Global sensitivity analysis for structural models by sparse grid integration

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Abstract
Global sensitivity indices based on variance can effectively apportion the output uncertainty to the inputs. How to efficiently and accurately perform the global sensitivity analysis is of great concern for researchers. In this work, the employment of sparse grid integration to the estimate of global sensitivity indices is discussed. The new method can be used for sensitivity analysis of the structural models involving independent variables or correlated variables, and can further decompose the variance contribution in the correlation cases. Advantage of the sparse grid integration in estimating integrals is well inherited by the new method, to ensure the accuracy while keeping the computational burden controllable. Numerical and engineering examples have been studied to test the applicability of the proposed method.
Keywords: Sensitivity analysis; variance; uncertainty; sparse grid integration

1. Introduction

Global sensitivity indices are playing an important role in identifying and representing uncertainties in engineering, and many researchers have proposed their own indices, as well as corresponding computing techniques [Borgonovo (2007); Sobol (2001)]. Among these different indices, the variance based ones have attracted increasing interests as they are able to capture the influence of the full range of variation of each input factor, and reflect the interaction effects among input factors. Variance based sensitivity analysis has been acknowledged as a versatile and effective tool in the uncertainty analysis.

In the real world, input factors of a model are often correlated to each other, and sometimes the correlation may have significant impacts on the sensitivity results [Borgonovo and Tarantola (2008); Kucherenko et al. (2012); Mara (2009)]. Xu and Gertner (2008) pointed out that the contribution of uncertainty to the output by an individual input factor should be decomposed into two parts: the uncorrelated part, which means this part is completely immune from the other input factors and is produced by this input factor “individually and independently”, and the correlated part, which means this part is produced by the correlation of this input variable with the others. Mara and Tarantola (2012) proposed a set of variance-based sensitivity indices to perform sensitivity analysis of models with correlated inputs, which can distinguish between the mutual dependent contribution and the independent contribution of an input to the model response variance.

An important task in the sensitivity analysis is to improve the computational efficiency, especially in engineering cases where the models involved usually take a long processing time [Saltelli et al. (2010)]. It is found that the variance based sensitivity indices can be viewed as nested expressions of expectation operator and variance operator. Enlightened by this feature, in this work a new method is developed for global sensitivity analysis by using the sparse grid integration (SGI) technique. The sparse grid technique, which is based on one-dimensional formulae and then extended to higher dimensions, has been extensively utilized for highly dimensional multivariate integration, as well as interpolation [Barthelmann et al. (2000); Gerstner and Griebel (2003); Gerstner and Griebel (1998); Smolyak (1963)]. To a certain extent, this technique avoids the "curse of dimension" of conventional integration algorithms, which means the computational cost grows exponentially with the dimension of the problem. In the SGI technique, multivariate quadrature formulae are constructed with combinations of tensor products of suitable one-dimensional formulae, thus the function evaluations needed and the numerical accuracy become independent of the dimension of the problem up to logarithmic factors. Existing literature has reported the high efficiency and accuracy of sparse grid applications.

2. Review on variance based sensitivity indices

2.1 Sensitivity analysis of model output with independent inputs

Let $Y = g(\boldsymbol{X})$ be the performance function of the model under investigation, with Y the output, $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^{\mathrm{T}}$ the vector of independent input variables, where X_i is the *i*th variable. Sobol (2001) proposed that the performance function can always be decomposed into summands of different dimensions, that is

$$g(\boldsymbol{X}) = g_0 + \sum_{i=1}^{n} g_i(X_i) + \sum_{i<j}^{n} g_{i,j}(X_i, X_j) + \cdots + g_{1,2,\ldots,n}(X_1, X_2, \ldots, X_n) \tag{1}$$

Sensitivity analysis based on variance is to quantify the contribution of an individual input variable to the output variance, and Sobol proposed the variance decomposition equation based on Eq.(1),

$$V(Y) = \sum_{i=1}^{n} V_i + \sum_{i=1, j>i}^{n} V_{ij} + \cdots + V_{1,2,\ldots,n} \tag{2}$$

V_i is the first order variance contribution of X_i, and can be formulated as

$$V_i = V_{X_i}(E_{\boldsymbol{X}_{-i}}(Y \mid X_i)) \tag{3}$$

where $\boldsymbol{X}_{-i}$ denotes the vector of all input variables except X_i, i.e. $\boldsymbol{X}_{-i} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)^{\mathrm{T}}$. V_{ij} and higher order variance terms in Eq.(2) denote the contribution to the output variance of variable interaction brought by the form of the performance function. When only the first order variance contribution is considered, the variance decomposition can be reformulated as

$$V(Y) = \sum_{i=1}^{n} V_i \tag{4}$$

The first order variance contribution V_i is also referred to as the main effect of X_i on the output variance, and it measures the first order effect of X_i on the output, ignoring the interactions between X_i and the other variables. When taking the interactions into consideration, the total contribution of X_i is measured by $E_{\boldsymbol{X}_{-i}}(V_{X_i}(Y \mid \boldsymbol{X}_{-i}))$. According to the known identity:

$$V_{\boldsymbol{X}_{-i}}(E_{X_i}(Y \mid \boldsymbol{X}_{-i})) + E_{\boldsymbol{X}_{-i}}(V_{X_i}(Y \mid \boldsymbol{X}_{-i})) = V(Y) \tag{5}$$

$V_{\boldsymbol{X}_{-i}}(E_{X_i}(Y \mid \boldsymbol{X}_{-i}))$ can be seen as the first order effect of $\boldsymbol{X}_{-i}$, thus $V(Y)$ minus $V_{\boldsymbol{X}_{-i}}(E_{X_i}(Y \mid \boldsymbol{X}_{-i}))$ should give the contribution of all terms in the variance decomposition which includes X_i.

To normalize the variance contribution, the main effect index is defined as [Saltelli et al. (2010)]

$$S_i^M = \frac{V_{X_i}(E_{\mathbf{X}_{-i}}(Y \mid X_i))}{V(Y)} \tag{6}$$

and the total effect index is defined as

$$S_i^T = \frac{E_{\mathbf{X}_{-i}}(V_{X_i}(Y \mid \mathbf{X}_{-i}))}{V(Y)} \tag{7}$$

2.2 Sensitivity analysis of model output with correlated inputs

The variance decomposition in Eq.(2) is proposed under the assumption of input independence, which thus means it may not hold when the input variables are correlated. For correlated variables, the main effect index and total effect index still reflect the variance contribution of variables, yet the difference from the independence case lies in that both indices are now composed by two parts. Take V_i for example, according to Xu and Gertner (2008), it can be divided into two parts, i.e. the variance contribution of the uncorrelated part of X_i, denoted by V_i^U, and the variance contribution of the correlated part with the other variables, denoted by V_i^C. It should be pointed out that the variance contribution of the correlated part of X_i is way different from the interaction contribution of X_i to the output variance. These two are absolutely different concepts, as the former comes from the correlation among the input variables, and the latter is produced by variable interactions in the performance function.

Thus, the main effect index, S_i^M, can be decomposed in the following way,

$$S_i^M = S_i^{MC} + S_i^{MU} \tag{8}$$

where S_i^{MC} denotes the correlated main contribution of X_i to the output variance, and S_i^{MU} denotes the uncorrelated main contribution. Similarly, the total effect index, S_i^T, can be decomposed as

$$S_i^T = S_i^{TC} + S_i^{TU} \tag{9}$$

where S_i^{TC} denotes the correlated total contribution of X_i to the output variance, and S_i^{TU} denotes the uncorrelated total contribution.

In fact, in the existing literature measuring the contributions of inputs to the output variance for cases involving correlated inputs is still a tricky issue. Researchers have proposed different variance based sensitivity indices based on different considerations, and it is difficult to judge which one is better. Agreement is highly needed to give an exact and unambiguous definition of the ANOVA for correlated inputs just as the one provided by Sobol decomposition when the inputs are independent. In this work, the sensitivity indices talked about are related to those in the work of Mara and Tarantola (2012).

3. Variance based sensitivity analysis with SGI

The SGI technique has been proven to be an effective tool in the uncertainty analysis, and it will be employed to perform the variance based sensitivity in this section. It should be pointed out that normally the SGI procedure is carried out in the independent space, which means it cannot be directly used when input variables are correlated. In this section, the procedure using SGI for sensitivity analysis of models with independent variables is first introduced. Afterwards, the use of SGI for variance based sensitivity analysis with correlated variables is discussed, which is partly based on the work of Mara and Tarantola (2012).

3.1 Algorithm based on SGI for sensitivity analysis of independent variables

For the performance function $Y = g(\boldsymbol{X})$ with independent variables, the expectation and variance of the output can be estimated by SGI according to the following formulae,

$$E(Y) = \int g(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \sum_{j=1}^{p} w_j g(\boldsymbol{x}_j) \tag{10}$$

$$V(Y) = \int (g(\boldsymbol{x}) - E(Y))^2 f_{\boldsymbol{X}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \sum_{j=1}^{p} w_j (g(\boldsymbol{x}_j) - E(Y))^2 \tag{11}$$

where the *n*-dimension quadrature point $\boldsymbol{x}_j$ and the associated weight w_j ($j = 1, 2, \ldots, p$) are obtained by the Smolyak algorithm [Gerstner and Griebel (2003); Gerstner and Griebel (1998)].

With the output variance obtained, $V_{X_i}(E_{\boldsymbol{X}_{-i}}(Y \mid X_i))$ and $E_{\boldsymbol{X}_{-i}}(V_{X_i}(Y \mid \boldsymbol{X}_{-i}))$, both of which can be seen as nested expressions of expectation operator and variance operator, need to be estimated to get the indices S_i^M and S_i^T.

Consider $V_{X_i}(E_{\boldsymbol{X}_{-i}}(Y \mid X_i))$ first. Keep X_i fixed, and treat $\boldsymbol{X}_{-i}$ as variables, then Y can be seen as the function of $\boldsymbol{X}_{-i}$. The inner expectation can be estimated by SGI as follows,

$$E_{\boldsymbol{X}_{-i}}(Y \mid X_i) = \int g(\boldsymbol{x}_{-i}, x_i) f_{\boldsymbol{X}_{-i}}(\boldsymbol{x}_{-i})\,\mathrm{d}\boldsymbol{x}_{-i} \approx \sum_{j=1}^{m} w_j g(\boldsymbol{x}_{-i}^j, x_i) \tag{12}$$

where $\boldsymbol{x}_{-i}^j$ denotes the *j*-th value of $\boldsymbol{X}_{-i}$. Apparently $E_{\boldsymbol{X}_{-i}}(Y \mid X_i)$ can be seen as a univariate function of X_i, and is redefined as

$$\varphi(X_i) = E_{\boldsymbol{X}_{-i}}(Y \mid X_i) \tag{13}$$

Thus,

$$V_{X_i}(E_{\boldsymbol{X}_{-i}}(Y \mid X_i)) = V(\varphi(X_i)) \tag{14}$$

The variance of the univariate function $\varphi(X_i)$ can be estimated by SGI as follows,

$$V(\varphi(X_i)) = \int (\varphi(x_i) - E(\varphi(x_i)))^2 f_{X_i}(x_i)\,\mathrm{d}x_i \approx \sum_{j=1}^{\tilde{m}} w_j (g(x_i^j) - E(\varphi(X_i)))^2 \tag{15}$$

where

$$E(\varphi(X_i)) = \int \varphi(x_i) f_{X_i}(x_i)\,\mathrm{d}x_i \approx \sum_{j=1}^{\tilde{m}} w_j g(x_i^j) \tag{16}$$

Finally, the main effect index can be obtained as

$$S_i^M = \frac{V_{X_i}(E_{\boldsymbol{X}_{-i}}(Y \mid X_i))}{V(Y)} = \frac{V(\varphi(X_i))}{V(Y)} \tag{17}$$

The total effect index S_i^T can be obtained in the similar manner.

3.2 Algorithm based on SGI for sensitivity analysis of correlated variables

When the input variables are correlated, the algorithm based on SGI in Section 3.1 is no longer applicable. In this section, the SGI technique is extended to the case of correlated variables to perform the variance based sensitivity analysis. In this work, the correlation matrix, which is a symmetric matrix composed of Pearson correlation coefficients, is adopted to measure the correlation between random variables.

3.2.1 Estimate of the response variance

Eq.(10) and Eq.(11) in Section 3.1 can be used to estimate the expectation and variance of the model output only when the variables are independent. Thus, for correlated variables there has to be a transformation from correlation space to independence space. In this work, we only consider the problems with normal distributions, i.e. $X_i \sim N(\mu_{X_i}, \sigma_{X_i}^2)$.

The covariance matrix of $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^{\mathrm{T}}$ is denoted as

$$\boldsymbol{C_X} = \begin{pmatrix} \sigma_{X_1}^2 & Cov(X_1, X_2) & \cdots & Cov(X_1, X_n) \\ Cov(X_2, X_1) & \sigma_{X_2}^2 & \cdots & Cov(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(X_n, X_1) & Cov(X_n, X_2) & \cdots & \sigma_{X_n}^2 \end{pmatrix} \tag{18}$$

where $Cov\left(X_i, X_j\right) = Cov\left(X_j, X_i\right) = \rho_{ij}\sigma_{X_i}\sigma_{X_j}$, as ρ_{ij} is the Pearson correlation coefficient of X_i and X_j. The joint PDF of $\boldsymbol{X}$ is

$$f\left(\boldsymbol{X}\right) = \left(2\pi\right)^{-\frac{n}{2}} \left|\boldsymbol{C_X}\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{X} - \boldsymbol{\mu_X}\right)^{\mathrm{T}} \boldsymbol{C_X}^{-1}\left(\boldsymbol{X} - \boldsymbol{\mu_X}\right)\right\} \tag{19}$$

where $\boldsymbol{\mu_X} = (\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n})^{\mathrm{T}}$ is the vector of input expectations, $\left|\boldsymbol{C_X}\right|$ and $\boldsymbol{C_X}^{-1}$ are the determinant value and reverse matrix of $\boldsymbol{C_X}$ respectively.

An orthogonal matrix $\boldsymbol{A}$ exists which would introduce random variable vector $\boldsymbol{U} = (U_1, U_2, \ldots, U_n)^{\mathrm{T}}$ by the following formula [Shi et al. (2009)]:

$$f(\boldsymbol{AU} + \boldsymbol{\mu_X}) = \left(2\pi\right)^{-\frac{n}{2}} (\lambda_1\lambda_2 \cdots \lambda_n)^{-\frac{1}{2}} \exp(-\frac{1}{2}\sum_{i=1}^{n}\frac{U_i^2}{\lambda_i}) \tag{20}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the latent roots of $\boldsymbol{C_X}$.

Thus the n-dimension correlated input variables $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^{\mathrm{T}}$ can be transformed to independent normal variables $\boldsymbol{U} = (U_1, U_2, \ldots, U_n)^{\mathrm{T}}$ by

$$\boldsymbol{U} = \boldsymbol{A}^T(\boldsymbol{X} - \boldsymbol{\mu_X}) \tag{21}$$

where the column vectors of $\boldsymbol{A}$ are the latent root vectors of $\boldsymbol{C_X}$, and the probability distribution of the independent normal variables can be obtained as $U_i \sim N(0, \lambda_i)$.

By Eq.(21), we further get

$$\boldsymbol{X} = \boldsymbol{AU} + \boldsymbol{\mu_X} \tag{22}$$

Substitute Eq.(22) into $Y = g(\boldsymbol{X})$, the performance function in the independence space can be obtained as follows,

$$Y = g(\boldsymbol{X}) = \phi(\boldsymbol{U}) \tag{23}$$

where $\phi(\bullet)$ denotes the mapping relationship between $\boldsymbol{U}$ and the output. Because distribution parameters of $\boldsymbol{U}$ in the dependence space have been obtained, the expectation and variance of the model output can be readily estimated by Eq.(10) and

Eq.(11).

3.2.2 Orthogonalization of correlated variables

For two normal variables, X_i and X_j, $E(X_i \mid X_j)$ can be used to define the marginal relationship between them. The value of $E(X_i \mid X_j)$ measures how much X_i is correlated to X_j, and equals $E(X_i)$ when the two variables are independent. In the same manner, $E(X_i \mid X_j, X_k)$ defines the correlation of X_i to X_j and X_k. Mara and Tarantola declared that for non-normal random variables, higher conditional moments are needed to characterize the correlation between the variables. Issues concerning correlated non-normal variables would become much complicated in most cases. In this work, the scope of research is confined to problems only involving normally distributed variables.

For a vector of correlated normal variables, $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^{\mathrm{T}}$, the following relationship holds,

$$f(X_1, X_2, ..., X_n) = f(X_1) f(X_2 \mid X_1) \cdots f(X_n \mid X_1, X_2, ..., X_{n-1}) \tag{24}$$

where $f(X_1, X_2, ..., X_n)$ is the joint PDF, $f(X_1)$ is the marginal PDF of X_1, $f(X_2 \mid X_1)$ is the marginal PDF of X_2 conditioned on X_1, $f(X_n \mid X_1, X_2, ..., X_{n-1})$ is the marginal PDF of X_n conditioned on $(X_1, X_2, ..., X_{n-1})^{\mathrm{T}}$. As we have talked about, $E(X_i \mid X_j)$ can quantify the marginal relationship between two correlated normal variables, thus the following equations hold,

$$\begin{aligned} \bar{X}_1 &= X_1 \\ \bar{X}_2 &= X_{2-1} = X_2 - E(X_2 \mid X_1) \\ \bar{X}_3 &= X_{3-12} = X_3 - E(X_3 \mid X_1, X_2) \\ &\cdots \\ \bar{X}_n &= X_{n-12\ldots(n-1)} = X_n - E(X_n \mid X_1, X_2, ..., X_{n-1}) \end{aligned} \tag{25}$$

Mara and Tarantola (2012) pointed out that the above transformation is one of Rosenblatt's for normal variables. A new set of variables $(\bar{X}_1, \bar{X}_2, ..., \bar{X}_n)^{\mathrm{T}}$ can be generated by the above transformation. In fact, these new variables are obtained by subtracting the correlation part from the original correlated variables, thus the new variables are independent from each other.

Besides, the orthogonalization from correlated variables to independent ones is not unique. This can be seen from the transformation in Eq.(25), which clearly depends on the ordering of variables. If we reorder the original input variables as $\boldsymbol{X} = (X_2, \ldots, X_n, X_1)$, which is valid except the corresponding correlation matrix needs to be modified accordingly, and apply the orthogonalization, a new set of independent variables will be obtained and denoted as $(\bar{X}_2, ..., \bar{X}_n, \bar{X}_1)$. It should be reminded that $(\bar{X}_2, ..., \bar{X}_n, \bar{X}_1)$ is different from $(\bar{X}_1, \bar{X}_2, ..., \bar{X}_n)$ initially obtained by orthogonalizing $(X_1, X_2, \ldots, X_n)$. In fact, through changing the ordering of the original correlated variables in sequence, a total of $n!$ sets of independent variables can be generated by the orthogonalization. However, among all these sets, only a total of n sets generated by cycling the orthogonalization are used in this work, i.e. $(\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n)$, $(\bar{X}_2, \ldots, \bar{X}_n, \bar{X}_1), \ldots, (\bar{X}_n, \ldots, \bar{X}_{n-2}, \bar{X}_{n-1})$.

3.2.3 Interpretations on sensitivity indices of the newly independent variables

It is necessary to find out the relationship between the sensitivity indices of the independent variables and those of correlated ones. Let us focus on the main effect index first, which is denoted as

$$\begin{aligned}
\bar{S}_1^M &= V[E(Y \mid \bar{X}_1)]/V(Y) \\
\bar{S}_2^M &= V[E(Y \mid \bar{X}_2)]/V(Y) \\
\bar{S}_3^M &= V[E(Y \mid \bar{X}_3)]/V(Y) \\
&\dots \\
\bar{S}_n^M &= V[E(Y \mid \bar{X}_n)]/V(Y)
\end{aligned} \tag{26}$$

Now consider the explanation of the sensitivity indices. Because $\bar{X}_1 = X_1$, $\bar{S}_1^M$ is the full marginal contribution of X_1 to the response variance, which means $\bar{S}_1^M = S_1^M$. $\bar{S}_2^M$ is the marginal contribution of X_2 to the response variance without its correlative contribution with X_1, as $\bar{X}_2$ is uncorrelated with X_1. In the same manner, $\bar{S}_3^M$ is the marginal contribution of X_3 to the response variance without its correlative contribution with $(X_1, X_2)^{\mathrm{T}}$. For the last new variable $\bar{X}_n$, $\bar{S}_n^M$ is the uncorrelated marginal contribution to the response variance, which means $\bar{S}_n^M = S_n^{MU}$. $\bar{X}_1$ keeps all the information concerning X_1 including its correlated part with the other variables while $\bar{X}_n$ only keeps the independent part of X_n excluding all of its correlated part.

Clearly, by the variance based sensitivity analysis on $(\bar{X}_1, \bar{X}_2, ..., \bar{X}_n)^{\mathrm{T}}$, the main effect index of X_1, i.e. S_1^M, as well as the uncorrelated main effect index of X_n, i.e. S_n^{MU}, can be obtained. Remember that a total of *n* sets of independent variables can be generated by orthogonalizing the original correlated variables in cycle. In the same way of obtaining S_1^M and S_n^{MU}, the indices S_2^M and S_1^{MU} can be obtained by the sensitivity analysis on $(\bar{X}_2, ..., \bar{X}_n, \bar{X}_1)$. Change the order of the original correlated variables, and perform orthogonalization to get the corresponding independent variables, which is further analyzed to get the sensitivity indices, the full and uncorrelated contributions of each correlated variable can be obtained. Similarly, the total effect index S_i^T of the correlated inputs can be thus estimated and decomposed.

3.2.4 Computational issues

In this section, the computational issues involved in the procedure of sensitivity analysis are addressed. Still, we take the analysis on $(\bar{X}_1, \bar{X}_2, ..., \bar{X}_n)^{\mathrm{T}}$ as an example. To analyze the sensitivity indices on $(\bar{X}_1, \bar{X}_2, ..., \bar{X}_n)^{\mathrm{T}}$, firstly the statistical characteristics have to be known.

By Eq.(25) we know the mean value and standard deviation of $\bar{X}_1$ are equal to those of X_1. For the *i*th (i >1) variable, the following equation holds according to Eq.(25),

$$\bar{X}_i = X_i - E(X_i \mid X_1, X_2, ..., X_{i-1}) \tag{27}$$

Only the first *i* variables are involved in Eq.(27), i.e. $\boldsymbol{X}_{1\sim i} = (X_1, X_2, ..., X_i)^{\mathrm{T}}$, of which the mean vector is $\boldsymbol{\mu}_{\boldsymbol{X}_{1\sim i}} = (\mu_{X_1}, \mu_{X_2}, \dots, \mu_{X_i})^{\mathrm{T}}$, and the corresponding covariance matrix can be taken from Eq.(18) as

$$\boldsymbol{C}_{\boldsymbol{X}_{1\sim i}} = \begin{pmatrix} \sigma_{X_1}^2 & Cov(X_1, X_2) & \cdots & Cov(X_1, X_i) \\ Cov(X_2, X_1) & \sigma_{X_2}^2 & \cdots & Cov(X_2, X_i) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(X_i, X_1) & Cov(X_i, X_2) & \cdots & \sigma_{X_i}^2 \end{pmatrix} \tag{28}$$

Now consider the conditional mean value in Eq.(27), $E(X_i \mid X_1, X_2, ..., X_{i-1})$. The mean vector and covariance matrix for $\boldsymbol{X}_{1\sim i}$ can be rewritten as $\boldsymbol{\mu}_{\boldsymbol{X}_{1\sim i}} = \begin{bmatrix} \mu_{X_i} \\ \boldsymbol{\mu}_{\boldsymbol{X}_{1\sim(i-1)}} \end{bmatrix}$ and $\boldsymbol{C}_{\boldsymbol{X}_{1\sim i}} = \begin{bmatrix} \boldsymbol{C}_{X_i X_i} & \boldsymbol{C}_{X_i \boldsymbol{X}_{1\sim(i-1)}} \\ \boldsymbol{C}_{\boldsymbol{X}_{1\sim(i-1)} X_i} & \boldsymbol{C}_{\boldsymbol{X}_{1\sim(i-1)} \boldsymbol{X}_{1\sim(i-1)}} \end{bmatrix}$, where

$$\boldsymbol{C}_{X_i X_i} = \sigma_{X_i}^2 \tag{29}$$

$$\boldsymbol{C}_{\boldsymbol{X}_{1\sim(i-1)} X_i}^T = \boldsymbol{C}_{X_i \boldsymbol{X}_{1\sim(i-1)}} = \left[Cov(X_i, X_1) \;\; Cov(X_i, X_2) \;\; \cdots \;\; Cov(X_i, X_{i-1})\right] \tag{30}$$

$$\boldsymbol{C}_{\boldsymbol{X}_{1\sim(i-1)} \boldsymbol{X}_{1\sim(i-1)}} = \begin{pmatrix} \sigma_{X_1}^2 & Cov(X_1, X_2) & \cdots & Cov(X_1, X_{i-1}) \\ Cov(X_2, X_1) & \sigma_{X_2}^2 & \cdots & Cov(X_2, X_{i-1}) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(X_{i-1}, X_1) & Cov(X_{i-1}, X_2) & \cdots & \sigma_{X_{i-1}}^2 \end{pmatrix} \tag{31}$$

The conditional distribution of X_i conditioned on $(X_1, X_2, ..., X_{i-1})^{\mathrm{T}}$ is still a normal distribution, and its mean value can be calculated in the following way [Shi et al. (2009)],

$$E(X_i \mid X_1, X_2, ..., X_{i-1}) = \mu_{X_i} + \boldsymbol{C}_{X_i \boldsymbol{X}_{1\sim(i-1)}} \boldsymbol{C}_{\boldsymbol{X}_{1\sim(i-1)} \boldsymbol{X}_{1\sim(i-1)}}^{-1} (\boldsymbol{X}_{1\sim(i-1)} - \boldsymbol{\mu}_{\boldsymbol{X}_{1\sim(i-1)}}) \tag{32}$$

From the above equation, it can be seen that $E(X_i \mid X_1, X_2, ..., X_{i-1})$ is in fact an expression containing the first (i–1) variables. Substitute Eq.(32) into Eq.(27), we can get an expression containing the first i variables. In other words, $\bar{X}_i$ can be viewed as a function of the first i correlated variables, of which the expression can be explicitly obtained. In the same way in Section 3.2.1, the mean value and standard deviation of $\bar{X}_i$ can be conveniently obtained using the SGI technique, which are denoted as $\mu_{\bar{X}_1}$ and $\sigma_{\bar{X}_i}$, respectively. After obtaining the mean and standard deviation of the independent variables, the sensitivity analysis still cannot be performed immediately, as the original performance function is a mapping of the output response with the original correlated variables, not with the new independent ones. Thus the performance function has to be rebuilt to describe the relationship between the output response and the new independent variables.

From Eq.(25), the following relationship holds,

$$\begin{aligned} &X_1 = \bar{X}_1 \\ &X_2 = \bar{X}_2 + E(X_2 \mid X_1) \\ &X_3 = \bar{X}_3 + E(X_3 \mid X_1, X_2) \\ &\cdots \\ &X_n = \bar{X}_n + E(X_n \mid X_1, X_2, ..., X_{n-1}) \end{aligned} \tag{33}$$

As discussed above, $E(X_2 | X_1)$ is an expression only containing X_1. Now substitute $X_1 = \bar{X}_1$ into $E(X_2 | X_1)$, then X_2 can be transformed into an expression only containing $\bar{X}_1$ and $\bar{X}_2$. Similarly, substitute the expression of X_1 and X_2 into $E(X_3 | X_1, X_2)$, then X_3 can be transformed into an expression only containing $\bar{X}_1$, $\bar{X}_2$ and $\bar{X}_3$. Repeat this process, the expression for X_n can be finally obtained, which should only involve $(\bar{X}_1, \bar{X}_2, ..., \bar{X}_n)^{\mathrm{T}}$. This process can be denoted as

$$\begin{aligned} X_1 &= \beta_1(\bar{X}_1) \\ X_2 &= \beta_2(\bar{X}_2) \\ X_3 &= \beta_2(\bar{X}_1, \bar{X}_2, \bar{X}_3) \\ &\cdots \\ X_n &= \beta_2(\bar{X}_1, \bar{X}_2, ..., \bar{X}_n) \end{aligned} \tag{34}$$

in which β_i denotes a mapping relationship.

Substitute Eq.(34) into the original performance function, a new performance function between the output response and the new independent variables can be obtained, which is denoted as

$$Y = \bar{g}(\bar{X}_1, \bar{X}_2, ..., \bar{X}_n) \tag{35}$$

According to the previous discussions, variance based sensitivity analysis can be easily performed for the performance function in Eq.(35). The sensitivity indices thus obtained are then used to interpret the main effect index, total effect index, as well as the decompositions, for the original correlated variables according to Section 3.2.3.

4. An automobile front axle

In the automobile engineering, the front axle is an important component that bears heavy loads. Due to the rigid requirements for its strength, stiffness and fatigue life, mechanical property of the front axle must be strictly tested before the mass production [Lu et al. (2012)]. Variance can act as an important index to measure the robustness of the front axle, considering the uncertainty existing in the structure. The I-beam structure is widely used in the design of front axle due to its high bending strength and light weight. Consider the I-beam structure shown in Figure 1. The maximum normal stress and shear stress are $\sigma = M / W_x$ and $\tau = T / W_\rho$ respectively, where M is the bending moment, T is the torque, W_x and W_ρ are the sectional factor and polar sectional factor given as

$$W_x = \frac{a(h-2t)^3}{6h} + \frac{b}{6h}\left[h^3 - (h-2t)^3\right] \tag{36}$$

$$W_\rho = 0.8bt^2 + 0.4\left[a^3(h-2t)/t\right] \tag{37}$$

Consider the static strength of the front axle, the performance function can be thus established as

$$g = \sigma_s - \sqrt{\sigma^2 + 3\tau^2} \tag{38}$$

where σ_s is the yielding stress, and σ_s =460 MPa according to the material property. In the real engineering, uncertainty is unavoidable in the manufacture process, and randomness exists in the external loads. In this example, the geometry parameters of the I-beam, i.e. a, b, t, m, and the loads M and T are taken as random variables. The probability distribution information is given in Table 1.

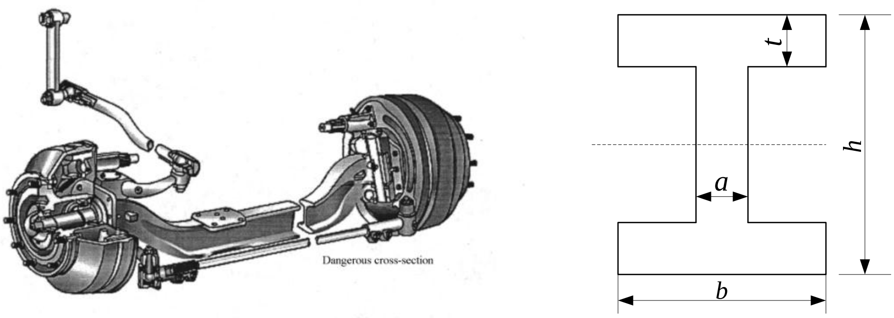

Figure 1. Sketch of the automobile front axle

Table 1. Distribution information of the inputs for the I-beam

Input (unit)	Distribution type	Mean	Standard deviation
a (mm)	Normal	12	0.06
b (mm)	Normal	65	0.325
t (mm)	Normal	14	0.07
h (mm)	Normal	85	0.425
M (N·mm)	Normal	3.5×10^6	1.75×10^4
T (N·mm)	Normal	3.1×10^6	1.55×10^4

By the above illustration, we may get the impression that the nonlinearity of the performance function is high, thus the interaction effect on the output variance should be noticeable. However, conclusions might be different if we perform the quantitative sensitivity analysis. First, consider the contribution of the inputs to the output variance under the assumption that the inputs are independent from each other. The results obtained by the proposed method and MCS are reported in Table 2.

When the inputs are independent, a total of 451 runs of the performance function are needed by the proposed method, and comparison with MCS shows the accuracy of the proposed method is acceptable. Another important feature in the sensitivity results is that, for each input, the main effect index is very close to the total effect index. It indicates the effect of interactions between inputs on the output variance is negligible, despite of the nonlinearity of the performance function. The independent inputs can be ranked as {t, T, b, a, h, M} in the descending order according to their contributions to the output variance.

Table 2 Sensitivity indices of the independent inputs for the I-beam

		a	b	t	h	M	T
S_i^M	Proposed method	0.112	0.171	0.418	0.033	0.0001	0.265
	MCS	0.115	0.172	0.410	0.032	0.0004	0.262
S_i^T	Proposed method	0.113	0.171	0.418	0.033	0.0001	0.265
	MCS	0.117	0.171	0.417	0.033	0.0001	0.269

As a matter of fact, in the real engineering it is inappropriate to assume the independence among the inputs. In most cases, one input is probably correlated to another, and such correlation may have notable effect on the output performance. Now consider the sensitivity analysis on the I-beam structure under the assumption of input correlation. Assume $\rho_{ab} = -0.3$, $\rho_{th} = -0.3$, and $\rho_{MT} = 0.4$. Since interactions between the inputs have little effect on the output variance, thus only the main effect index is estimated and decomposed by the proposed method, and shown in Figure 2.

The sensitivity analysis results in Figure 2 can provide helpful interpretations of the model. Still, the input *t* and *T* contribute the most to the output variance, as is the case when the inputs are independent. Thus, these two inputs should be carefully controlled if we want to reduce the output variation, especially the former one. Besides, *M* becomes more important than *h* after the correlation is introduced. It can also be noticed that the contributions of the correlated part by the first four inputs are negative, which is caused by the negative correlation coefficient. In this example, the inputs can be seen as independent pairs of dependent variables, e.g. the input *a* is only correlated to *b* and independent from the rest. As a result, the correlated contributions of two correlated inputs are equal, e.g. $S_a^{MC} = S_b^{MC}$. From Figure 2 it can be clearly seen that correlated part of the inputs plays an important role in contributing to the output variance, sometimes even more significant than the uncorrelated part. With the proposed method, more information has been explored, which can be referred to by analysts to improve the model performance.

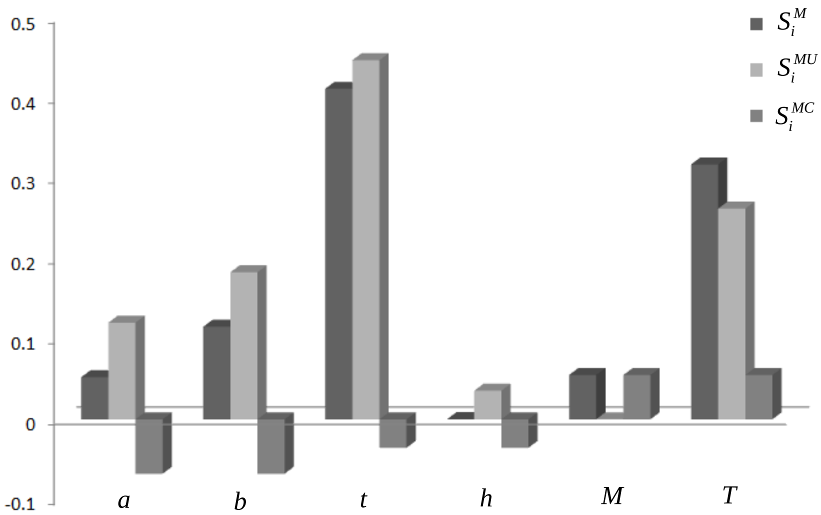

Figure 2 Sensitivity indices of the correlated inputs for the I-beam

5. Conclusions

In this work, application of the SGI technique to the global sensitivity analysis is discussed. Sensitivity analysis under the case of independent variables is much different from that of correlated variables, as in the latter case both the computation and interpretation are more complicated. When the variables are independent, both the main effect index and total effect index can be estimated by the two-stage use of the SGI technique. The whole procedure is considerably clear and simple. When it comes to the correlated variables, necessary steps need to be taken before the sensitivity results can be obtained. The whole procedure can be generalized as follows: (1) estimate the output variance with the SGI technique, (2) orthogonalize the correlated variables to independent variables, (3) estimate the statistical information of the independent variables by SGI, as well as the mapping between these new variables and the output, and (4) perform the sensitivity analysis with the SGI technique for the new performance function and independent variables. The sensitivity results are then used to interpret the contributions of the original correlated variables to the output variance. The proposed method inherits the merits of the SGI technique, and can estimate the integrals involved in the sensitivity analysis with acceptable accuracy, while keeping the computational burden under control. Applications to the examples have shown that the proposed method can be seen as a viable choice for sensitivity analysis of engineering models.

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