Adaptive Precise Integration BEM for Solving Transient Heat Conduction Problems

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Abstract

A combined approach of boundary element method (BEM) and precise integration method (PIM) is presented for solving transient heat conduction problems with variable thermal conductivity. The boundary integral equation is derived by means of the Green's function for the Laplace equation. As a result, three domain integrals are involved in the integral equation. The radial integration method is used to transform the domain integrals into the boundary integrals. After discretization the solved domain by the BEM, a system of ordinary differential equations (ODEs) can be obtained. Adaptive PIM can solve efficiently ODEs and improve greatly the computational efficiency. Numerical examples show that the present approach can obtain satisfactory performance even for very large time step size. In addition, the results are independent of the time step size when the integral of free term can be analytically integrated, here, the free term is formed by boundary conditions and heat sources.

Keywords: Adaptive precise integration method, Radial integration method, Boundary element method, Transient heat conduction

Introduction

It is generally known that the finite difference method (FDM) is used to solve the transient heat conduction problems. However, the result of FDM is unstable when change the time step size. The precise integration method (PIM) [Zhong (1994)] can obtain stable and accurate results for different time step sizes. Particularly, the results are independent of the time step size when the free term can be divided into the functions of space and time and the time-related integral can be integrated analytically. Up to now, the PIM in conjunction with the finite element method (FEM) has been applied to conduct the transient heat transfer analysis [Cheng et al. (2004)], the transient forced vibration analysis of beams [Tang (2008)] and the sensitivity analysis and optimization problems [Xu et al. (2011)]. In addition, the method combining the PIM with meshless local Petrov–Galerkin method has been applied to the transient heat conduction problems [Li et al. (2011)].

Compared with FDM, FEM and the meshless method, BEM is very robust for solving the linear and homogeneous heat conduction problems [Song and Li (2003)]. However, BEM is still a challenge for solving nonlinear problems such as variable thermal conductivity problems. The main reason is that the fundamental solution of the problem obtains extremely difficult. Fortunately, we can use the fundamental solution of the linear problem to solve the nonlinear problem, whereas domain integrals are involved in resulting integral equations.

Generally, there are mainly two methods which can transform the domain integrals into the boundary integrals. The first one is the dual reciprocity method (DRM) [Nardini and Brebbia (1983)]. The deficiency of the method is that the particular solutions may be difficult to obtain for some complicated problems. In addition, even for known heat sources term, the method still requires an approximation of the known function. The second one is the radial integration method (RIM) [Gao (2002)]. The RIM not only can transform any complicated domain integral into the boundary without using particular solution, but also can remove various singularities appearing in domain integrals. The method combining the RIM with the BEM is called the radial integration boundary element method (RIBEM).

The RIBEM has been widely applied to many fields including the crack analysis in functionally graded materials [Zhang (2011)], the heat transfer problems [AL-Jawary and Wrobel (2012); Yu et al. (2014a; 2014b; 2014c;)] and the viscous flow problems [Peng (2013)]. The RIBEM still exists a problem, which solved results are sensitive for different time step sizes when the problems are transient. The RIBEM and the PIM have been combined to solve transient heat conduction problems [Yu et al. (2014c)].

In this paper, an adaptive technique is introduced in the present method to improve the computational efficiency without affecting accuracy. First of all, we discretize the space domain by using the RIBEM to obtain a system of ordinary differential equations (ODEs) with respect to time, and then solve the ODEs by the PIM. Finally, two numerical examples are presented to validate the proposed method.

Governing Equation

Considering a two-dimensional bounded domain Ω with heat source and a spatially variable heat conductivity, the governing equation for transient heat conduction problems in isotropic media can be expressed as

$$\frac{\partial}{\partial x_i}\left(k(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial x_i}\right)+f(\mathbf{x},t)=\rho c\left(\frac{\partial T(\mathbf{x},t)}{\partial t}\right) \qquad \mathbf{x}\in\Omega \tag{1}$$

where $\mathbf{x}=(x_1,x_2)$, $T(\mathbf{x},t)$ is the temperature at point $\mathbf{x}\in\Omega$ and at time t, $k(\mathbf{x})$ is the thermal conductivity, $f(\mathbf{x},t)$ is a known heat source, ρ is the density and c is the specific heat. The repeated subscript i denotes the summation through its range which is 2 for two-dimensional problem.

The initial condition is $T(\mathbf{x},0)=T_0$, where T_0 is a prescribed function. The boundary conditions are

$$T(\mathbf{x},t)=\overline{T}(\mathbf{x},t) \quad \mathbf{x}\in\Gamma_1 \tag{2}$$

$$-k\frac{\partial T}{\partial x_i}n_i=\overline{q}(\mathbf{x},t) \quad \mathbf{x}\in\Gamma_2 \tag{3}$$

where $\Gamma=\partial\Omega$, $\Gamma_1\cup\Gamma_2=\Gamma$, $\Gamma_1\cap\Gamma_2=\varnothing$, n_i is the i-th component of the outward normal vector $\mathbf{n}$ to the boundary Γ, $\overline{T}$ and $\overline{q}$ are prescribed temperature history and heat flux on the boundary, respectively.

Implementation of RIBEM

Boundary-domain Integral Equation

To derive the boundary integral equation, a weight function G is introduced to Eq. (1) and the following domain integrals can be written as

$$\int_\Omega G\frac{\partial}{\partial x_i}\left(k(\mathbf{x})\frac{\partial T}{\partial x_i}\right)\mathrm{d}\Omega+\int_\Omega Gf\mathrm{d}\Omega=\rho c\int_\Omega G\frac{\partial T}{\partial t}\mathrm{d}\Omega \tag{4}$$

Using Gauss' divergence theorem, the first domain integral can be manipulated as

$$\begin{aligned}&\int_\Omega G(\mathbf{x},\mathbf{y})\frac{\partial}{\partial x_i}\left(k(\mathbf{x})\frac{\partial T}{\partial x_i}\right)\mathrm{d}\Omega=\int_\Gamma G(\mathbf{x},\mathbf{y})k(\mathbf{x})\frac{\partial T}{\partial x_i}n_i\mathrm{d}\Gamma-\int_\Gamma k(\mathbf{x})T\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial x_i}n_i\mathrm{d}\Gamma+\\&\int_\Omega T\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial x_i}\frac{\partial k(\mathbf{x})}{\partial x_i}\mathrm{d}\Omega+\int_\Omega k(\mathbf{x})T\frac{\partial}{\partial x_i}\left(\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial x_i}\right)\mathrm{d}\Omega\end{aligned}\tag{5}$$

If Green's function ($-\ln r/2\pi$) is acted as the weight function G, the last domain integral in Eq. (5) can be written as

$$\int_\Omega k(\mathbf{x})T\frac{\partial}{\partial x_i}\left(\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial x_i}\right)\mathrm{d}\Omega=-k(\mathbf{y})T(\mathbf{y})\tag{6}$$

where $r(\mathbf{x},\mathbf{y})$ is the distance between the source point $\mathbf{y}$ and the field point $\mathbf{x}$. Substituting the equation into Eqs. (4) and (5), it follows that

$$\begin{aligned}&\tilde{T}(\mathbf{y},t)=-\int_\Gamma G(\mathbf{x},\mathbf{y})q(\mathbf{x},t)\mathrm{d}\Gamma-\int_\Gamma\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}}\tilde{T}(\mathbf{x},t)\mathrm{d}\Gamma+\int_\Omega G(\mathbf{x},\mathbf{y})f(\mathbf{x},t)\mathrm{d}\Omega+\\&\int_\Omega V(\mathbf{x},\mathbf{y})\tilde{T}(\mathbf{x},t)\mathrm{d}\Omega-\rho c\int_\Omega\frac{G(\mathbf{x},\mathbf{y})}{k(\mathbf{x})}\frac{\partial\tilde{T}(\mathbf{x},t)}{\partial t}\mathrm{d}\Omega\end{aligned}\tag{7}$$

where $q(\mathbf{x},t)=-k(\mathbf{x})\partial T(\mathbf{x},t)/\partial\mathbf{n}$, $\tilde{k}(\mathbf{x})=\ln k(\mathbf{x})$, $\tilde{T}(\mathbf{x},t)=k(\mathbf{x})T(\mathbf{x},t)$, $V(\mathbf{x},\mathbf{y})=(\partial G(\mathbf{x},\mathbf{y})/\partial x_i)(\partial\tilde{k}(\mathbf{x})/\partial x_i)$ in which $q(\mathbf{x},t)$ is the heat flux, $\tilde{T}(\mathbf{x},t)$ and $\tilde{k}(\mathbf{x})$ are the normalized temperature and thermal conductivity, respectively. Eq. (7) is valid only for internal points. For boundary points, a similar integral equation can be obtained by letting $\mathbf{y}\to\Gamma$ as is done in the conventional BEM such as

$$\begin{aligned}&c(\mathbf{y})\tilde{T}(\mathbf{y},t)=-\int_\Gamma G(\mathbf{x},\mathbf{y})q(\mathbf{x},t)\mathrm{d}\Gamma-\int_\Gamma\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}}\tilde{T}(\mathbf{x},t)\mathrm{d}\Gamma+\int_\Omega G(\mathbf{x},\mathbf{y})f(\mathbf{x},t)\mathrm{d}\Omega+\\&\int_\Omega V(\mathbf{x},\mathbf{y})\tilde{T}(\mathbf{x},t)\mathrm{d}\Omega-\rho c\int_\Omega\frac{G(\mathbf{x},\mathbf{y})}{k(\mathbf{x})}\frac{\partial\tilde{T}(\mathbf{x},t)}{\partial t}\mathrm{d}\Omega\end{aligned}\tag{8}$$

where

$$c(\mathbf{y})=\begin{cases}1 & , \quad \mathbf{y}\in\Omega\\ \dfrac{\varphi(\mathbf{y})}{2\pi} & , \quad \mathbf{y}\in\Gamma\end{cases}\tag{9}$$

$\varphi(\mathbf{y})$ is the interior angle at a point $\mathbf{y}$ of the boundary Γ. Particularly, $c(\mathbf{y})=0.5$ if $\mathbf{y}$ is a smooth point on the boundary.

Transformation of Domain Integrals to the Boundary by RIM

In general, the heat source $f(\mathbf{x},t)$ is a known function. In this circumstances, RIM [Gao (2002)] can be directly used to transform the first domain integral in Eq. (8) into the boundary as follows:

$$\int_\Omega G(\mathbf{x},\mathbf{y})f(\mathbf{x},t)\mathrm{d}\Omega(\mathbf{x})=\int_\Gamma\frac{1}{r(\mathbf{z},\mathbf{y})}\frac{\partial r}{\partial n}F^A(\mathbf{z},\mathbf{y},t)\mathrm{d}\Gamma(\mathbf{z})\tag{10}$$

where the radial integral F^A can be expressed as $F^A(\mathbf{z},\mathbf{y},t)=\int_0^{r(\mathbf{z},\mathbf{y})} G(\mathbf{x},\mathbf{y})\, f(\mathbf{x},t)\xi \mathrm{d}\xi$.

For the last two domain integrals in Eq. (8), the RIM formulation cannot be directly used because $\tilde{T}$ and $\partial\tilde{T}/\partial t$ are unknown. To solve this problem, $\tilde{T}$ and $\partial\tilde{T}/\partial t$ are approximated by the combination of the radial basis functions (RBFs) and the polynomials in terms of global coordinates [Zhang (2011)]. Thus, $\tilde{T}$ and $\partial\tilde{T}/\partial t$ are respectively expressed as

$$\tilde{T}=\sum_{i=1}^{N}\alpha_i\phi_i(R)+a_1x_1+a_2x_2+a_3 \tag{11}$$

$$\frac{\partial\tilde{T}}{\partial t}=\sum_{i=1}^{N}\beta_i\phi_i(R)+b_1x_1+b_2x_2+b_3 \tag{12}$$

and the following equilibrium conditions have to be satisfied:

$$\sum_{i=1}^{N}\alpha_i=\sum_{i=1}^{N}\alpha_i x_{1,i}=\sum_{i=1}^{N}\alpha_i x_{2,i}=0 \tag{13}$$

$$\sum_{i=1}^{N}\beta_i=\sum_{i=1}^{N}\beta_i x_{1,i}=\sum_{i=1}^{N}\beta_i x_{2,i}=0 \tag{14}$$

where N is the total number of boundary and interior nodes, $R=r(\mathbf{x},\ \mathbf{x}_i)$ is the distance from the *i*-th application point $\mathbf{x}_i=(x_{1,\,i},x_{2,\,i})$ to the field point $\mathbf{x}$ and $\phi(R)$ is the RBF. In this paper, the compactly supported fourth-order spline RBF is adopted, i.e.,

$$\phi_i(R)=\begin{cases}1-6\left(\dfrac{R}{d_i}\right)^2+8\left(\dfrac{R}{d_i}\right)^3-3\left(\dfrac{R}{d_i}\right)^4 & 0\le R<d_i\\ 0 & d_i\le R\end{cases} \tag{15}$$

in which d_i is radius of the supported region at the *i*-th point.

The coefficients α_i, a_1, a_2 and a_3 in Eq. (11) can be determined by collocating the application point $\mathbf{x}_i$ in Eq. (11) at all nodes. A set of algebraic equations can be written in the matrix form as $\tilde{\mathbf{T}}_\alpha=\boldsymbol{\phi\alpha}$, where $\boldsymbol{\alpha}=\{\alpha_1,\alpha_2,\cdots,\alpha_N,a_1,a_2,a_3\}^{\mathrm{T}}$, $\tilde{\mathbf{T}}_\alpha=\{\tilde{T}_1,\ \tilde{T}_2,\ \cdots,\ \tilde{T}_N,\ 0,\ 0,\ 0\}^{\mathrm{T}}=\{\{\tilde{\mathbf{T}}\}^{\mathrm{T}},\mathbf{0}\}^{\mathrm{T}}$. If no two nodes share the same coordinates, the matrix $\boldsymbol{\phi}$ is invertible and thereby $\boldsymbol{\alpha}=\boldsymbol{\phi}^{-1}\tilde{\mathbf{T}}_\alpha$. According to $\tilde{\mathbf{T}}_\alpha=\{\{\tilde{\mathbf{T}}\}^{\mathrm{T}},\mathbf{0}\}^{\mathrm{T}}$, the matrix $\boldsymbol{\phi}^{-1}$ can be expressed in the block form as $\left[(\tilde{\boldsymbol{\phi}}_1)_{(N+3)\times N},(\tilde{\boldsymbol{\phi}}_2)_{(N+3)\times 3}\right]$. Then $\boldsymbol{\alpha}$ can be rewritten as $\boldsymbol{\alpha}=\tilde{\boldsymbol{\phi}}_1\tilde{\mathbf{T}}$. Similarly, the coefficients in Eq. (12) can also be simply expressed as $\boldsymbol{\beta}=\tilde{\boldsymbol{\phi}}_1\dot{\tilde{\mathbf{T}}}$, where $\boldsymbol{\beta}=\{\beta_1,\beta_2,\cdots,\beta_N,b_1,b_2,b_3\}^{\mathrm{T}}$, $\dot{\tilde{\mathbf{T}}}=\{\partial\tilde{T}_1/\partial t,\ \partial\tilde{T}_2/\partial t,\ \cdots,\ \partial\tilde{T}_N/\partial t\}$.

Substituting Eqs. (11) and (12) into the last two domain integrals in Eq. (8), then transforming it into the boundary integrals by RIM, a pure boundary integral equation can be obtained as follows [Yu et al. (2014b)]:

$$c(\mathbf{y})\tilde{T}(\mathbf{y})=-\int_\Gamma G(\mathbf{x},\mathbf{y})q(\mathbf{x})\mathrm{d}\Gamma-\int_\Gamma\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}}\tilde{T}(\mathbf{x})\mathrm{d}\Gamma+\int_\Gamma\frac{1}{r}\frac{\partial r}{\partial\mathbf{n}}F^A\mathrm{d}\Gamma+\mathbf{V}_{\mathrm{y}}\tilde{\mathbf{T}}-\mathbf{C}_{\mathrm{y}}\dot{\tilde{\mathbf{T}}} \tag{16}$$

where $\mathbf{V}_{\mathrm{y}}$ and $\mathbf{C}_{\mathrm{y}}$ are the boundary integral terms corresponding to the last two domain integrals in Eq. (8).

System of Differential Equations

Assuming that the boundary Γ is discretized into N_b linear elements and the region is distributed N_I internal nodes, the total number of nodes is $N=N_b+N_I$. Eq. (16) can be conveniently expressed in the following matrix form:

$$\mathbf{C}_\alpha \tilde{\mathbf{T}}_b = \mathbf{G}_b\mathbf{Q}_b - \mathbf{H}_b\tilde{\mathbf{T}}_b + \mathbf{f}_b + \mathbf{V}_b\tilde{\mathbf{T}} - \mathbf{C}_b\dot{\tilde{\mathbf{T}}} \qquad on\ \Gamma \tag{17}$$

$$\tilde{\mathbf{T}}_I = \mathbf{G}_I\mathbf{Q}_b - \mathbf{H}_I\tilde{\mathbf{T}}_b + \mathbf{f}_I + \mathbf{V}_I\tilde{\mathbf{T}} - \mathbf{C}_I\dot{\tilde{\mathbf{T}}} \qquad in\ \Omega \tag{18}$$

where

$\mathbf{C}_\alpha = k\,\mathbf{diag}\left\{c(\mathbf{y}_1),\, c(\mathbf{y}_2),\, \cdots,\, c(\mathbf{y}_{N_b})\right\}$, $\mathbf{Q}_b = \left\{-k\partial T_1/\partial n,\ -k\partial T_2/\partial n,\ \cdots,\ -k\partial T_{N_b}/\partial n\right\}^{\mathrm{T}}$, $\tilde{\mathbf{T}}_b = \left\{\tilde{T}_1,\, \tilde{T}_2,\, \cdots,\, \tilde{T}_{N_b}\right\}^{\mathrm{T}}$, $\tilde{\mathbf{T}}_I = \left\{\tilde{T}_{I1},\, \tilde{T}_{I2},\, \cdots,\, \tilde{T}_{N_I}\right\}^{\mathrm{T}}$, $\mathbf{f}_b = \left\{f_1, f_2, \cdots, f_{N_b}\right\}^{\mathrm{T}}$, $\mathbf{f}_I = \left\{f_{I1}, f_{I2}, \cdots, f_{N_I}\right\}^{\mathrm{T}}$. The matrices $\mathbf{G}_b$, $\mathbf{H}_b$, $\mathbf{G}_I$ and $\mathbf{H}_I$ correspond to the coefficients of boundary integrals and $\mathbf{f}_b$, $\mathbf{V}_b$, $\mathbf{C}_b$, $\mathbf{f}_I$, $\mathbf{V}_I$ and $\mathbf{C}_I$ refer to the coefficients of domain integrals term.

After the application of boundary conditions and elimination the unknown heat flux quantity, a system of ordinary differential equations is obtained only relation to temperature as follows [Yu et al. (2014b)]:

$$\dot{\tilde{\mathbf{T}}}_u(t) = \mathbf{B}_u\tilde{\mathbf{T}}_u(t) + \mathbf{F}(t) \tag{19}$$

Adaptive Precise Integration Method

The general solution of Eq. (19) can be written as

$$\tilde{\mathbf{T}}_u(t_{k+1}) = \mathbf{E}\tilde{\mathbf{T}}_u(t_k) + \int_0^{\Delta t}\exp(\mathbf{B}_u(\Delta t - \xi))\mathbf{F}(t_k + \xi)\mathrm{d}\xi \tag{20}$$

where $\mathbf{E}=\exp(\mathbf{B}_u\Delta t)$ and $t_k = k\Delta t$. The matrix $\mathbf{E}$ can be rewritten as $\mathbf{E}=[\exp(\mathbf{B}_u\Delta t/m)]^m$, where m is an integer. Now, $m=2^M$ is selected, where M is an integer. The following truncated Taylor series expansion can be used:

$$\exp(\mathbf{B}_u\eta) \approx \mathbf{I} + \mathbf{B}_u\eta + (\mathbf{B}_u\eta)^2/2! + \cdots + (\mathbf{B}_u\eta)^p/p! = \mathbf{I} + \mathbf{E}_a \tag{21}$$

where $\eta = \Delta t/m$, $\mathbf{I}$ is the identity matrix. How to compute the matrix $\mathbf{E}$ has been detailedly shown in literature [Zhong (1994)].

The main factor of influence computation efficiency is how to select a optimal M and p. Because the most of the computational cost of PTI is the times of the matrix multiplications (TMM), where $\mathrm{TMM}=M+p-1$. The optimal selection of TMM is shown in literature [Chen et al. (2004)] for different prescribed error tolerance. In addition, in Eq. (20), the function $\mathbf{F}(t_k+\xi)$ is formed by the known temperature boundary conditions, heat flux boundary conditions or heat sources. In this article, the term $\int_0^{\Delta t}\exp(\mathbf{B}_u(\Delta t - \xi))\mathbf{F}(t_k+\xi)\mathrm{d}\xi$ in Eq. (20) is analytically integrated for all numerical example.

Finally, true temperature $T(\mathbf{x},t)$ can be computed by using $T(\mathbf{x},t) = \tilde{T}(\mathbf{x},t)/k(\mathbf{x})$.

Numerical Examples

To check the convergence of the proposed method, the root mean square (RMS) error is given by

$$RMS = \sqrt{\sum_{i=1}^{N}(T_{numerical,i} - T_{exact,i})^2 / \sum_{i=1}^{N} T_{exact,i}^2} \tag{22}$$

where $T_{numerical,i}$ and $T_{exact,i}$ are the numerical solution and the exact solution of the i-th node, respectively. For comparison, two examples are also computed by using the RIBEM, which use the finite difference technique to simulate the derivative of temperature with respect to time (it will be abbreviated to RIBEM-FD) [Yu et al. (2014b)].

Example 1: In this example, a square plate $\Omega = [1,2]^2$ is considered with $k(\mathbf{x}) = x_1 + x_2$, ρ=1 and $c = 1$. The initial condition and the heat source are T_0=x_1^2+x_2^2 and $f(\mathbf{x},t) = -6(x_1 + x_2) + 10\cos(10t)$, respectively. The boundary conditions are given by $T(x_1,1,t) = x_1^2 + 1 + \sin(10t)$, $T(2,x_2,t) = 4 + x_2^2 + \sin(10t)$, $T(x_1,2,t) = x_1^2 + 4 + \sin(10t)$, $T(1,x_2,t) = 1 + x_2^2 + \sin(10t)$. The exact solution of the problem is $T(\mathbf{x},t) = x_1^2 + x_2^2 + \sin(10t)$. The plate is discretized into 20 equally space linear boundary elements and distributed uniformly 16 internal nodes.

Table 1. The value of TMM for different ε_p

Δt	TMM $\varepsilon_p = 10^{-5}$	TMM $\varepsilon_p = 10^{-6}$	TMM $\varepsilon_p = 10^{-7}$	TMM $\varepsilon_p = 10^{-8}$	TMM $\varepsilon_p = 10^{-9}$	TMM $\varepsilon_p = 10^{-10}$	TMM $\varepsilon_p = 10^{-11}$	TMM $\varepsilon_p = 10^{-12}$
0.2	15	15	16	16	16	16	17	18
5	19	19	20	20	20	20	21	22

Table 1 shows the optimal value of TMM for different time step sizes and computational error tolerance. Comparison with the general selection TMM=23, the adaptive PTI improves the computational efficiency greatly. For different time step size, it can be seen from Figure 1 that the RMS errors of the PIBEM are highly coincident, but the errors of the RIBEM-FD emerge a big fluctuation.

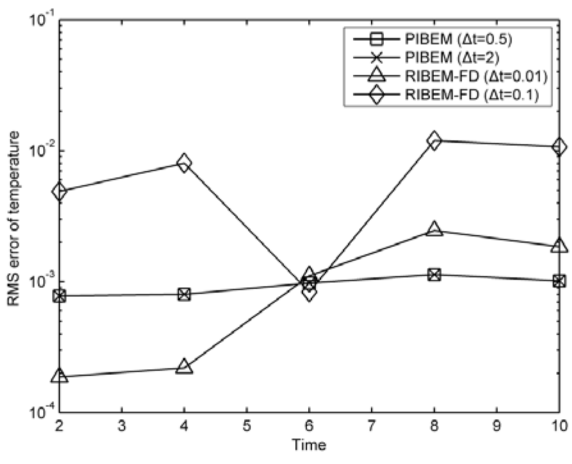

Figure 1. RMS error of temperature with $\varepsilon_p = 10^{-5}$ for example 1.

Example 2: In this example, we consider a concave geometry with $k(\mathbf{x}) = \exp(x_1)$, $\rho = c = 1$ and $0 < t \le 1$. The initial temperature and the heat source are $T_0 = 0$ and $f(x,t) = 10$, respectively. The time-dependent temperature condition is $T(0, y, t) = 60t$ for the left boundary and the other boundaries are insulated. The geometry and computational model of the BEM can be seen in Figure 2 with 36 boundary elements and 13 internal nodes. The problem is also computed using the FEM software ANSYS, which the results are considered as the reference solutions T_{exact} in Eq. (22). The solved domain is uniformly discretized into 832 4-noded elements. Table 2 shows the optimal value of TMM for different time step sizes and computational error tolerance. It can be seen from Figure 3 that the solutions of PIBEM are very stable and accurate than the solutions of RIBEM-FD for the different time step size.

Table 2. The value of TMM for different ε_p

Δt	TMM $\varepsilon_p = 10^{-5}$	TMM $\varepsilon_p = 10^{-6}$	TMM $\varepsilon_p = 10^{-7}$	TMM $\varepsilon_p = 10^{-8}$	TMM $\varepsilon_p = 10^{-9}$	TMM $\varepsilon_p = 10^{-10}$	TMM $\varepsilon_p = 10^{-11}$	TMM $\varepsilon_p = 10^{-12}$
0.001	9	9	10	10	10	10	11	12
0.2	17	17	18	18	18	18	19	20

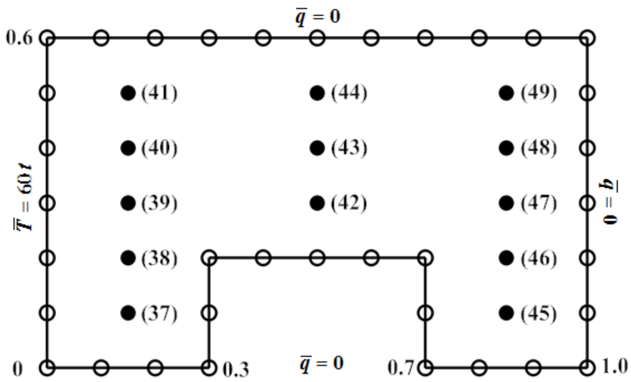

Figure 2. Computational model of the BEM for example 2.

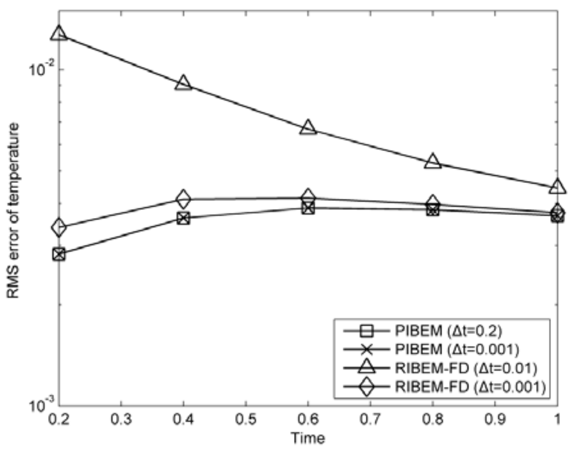

Figure 3. RMS error of temperature with $\varepsilon_p = 10^{-5}$ for example 2.

Conclusions

In this paper, the adaptive PIM is introduced into the RIBEM for solving the transient heat conduction problems with variable thermal conductivity. For the RIBEM-FD, the sensitive results are caused by the finite difference method to solve the derivative of temperature with respect to time. The PIBEM can perfectly solve the problem. Numerical examples show the PIBEM with adaptive technique can obtain the stable and accurate results for a big time step size and improve efficiency, whereas only in the case of a small time step the RIBEM-FD can obtain accurate results.

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