A multi-level meshless method based on an implicit use of the method of fundamental solutions

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Abstract

A new version of the Method of Fundamental Solutions is proposed. Instead of locating external point sources, an external boundary should be defined, and an extension of the original solution is created by enforcing the original boundary conditions for the extended problem. This leads to a better conditioned problem than the traditional Method of Fundamental Solutions. To numerically solve the extended problem, a quadtree-based multi-level finite volume method is used, which is quite economical from the computational points of view. In addition to it, the problem of large, dense and ill-conditioned systems is completely avoided.

Keywords: Meshless Methods, Method of Fundamental Solutions, Multi-level Methods, Quadtrees

Introduction

Due to its simplicity and accuracy, the Method of Fundamental Solutions (MFS, see e.g. [Alves et al. 2002]) has become quite popular among the meshless methods. This approach can be applied easily, if the fundamental solution of the original partial differential equation is known (or the original problem can be converted to such a problem). In its traditional form, the approximate solution is sought as a linear combination of the fundamental solution shifted to some external source points:

$$u(x) \approx \sum_{j=1}^{N} \alpha_j \Phi(x - \tilde{x}_j), \tag{1}$$

where $\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N$ are predefined *source points* located in the exterior of the domain Ω of the original partial differential equation, Φ denotes the fundamental solution. In case of the familiar second-order partial differential equations, Φ has a (weak) singularity at the origin, so that the approximate solution (1) has singularities at the source points but remains smooth inside the domain. The a priori unknown co-efficients $\alpha_1, \alpha_2, ..., \alpha_N$ can be computed by enforcing the boundary conditions at some *boundary collocation points* $x_1, x_2, ..., x_N$:

$$\begin{aligned} &\sum_{j=1}^{N} \alpha_j \Phi(x_k - \tilde{x}_j) = u_k \qquad (x_k \in \Gamma_D) \\ &\sum_{j=1}^{N} \alpha_j \frac{\partial \Phi}{\partial n_k}(x_k - \tilde{x}_j) = v_k \quad (x_k \in \Gamma_N) \end{aligned} \tag{2}$$

Here Γ_D and Γ_N denote the Dirichlet part (and the Neumann part, respectively) of the boundary of the domain Ω; n_k is the outward normal unit vector at the point x_k;

u_k , v_k are the Dirichlet and Neumann boundary conditions at the boundary collocation points.

Generally speaking, the MFS has excellent convergence properties (see [Li (2005)]); however, it is well known that the matrix of the system (2) is fully populated, non-selfadjoint and severely ill-conditioned, especially when the sources are located far from the boundary. On the other hand, if they are located too close to the boundary, numerical singularities are generated. In addition to it, the proper definition of the location of the sources can hardly be automatized. Therefore it is popular to allow the source points and the boundary collocation points to coincide, which needs special tricks to handle the problem of appearing singularities.

To circumvent these difficulties, a lot of approaches have been developed. In the boundary knot method [Chen (2002)], [Chen et al. (2005)], nonsingular general solutions are used instead of fundamental solutions. It is also possible to use fundamental solutions concentrated to *lines* rather than discrete points, see [Gáspár (2013b)]. In both approaches, the solution is approximated by nonsingular functions, thus, the problem of singularity is avoided. However, though they have especially good accuracy, the resulting linear system is severely ill-conditioned, which can cause serious computational difficulties. The problem is more difficult, if the original fundamental solution is used. Utilizing some boundary mesh structure, the appearing singular integrals can be evaluated analytically, see [Young et al. (2005)]. The desingularization can also be carried out by solving some auxiliary Dirichlet subproblem, as in the Modified Method of Fundamental Solutions (MMFS, see [Šarler (2008; 2009)]), or in the Boundary Distributed Source method (BDS, see [Liu (2010)]). It is also possible to combine the above approach with the use of approximate fundamental solutions which have no singularity at the origin; such an approximate fundamental solution may be the fundamental solution of the fourth-order Laplace-Helmholtz operator $\Delta(\Delta - c^2 I)$ with a sufficiently large scaling parameter *c*, (the Regularized Method of Fundamental Solutions, see [Gáspár (2013a; 2013b)]). A further possibility is that, in contrast to Eq. (1), the approximate solution is sought as a linear combination of the *normal derivatives* of Φ (dipole formulation), which can be considered a discretization of an indirect boundary integral equation based on a double layer potential and so forth.

Most of the above methods are indirect in the sense that they convert the original problem to the determination of some coefficients of the linear combination (i.e. the strengths of the point sources in the original MFS-formulation). These coefficients control the values (and their derivatives) at the collocation points, thus control the whole approximate solution inside the domain.

In this paper, we present a technique in which the values at the collocation points are controlled by the values at some external boundaries. In another words, an extension (continuation) of the solution is computed directly by prescribing certain, a priori unknown 'external boundary conditions'. If the external boundary is located sufficiently close to the original boundary, this results in much better conditioned problem. The external conditions are updated iteratively. During the iteration procedure, the original problem should be solved in a larger domain, which can economically be performed by using quadtree-based multi-level tools. In addition to this, the problem of large, fully populated and ill-conditioned matrices is completely avoided.

Approximation of the solution by external boundary conditions

To illustrate the above outlined idea, consider the 2D model problem with pure Dirichlet boundary conditions:

$$\Delta U = 0 \quad \text{in}\, \Omega, \qquad U|_{\Gamma} = u \tag{3}$$

defined on a circle $\Omega := \Omega_R := \{x \in \mathbf{R}^2 : \| x \| < R\}$ with boundary $\Gamma_R := \partial\Omega_R$. Let us express the function u in terms of (complex) Fourier series:

$$u(t) = \sum_k \alpha_k \cdot e^{ikt}, \tag{4}$$

where, for the sake of simplicity, we used polar coordinated. Then the solution of (3) is expressed as:

$$U(r,t) = \sum_k \alpha_k \cdot \left(\frac{r}{R}\right)^{|k|} \cdot e^{ikt} \tag{5}$$

Now consider a larger circle $\tilde{\Omega} := \Omega_{R+\delta}$ with boundary $\tilde{\Gamma} := \partial\Omega_{R+\delta}$, where $\delta > 0$. Then Eq. (5) defines a harmonic continuation of U to $\tilde{\Omega}$. On the boundary this yields:

$$\tilde{u}(t) = \sum_k \alpha_k \cdot \left(1 + \frac{\delta}{R}\right)^{|k|} \cdot e^{ikt}, \tag{6}$$

provided that the series is convergent in a proper function space. (This is sometimes not the case due to the exponentially increasing factor $\left(1 + \frac{\delta}{R}\right)^{|k|}$.)

Conversely, if U is prescribed along the external boundary $\tilde{\Gamma}$:

$$\Delta U = 0 \quad \text{in}\, \tilde{\Omega}, \qquad U|_{\tilde{\Gamma}} = \tilde{u}, \tag{7}$$

where $\tilde{u}(t) = \sum_k \beta_k \cdot e^{ikt}$, then the restriction of U to Γ defines a (much more smooth) function:

$$u(t) = \sum_k \beta_k \cdot \left(1 + \frac{\delta}{R}\right)^{-|k|} \cdot e^{ikt} \tag{8}$$

The operator $\tilde{A} : H^{1/2}(\tilde{\Gamma}) \to H^{1/2}(\Gamma)$ is always bounded and $\| \tilde{A} \| < 1$, but it is not regular, since the inverse operator is not bounded. However, the discretized operators might have uniformly bounded inverses, if the distance δ itself depends on the discretization. Let us discretize the above problems by cutting the Fourier series (6) and (8) at a maximal index N. Define the distance δ of the original and the external boundary to be inversely proportional to N, i.e. $\delta := \delta_0 \cdot \frac{2R\pi}{N}$. Then Eqs. (4), (6) are rewritten as:

$$u_N(t) = \sum_{|k|\le N} \alpha_k \cdot e^{ikt}, \qquad \tilde{u}_N(t) = \sum_{|k|\le N} \alpha_k \cdot \left(1+\frac{2\pi\delta_0}{N}\right)^{|k|} \cdot e^{ikt} \qquad (9)$$

Since $1 \le \left(1+\frac{2\pi\delta_0}{N}\right)^{|k|} \le \left(1+\frac{2\pi\delta_0}{N}\right)^{N} \le e^{2\pi\delta_0}$, the operators $\tilde{A}_N^{-1}$ defined by $\tilde{A}_N \tilde{u}_N = u_N$ are uniformly bounded, moreover, the condition numbers are also bounded: $\mathrm{cond}(\tilde{A}_N) \le e^{2\pi\delta_0}$, independently of *N*. Therefore the 'discrete' problem:

- find $\tilde{u}_N$ in such a way that the solution of the Dirichlet problem

$$\Delta U = 0 \quad \text{in } \tilde{\Omega}, \qquad U|_{\tilde{\Gamma}} = \tilde{u}_N, \qquad (10)$$

  satisfies the original 'inner' boundary condition $U|_{\Gamma} = u_N$;

is now a well-conditioned problem, independently on *N* (which characterizes the fineness of the discretization).

Based on the above considerations, in order to solve the Dirichlet problem (3), it is sufficient to solve the extended problem

$$\Delta U = 0 \quad \text{in } \tilde{\Omega}, \qquad U|_{\tilde{\Gamma}} = \tilde{u}, \qquad (11)$$

where the boundary condition $\tilde{u}$ should be chosen in such a way that the original boundary condition

$$U|_{\Gamma} = u, \qquad (12)$$

is satisfied. That is, the approximate solution of the original boundary value problem is controlled by the external boundary condition $\tilde{u}$ rather than external sources as in the case of the traditional method of fundamental solutions. It is expected that, in order to enforce the boundary condition (12), the simple, traditional iterative methods e.g. the Seidel iteration or the simplest Richardson iteration

$$\tilde{u}_{n+1} := \tilde{u}_n - \omega \cdot \left(U_n|_{\Gamma} - u\right) \qquad (13)$$

can be applied, where $\omega > 0$ is a predefined iteration parameter. In the above model problem, this remains the case.

The external boundary $\tilde{\Gamma}$ can be defined in a flexible way. The approach can easily be generalized for mixed boundary conditions and 3D problems as well.

Solution of the extended problem

In principle, the extended problem (11) can be solved by an arbitrary method. Since the external boundaries are not predefined, it is worth defining them in such a way that the extended problem (11) could easily be handled. In the next examples, the extended problem is solved by the regularized method of fundamental solutions [Gáspár (2013a)], which does not result in an optimal technique from a computational point of view, however, it demonstrates well the main idea of the approach.

Example 1. Let Ω be the unit circle and discretize the boundary Γ by the boundary collocation points $x_1, x_2, ..., x_N \in \Gamma$ in an equidistant way. Let the external boundary

$\tilde{\Gamma}$ be a concentric circle with radius $1+\delta$, where $\delta := \delta_0 \cdot \frac{2\pi}{N}$ (δ_0 is a constant of proportionality). That is, the distance of the original and the external boundary is proportional to the characteristic distance of the boundary collocation points. Consider the pure Dirichlet problem (3) with the simple test solution

$$U(x, y) = y \tag{13}$$

(using the more familiar notations *x*, *y* for the space variables). The extended problem (11) is (approximately) solved by the regularized method of fundamental solution (see [Gáspár (2013a)]), using the truncated fundamental solution:

$$\Phi(x) := \begin{cases} \frac{1}{2\pi} \log \| x \| & \text{if } \| x \| \geq \varepsilon \\ \\ \frac{1}{2\pi} \log \varepsilon & \text{if } \| x \| < \varepsilon \end{cases} \tag{14}$$

In this example, the truncation parameter ε was set to the value $\varepsilon := \frac{1}{4} \cdot \frac{2\pi(1+\delta)}{N}$.

The collocation points on the external boundary $\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N \in \tilde{\Gamma}$ were defined from the original collocation points $x_1, x_2, ..., x_N$ by shifting them in the outward normal direction. The extended solution is sought in the form:

$$U(x) := \sum_{j=1}^{N} \tilde{\alpha}_j \Phi(x - \tilde{x}_j), \tag{15}$$

where the coefficients $\tilde{\alpha}_j$ are determined by the system of equations:

$$\sum_{j=1}^{N} \tilde{\alpha}_j \Phi(\tilde{x}_k - \tilde{x}_j) =: \sum_{j=1}^{N} \tilde{\alpha}_j \tilde{A}_{kj} = \tilde{u}_k \qquad (k = 1,2,..., N) \tag{16}$$

In short:

$$\tilde{\mathbf{A}}\tilde{\alpha} = \tilde{\mathbf{u}} \tag{17}$$

(where the matrix $\tilde{\mathbf{A}}$ and the column vectors $\tilde{\alpha}, \tilde{\mathbf{u}}$ are assembled from the entries defined in Eq. (16)).

At the original boundary collocation points:

$$\sum_{j=1}^{N} \tilde{\alpha}_j \Phi(x_k - \tilde{x}_j) =: \sum_{j=1}^{N} \tilde{\alpha}_j A_{kj} = u_k \qquad (k = 1,2,..., N), \tag{18}$$

In short:

$$\mathbf{A}\tilde{\alpha} = \mathbf{u} \tag{19}$$

Eliminating the vector of coefficients $\tilde{\alpha}$, we have:

$$\mathbf{A}\tilde{\mathbf{A}}^{-1}\tilde{\mathbf{u}} = \mathbf{u} \tag{20}$$

Remark: In the traditional method of fundamental solutions, the coefficients $\tilde{\alpha}$ are determined by enforcing the original boundary conditions at the original boundary collocation points, i.e. by solving Eq. (19) only. However, the matrix $\mathbf{A}$ is generally much more ill-conditioned than $\mathbf{A}\tilde{\mathbf{A}}^{-1}$. That is, the original boundary conditions are more easily controllable by the external boundary conditions than by the strengths of the external sources. If the distance δ is small enough, then the matrix $\mathbf{A}\tilde{\mathbf{A}}^{-1}$ is diagonally dominant, and the Richardson iteration

$$\tilde{\mathbf{u}}_{n+1} := \tilde{\mathbf{u}}_n - \omega \cdot \left(\mathbf{A}\tilde{\mathbf{A}}^{-1}\tilde{\mathbf{u}}_n - \mathbf{u} \right) \tag{21}$$

is convergent for a sufficiently small iteration parameter $\omega > 0$.

In this example, the iteration parameter was set to $\omega := 1$. Table 1 shows the condition numbers of the matrices $\mathbf{A}$ and $\mathbf{A}\tilde{\mathbf{A}}^{-1}$ as well as the relative L_2-errors of the approximate solution (in %, after 5 Richardson iterations (21)) at the boundary collocation points with different numbers of boundary collocation points (*N*). The constant of proportionality δ_0 was set to $\delta_0 := 2$. The results demonstrate that the system (20) is really much better conditioned than the system (19) obtained by the classical method of fundamental solutions.

Table 1. Results of Example 1

N	16	32	64	128	256	512
cond($\mathbf{A}$)	478	1.1E+3	4.1E+3	1.1E+4	2.8E+4	6.2E+4
cond($\mathbf{A}\tilde{\mathbf{A}}^{-1}$)	46	89	139	180	207	226
Rel. L_2-error (after 5 Richardson iterations)	0.4528	0.0317	0.0012	3.25E–5	1.54E–5	2.44E-5

Example 2. The difference between the test problems of Example 1 and Example 2 is that now a mixed boundary condition is prescribed:

$$U|_{\Gamma_D} = u, \qquad \frac{\partial U}{\partial n}|_{\Gamma_N} = v, \tag{22}$$

A half of the boundary was considered to be the Dirichlet part Γ_D, and the remaining part was treated as a Neumann boundary Γ_N. The boundary conditions were defined to be consistent with the test solution (13). In principle, it is possible to use the same strategy as earlier, i.e. to control the original mixed boundary by pure Dirichlet condition on the external boundary, but this seemed to result in slow convergence. Instead, let us control the original mixed boundary by a similar mixed boundary condition on the external boundary as shown in the followings.

The extended problem:

$$\Delta U = 0 \quad \text{in } \tilde{\Omega}, \qquad U|_{\tilde{\Gamma}_D} = \tilde{u}, \quad \frac{\partial U}{\partial n}|_{\tilde{\Gamma}_N} = \tilde{v} \tag{23}$$

is solved again by a version of the regularized method of fundamental solutions, assuming the approximate solution in the following form

$$U(x) := \sum_{j=1}^{N} \tilde{\alpha}_j \Phi(x - \tilde{x}_j), \tag{24}$$

where Φ denotes again the truncated fundamental solution (14). The Dirichlet boundary condition is treated as earlier, but the proper treatment of the Neumann condition needs a desingularization procedure [Šarler (2008; 2009)], [Liu (2010)], [Gáspár (2013a; 2013b).] The normal derivatives of U are expressed as:

$$\frac{\partial U}{\partial n}(x) = \sum_{j=1}^{N} \tilde{\alpha}_j \frac{\partial \Phi}{\partial n}(x - \tilde{x}_j) \tag{24}$$

Consequently, the boundary values of the external boundary satisfy

$$\begin{aligned} &\sum_{j=1}^{N} \tilde{\alpha}_j \Phi(\tilde{x}_k - \tilde{x}_j) =: \sum_{j=1}^{N} \tilde{\alpha}_j \tilde{A}_{kj} = \tilde{u}_k \qquad (\tilde{x}_k \in \tilde{\Gamma}_D) \\ &\sum_{j=1}^{N} \tilde{\alpha}_j \frac{\partial \Phi}{\partial n_k}(\tilde{x}_k - \tilde{x}_j) =: \sum_{j=1}^{N} \tilde{\alpha}_j \tilde{B}_{kj} = \tilde{v}_k \qquad (\tilde{x}_k \in \tilde{\Gamma}_N) \end{aligned} \tag{25}$$

Note that the diagonal entries $\tilde{B}_{kk}$ should be defined in a special way (by solving a pure Dirichlet subproblem in the extended domain) due to the desingularization procedure, see [Liu (2010)], [Gáspár (2013a)] for details.

The original boundary conditions can be enforced by the following system of equations:

$$\begin{aligned} &\sum_{j=1}^{N} \tilde{\alpha}_j \Phi(x_k - \tilde{x}_j) =: \sum_{j=1}^{N} \tilde{\alpha}_j A_{kj} = u_k \qquad (x_k \in \Gamma_D) \\ &\sum_{j=1}^{N} \tilde{\alpha}_j \frac{\partial \Phi}{\partial n_k}(x_k - \tilde{x}_j) =: \sum_{j=1}^{N} \tilde{\alpha}_j B_{kj} = v_k \qquad (x_k \in \Gamma_N) \end{aligned} \tag{26}$$

Let us build up the following matrices and vectors:

$$\begin{aligned} &\tilde{C}_{kj} := \tilde{A}_{kj}, \quad C_{kj} := A_{kj}, \quad \tilde{w}_k := \tilde{u}_k, \quad w_k := u_k \qquad (x_k \in \Gamma_D) \\ &\tilde{C}_{kj} := \tilde{B}_{kj}, \quad C_{kj} := B_{kj}, \quad \tilde{w}_k := \tilde{v}_k, \quad w_k := v_k \qquad (x_k \in \Gamma_N) \end{aligned} \tag{26}$$

Then we have:

$$\tilde{\mathbf{C}}\tilde{\alpha} = \tilde{\mathbf{w}}, \qquad \mathbf{C}\tilde{\alpha} = \mathbf{w} \tag{27}$$

Eliminating the vector of coefficients $\tilde{\alpha}$, we have:

$$\mathbf{C}\tilde{\mathbf{C}}^{-1}\tilde{\mathbf{w}} = \mathbf{w}\,, \tag{28}$$

Once the external boundary conditions $\tilde{\mathbf{w}}$ have been computed, the coefficients can also be computed by $\tilde{\alpha} := \tilde{\mathbf{C}}^{-1}\tilde{\mathbf{w}}$. Thus, the approximate solution on the original boundary $\mathbf{A}\tilde{\alpha} = \mathbf{u}$, which makes it possible to directly compute the L_2-error of approximation at the boundary collocation points (referred to as ‘direct solution’ in

Table 2). The iteration parameter was set again to $\omega := 1$. Table 2 shows the condition numbers of the matrices $\mathbf{C}$ and $\mathbf{C}\tilde{\mathbf{C}}^{-1}$ as well as the relative L_2-errors of the approximate solution (in %, after 5 Richardson iterations) at the boundary collocation points with different numbers of boundary collocation points (N). The constant δ_0 was set to $\delta_0 := 2$. The results show that the method still works in case of mixed boundary conditions.

Finally note that the extended solution can be computed also in a quite economical way based on a non-uniform cell system and multi-level tools. This is outlined in the next section.

Table 2. Results of Example 2

N	16	32	64	128	256	512
cond($\mathbf{C}$)	322	1.1E+3	4.1E+3	1.2E+4	2.9E+4	6.5E+4
cond($\mathbf{C}\tilde{\mathbf{C}}^{-1}$)	65	218	1.0E+3	5.3E+3	2.4E+3	1.0E+5
Rel. L_2-error (direct solution)	0.0407	0.0077	0.0026	0.0014	0.0010	0.0008
Rel. L_2-error (after 5 Richardson iterations)	0.8515	0.2100	0.0526	0.0137	0.0036	0.0012

Multi-level solution using quadtree-based cell systems

From a computational point of view, the realization of the above methods is far from being optimal. However, if the extended problem (11) - (12) or (23) is handled directly, this makes it possible to use the much more economical multi-level techniques. Here a quadtree- (QT-) based finite volume method is used (see [Gáspár (2000)]). (The natural 3D generalization is based on the octtree cell system.) Strictly speaking, this is a domain type method; however, the generation of the cell system is performed entirely on the basis of the boundary collocation points in a completely automatic way, so that it can be considered a meshless method. The computational cost as well as the memory requirement is typically $\mathcal{O}(N \cdot \log N)$ only.

Thus, the solution algorithm is as follows.

- Generate a quadtree cell system by the boundary collocation points x_1, x_2,…, x_N. This results in a nested cell system with automatically created local refinements at the boundary collocation points. By additional subdivisions, it is possible to assure that the ratio of the sizes of the neighbouring cells is at most 2, i.e. no abrupt changes occur in cell sizes (regularization of the QT-cell system).
- Shift the points x_1, x_2,…, x_N in the outward normal direction with the distance δ. Determine the leaf cells of the QT-cell system which contain these points. (These cells have typically larger sizes than the finest cells containing

the boundary collocation points.) Define the external boundary points $\tilde{x}_1$, $\tilde{x}_2$, …, $\tilde{x}_N$ to be these cell centers.

- Using simple cell-centered finite volume schemes, solve the extended problem and update the external boundary condition by e.g. a Richardson iteration. Repeat this step until convergence.

The solution procedure can be embedded in a natural multi-level context (see [Gáspár (2000; 2004)] for details).

Example 3. Let Ω be a circle contained in the unit square $[0,1]\times[0,1]$ centered at the point $[0.5, 0.5]$ with radius $R := 0.30$. Discretize the boundary Γ by the boundary collocation points $x_1, x_2, ..., x_N \in \Gamma$ in an equidistant way. A regular QT-cell system was generated by recursively subdividing the unit square based on the boundary collocation points. The maximal subdivision level was 8, i.e. the finest cell size was 1/256. The collocation points of the external boundary $\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N \in \tilde{\Gamma}$ were defined as the cell centers of the boundary collocation points shifted in the outward normal direction with distance $\delta := \dfrac{2R\pi}{N}$. Figure 1. illustrates the QT-cell system and the external boundary points generated by 32 boundary collocation points.

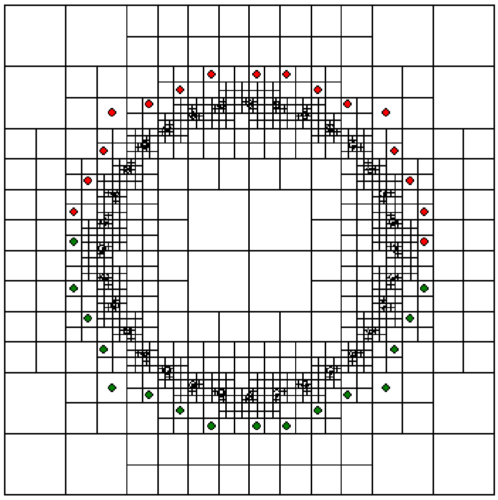

Figure 1. Quadtree cell system and external boundary points generated by 32 boundary collocation points

The test solution was as follows:

$$U(x, y) = -x + 2y + \frac{1}{2} \tag{29}$$

(using the more familiar notations *x*, *y* for the space variables). Mixed boundary conditions were prescribed: on a half of the boundary, Dirichlet boundary condition was supposed, while the remaining part of the boundary was considered as Neumann boundary. Along the boundary of the initial unit square, a separate boundary condition can be prescribed independently of the original boundary conditions; in this example, a homogeneous Dirichlet boundary condition was imposed. The method gives the approximate solutions in the interior and the exterior of the original domain at the same time. The Dirichlet data at the external boundary points were updated by Richardson iteration (13).

Another variant of the method was also tested. Here mixed boundary conditions were prescribed also along the external boundary. That is, the original Neumann boundary condition was controlled by an external Neumann boundary condition, which was updated by Richardson iteration as well:

$$\tilde{v}_{n+1} := \tilde{v}_n - \omega \cdot \left(\frac{\partial U_n}{\partial n} |_\Gamma - v \right) \tag{30}$$

At the Neumann part, the external boundary was supposed to be the union of circles centered at the external Neumann points; the radii were defined to be proportional to the characteristic distance of the external Neumann boundary points. This boundary condition was implemented on the (coarser) QT-cells containing the external Neumann points only. (The role of external boundaries is only to control the original boundary conditions at the original boundary collocation points, therefore the solution at the external boundaries is allowed to be less exact than at the original boundary.) Table 3 shows the relative L_2-errors (in %) at the original boundary collocation points in both cases. Here ‘Method 1’ refers to the method which controls the mixed boundary conditions via external Dirichlet boundary condition, while ‘Method 2’ corresponds to the external mixed boundary conditions. It can be clearly seen that the exactness of the two variants is the essentially the same: however, the second variant has proved faster.

Table 3. Relative L_2-errors using QT-cell system, Example 3

N	16	32	64	128	256
Rel. L_2-error (Method 1)	0.7035	0.1606	0.0918	0.0338	0.005
Rel. L_2-error (Method 2)	0.7036	0.1606	0.0913	0.0337	0.004

For illustration, Figure 2 shows the approximate solution on the QT-cell system with 32 boundary collocation points. Along the boundary of the initial rectangle of the QT-subdivision, a homogeneous Dirichlet boundary condition was prescribed. In the vicinity of the external boundary, the solution is less smooth than the interior of the original domain, similarly to the case of the traditional method of fundamental solutions. However, the irregularity is much less, due to the fact that the solution is controlled by external boundary condition rather than the strengths of the external singularities; moreover, the cell system is allowed to be coarser here than in the vicinity of boundary collocation points.

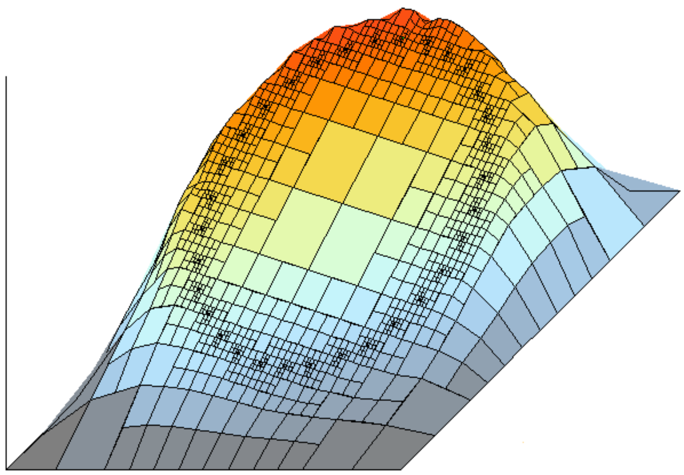

Figure 2. Approximate test solution on a quadtree cell system generated by 32 boundary collocation points

Conclusions

In this paper, the original idea of the Method of Fundamental Solution has been extended in the sense that the approximate solution was sought as a solution of an extended problem with an external boundary. This often results in a well-conditioned problem provided that the external boundary is located sufficiently close to the original boundary (depending also on the discretization). The method controls the values along the original boundary via the external boundary conditions. These external boundary conditions are adjusted iteratively using familiar, simple iterative techniques. The external boundary can be defined in a flexible way. In the vicinity of the external boundary, the approximate solution is allowed to be less exact than along the original boundary, which makes it possible to apply a coarser discretization at the external boundaries. The numerical benefit of the approach is that the extended problem can be handled by the quite economical quadtree-based multi-level method. Moreover, the problem of large, dense and ill-conditioned systems of equations is also avoided.

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