\Lambda)$. The equivalent systems in standard singular perturbation form can be obtained.

$$\begin{aligned}
\dot{y} &= f(y,\Lambda), \\
\sqrt{\varepsilon}\dot{\Lambda} &= \theta + h(y,\Lambda) + S(\Lambda) && := g_1(y,\theta,\Lambda), \\
\sqrt{\varepsilon}\dot{\theta} &= h(y,\Lambda) - \tfrac{\mathrm{d}S}{\mathrm{d}\Lambda}(\theta + h(y,\Lambda) + S(\Lambda)) && := g_2(y,\theta,\Lambda).
\end{aligned}$$

Setting $\varepsilon = 0$ leads to the reduced problem in form of differential algebraic equations

$$\begin{aligned}
\dot{y} &= f(y,\Lambda), \\
0 &= \theta + h(y,\Lambda) + S(\Lambda) && := g_1(y,\theta,\Lambda), \\
0 &= h(y,\Lambda) - \tfrac{\mathrm{d}S}{\mathrm{d}\Lambda}(\theta + h(y,\Lambda) + S(\Lambda)) && := g_2(y,\theta,\Lambda).
\end{aligned}$$

with the column matrices $\lambda = [\Lambda, \theta]$, $g = [g_1, g_2]$ and $f(y,\lambda) = [y_2, M^{-1}(F - G^{\mathsf{T}})\Lambda, 1]$ the problems can be written conveniently. The differential algebraic equation can be represented in the following form:

Definition 3 (Differential algebraic equation)**.**

$$\begin{aligned}
\dot{y} &= f(y,\lambda), \\
0 &= g(y,\lambda), \\
y(0) &= y_0^0.
\end{aligned} \tag{7}$$

The viscoelastic approximation reads as:

Definition 4 (Viscoelastic description in singular perturbation standard form)**.**

$$\begin{aligned}
\dot{y} &= f(y,\lambda), \\
\sqrt{\varepsilon}\dot{\lambda} &= g(y,\lambda), \\
y(0) &= y_0^0 + \sqrt{\varepsilon}y_1^0 + \sqrt{\varepsilon}^2 y_2^0 + \dots, \quad \lambda(0) = \lambda_0^0 + \sqrt{\varepsilon}\lambda_1^0 + \sqrt{\varepsilon}^2\lambda_2^0 + \dots .
\end{aligned} \tag{8}$$

3. Proof of convergency

In the underlying form, standard singular perturbation approaches can be used in order to obtain an estimate of the distance of the solution of the problems given in definition 3 and definition 4.
In order to construct a solution of the initial value problem eq. (8) in form of an infinite asymptotic power series expansion the following theorem by Hairer and Wanner [1] can be applied.

Theorem 1. *Let f and g be sufficiently smooth functions. Consider the initial value problem given in eq.* (8)

$$\begin{aligned}
\dot{y} &= f(y,\lambda), \\
\sqrt{\varepsilon}\dot{\lambda} &= g(y,\lambda), \\
y(0) &= y_0^0 + \sqrt{\varepsilon}y_1^0 + \sqrt{\varepsilon}^2 y_2^0 + \dots, \qquad \lambda(0) = \lambda_0^0 + \sqrt{\varepsilon}\lambda_1^0 + \sqrt{\varepsilon}^2\lambda_2^0 + \dots .
\end{aligned}$$

Introducing the time scale $\tau = \frac{t}{\sqrt{\varepsilon}}$ enables the construction of the solutions in form of an infinite asymptotic series expansion according to

$$y(t) = \sum_{j=0}^{\infty}\sqrt{\varepsilon}^j y_j(t) + \sqrt{\varepsilon}\sum_{j=0}^{\infty}\sqrt{\varepsilon}^j\eta_j(\tau), \quad \lambda(t) = \sum_{j=0}^{\infty}\sqrt{\varepsilon}^j\lambda_j(t) + \sum_{j=0}^{\infty}\sqrt{\varepsilon}^j\zeta_j(\tau). \tag{9}$$

The functions $\eta_j(\tau)$ and $\zeta_j(\tau)$ satisfy the conditions

$$\|\eta_j(\tau)\| \le K_j \mathrm{e}^{-\kappa_j\tau}, \quad \|\zeta_j(\tau)\| \le C_j \mathrm{e}^{-\kappa_j\tau}.$$

The proof of this theorem is given in [1]. Instead of the infinite power series expansion, the truncated power series expansion will be used instead. Special interest is devoted to the series truncated at $N = 0$ since the resulting zeroth approximation corresponds to the differential algebraic equation. Thus the target is to find an estimation of the error made when using the truncated series expansion instead of the full series expansion. This question is answered by the following theorem from Hairer and Wanner [1].

Theorem 2. *Let f and g be sufficiently smooth functions. Consider the viscoelastic formulation in form of initial value problem (8). Suppose that the logarithmic norm $\mu(g_\lambda) < -1$ holds in an ε independent neighborhood of the solution $y_0(t)$, $\lambda_0(t)$ of the differential algebraic equation (7) with the initial condition $y_0(0) = y_0^0$, satisfying the constraint equation, on the interval $0 < t < T$. If the initial values y_0^0 and λ_0^0 lie in this neighborhood, then the initial value problem (8) has a unique solution for ε sufficiently small and for $0 < t < T$, which is of the form*

$$y(t) = y_{tr}(t) + \mathcal{O}(\sqrt{\varepsilon}^{N+1}) = \sum_{j=0}^{N} \sqrt{\varepsilon}^j y_j(t) + \sqrt{\varepsilon} \sum_{j=0}^{N} \sqrt{\varepsilon}^j \eta_j(\tfrac{t}{\sqrt{\varepsilon}}) + \mathcal{O}(\sqrt{\varepsilon}^{N+1}), \tag{10}$$

$$\lambda(t) = \lambda_{tr}(t) + \mathcal{O}(\sqrt{\varepsilon}^{N+1}) = \sum_{j=0}^{N} \sqrt{\varepsilon}^j \lambda_j(t) + \sum_{j=0}^{N} \sqrt{\varepsilon}^j \zeta_j(\tfrac{t}{\sqrt{\varepsilon}}) + \mathcal{O}(\sqrt{\varepsilon}^{N+1}). \tag{11}$$

The coefficient functions $\eta_j(\tau)$ and $\zeta_j(\tau)$ satisfy $\|\eta_j(\tau)\| \le K_j \mathrm{e}^{-\kappa_j \tau}$ and $\|\zeta_j(\tau)\| \le C_j \mathrm{e}^{-\kappa_j \tau}$. The error between the solution of the differential algebraic equation (7), which corresponds to the truncated series at $N = 0$, and the viscoelastic formulation (8) can be estimated above according to

$$\|y - y_0\| \le M_1 \sqrt{\varepsilon}, \quad \|\lambda - \lambda_0\| \le M_2 \sqrt{\varepsilon}.$$

The proof of this theorem is given in Hairer and Wanner [1]. Thus the solution of the viscoelastic contact formulation is in an $\mathcal{O}(\sqrt{\varepsilon})$ vincinity of the solution of the differential algebraic equation. The request that the logarithmic norm $\mu(g_\lambda) < -1$ leads to the following condition on the eigenvalues of the matrix $-GM^{-1}G^{\mathsf{T}}$:

$$\lambda_{max}^{-GM^{-1}G^{\mathsf{T}}} \le -\tfrac{2c}{d^2}, \tag{12}$$

where $\lambda_{max}^{-GM^{-1}G^{\mathsf{T}}}$ denotes the maximum eigenvalue of the matrix $-GM^{-1}G^{\mathsf{T}}$.

4. Numerical experiments

In order to confirm the theoretical results, numerical experiments were carried out. Therefore a classical mechanical system of a skate sliding down an inclined plane under the influence of gravity is considered. The model is shown in fig. 1. Mathematically the constraint equation is given by the demand, that the velocity of the contact point is always parallel

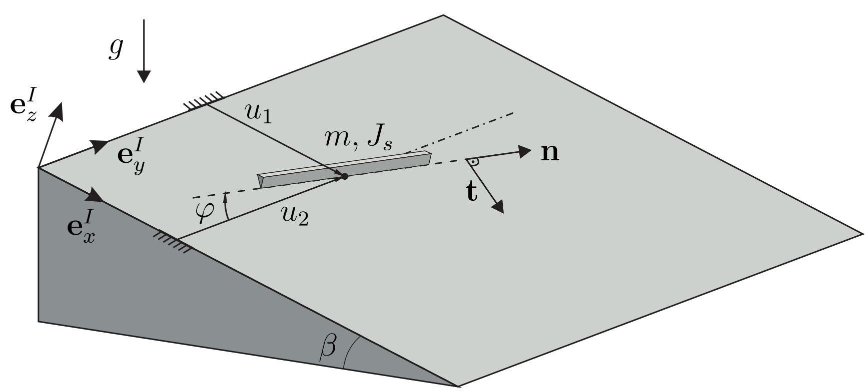
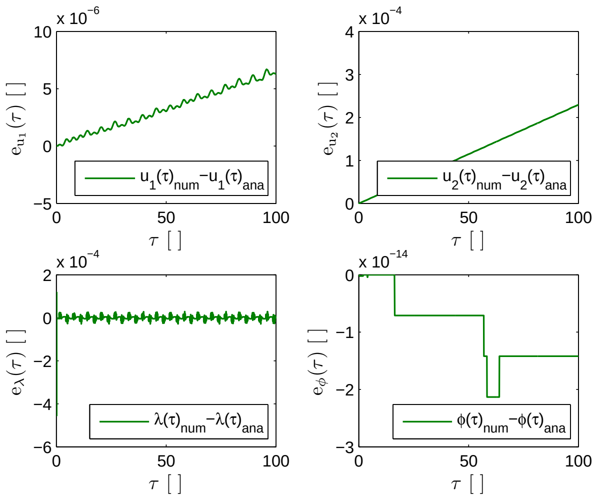


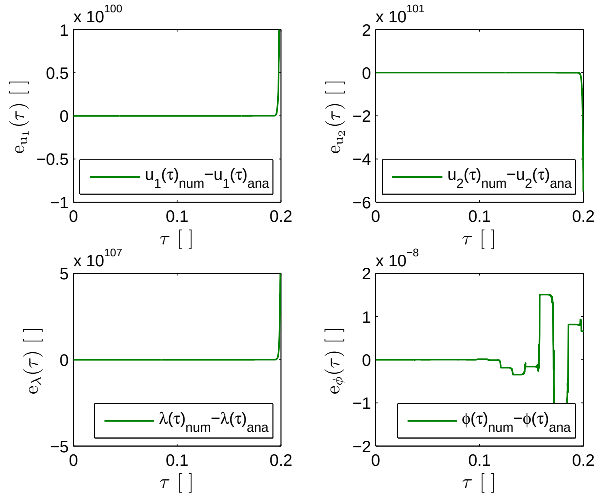
Figure 1: disk rolling on a flat support

to the skid, which can be expressed in the following fashion

$$\mathbf{v} \cdot \mathbf{t} = 0,$$



(a) Condition $\lambda_{max}^{-GM^{-1}G^{\mathsf{T}}} \leq -\frac{2c}{d^2}$ fulfilled.



(b) Condition $\lambda_{max}^{-GM^{-1}G^{\mathsf{T}}} \leq -\frac{2c}{d^2}$ hurt.

Figure 2: Deviations of the solutions of differential algebraic equation and viscoelastic formulation in case (12) is fulfilled (left) and not fulfilled (right), where the deviation grows exponentially fast.

where $\mathbf{v}$ denotes the velocity of the contact point and $\mathbf{t}$ the vector perpendicular to the skid. This finally results in the scalar constraint equation

$$0 = \underbrace{\begin{bmatrix} -\sin\varphi & \cos\varphi & 0 \end{bmatrix}}_{G(q)} \underbrace{\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{\varphi} \end{bmatrix}}_{\dot{q}}.$$

5. Discussion and conclusion

Convergency of the viscoelastic description of contact forces is proven for nonholonomic constraints in general form. The proof is performed for the principal damping exponent. The solutions of the DAE and the corresponding ODE converge if the condition

$$\mu(GM^{-1}G^{\mathsf{T}}) > \tfrac{2c}{d^2}$$

is fulfilled. Numerical experiments were made to verfiy the statement of the proof. They confirm the optimum performance for this choice of the viscoelastic parameters. In the future the described approach will enable a consistent modelling of sticking, sliding and rolling contacts in multibody dynamics.

References

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