The influence of the parameter h and a new modified method of Homotopy analysis method for initial value problems

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Abstract

In this paper, we pay more attention to the embedding parameter h, which has an influence on the convergence region of solution series in Homotopy analysis method (HAM). We use some theorems to give the concrete influence and proof. Then we introduce a new modified method of HAM called the Piecewise homotopy analysis method (PHAM). Futhermore, examples such as NLS equation, Ricatti equation and Duffing equation are presented to illustrate the main results.

Keywords: Homotopy analysis method; the convergence region; Cauchy-Kowalevskaya theorem; Piecewise homotopy analysis method

Introduction

In 1992, Liao [1] introduced a powerful method known as the Homotopy analysis method (HAM) to solve the nonlinear problems. Compared with perturbation techniques, on one hand, the validity of HAM is independent of whether or not there exist small parameters in the considered equation. And on the other hand, HAM has an embedding parameter to control the convergence region of the series, namely convergence-control parameter. That is to say, we can adjust and control the convergence region of the series by assigning the convergence-control parameter a proper value. In recent 20 years, this method has been successfully applied to solve many types of nonlinear problems [2-6]. In these papers, absolutely all of them note that the convergence region of solution series depend upon the convergence-control parameter [7]. For some special cases, Liao only shows some special equations' convergence region, such as Ricatti equation, Blasius equation [2] and so on. Liu [8] make contributions to giving the essence of the generalized Taylor theorem, which is the key of HAM. However, the influence that the closer the value of the convergence-control parameter is to zero, the larger the convergence region is to infinite usually known as one common things without giving a strict proof.

Inspired by Abdelrazec's [9] idear to prove the convergence of the Adomian decomposition method, we aim to the convergence-control parameter to show the detail influence and verify it on the NLS equation with cubic nonlinearity. On the basis of the convergence of HAM, we propose a new analysis method called Piecewise homotopy analysis method. To our surprise, this method has a lot of benefits than HAM under the application of some examples.

Mathematical Formulation

Consider the abstract initial-value problem

$$\begin{cases}(1-q)L[\phi(t,q)-u_0(t)]=hqH(t)[L(\phi(t,q))-N(\phi(t,q))], & t>0,\\ u_0(0)=f\end{cases} \tag{1}$$

where the operator L is linear and N is nonlinear, N(u) is analytic near the initial data f. Without loss of generality, we take $\mathrm{L}=\frac{\partial}{\partial \mathrm{t}}$, $\mathrm{u}_0(\mathrm{t})=\mathrm{f}$, $\mathrm{H}(\mathrm{t})=-1$, the part of $\mathrm{t}>0$ in formula (1) could be transformed into

$$(1-q)\frac{\partial}{\partial t}\phi(t,q)+hq[\frac{\partial}{\partial t}\phi(t,q)-N(\frac{\partial}{\partial t}\phi(t,q))]=0.$$

Simplify the above formulation, we have

$$\frac{\partial}{\partial t}\phi(t,q)=mN[\phi(t,q)] \tag{2}$$

with $\phi(0,q)=f$, where $m=\dfrac{hq}{hq-q+1}$. Moreover, Eq. (2) could be reformulated as

$$\phi(t,q)=f+m\int_0^t N[\phi(s,q)]ds \tag{3}$$

with $\phi(0,q)=f$.

Convergence Analysis

Lemma (Cauchy's estimate). Suppose that $\phi(t)$ is differentiable in $\{t\in \mathrm{R}:|t-\mathrm{t}_0|<\mathrm{T}\}$, and to any $\hat{\mathrm{t}}\in(0,\mathrm{T})$, there exist a $\mathrm{M}>0$, such that $\phi(t)\le M$ on $C:|t-t_0|=\hat{t}$, then

$$\left|\phi^{(k)}(t_0)\right|\le\frac{Mk!}{\hat{t}^k},\forall k\ge 0.$$

Proof. From the Cauchy integral formula, we obtain

$$\phi^{(k)}(t_0)=\frac{k!}{2\pi i}\int_C\frac{\phi(t)}{(t-t_0)^{k+1}}dt,$$

then

$$\left|\phi^{(k)}(t_0)\right|=\frac{k!}{2\pi}\left|\int_C\frac{\phi(t)}{(t-t_0)^{k+1}}dt\right|\le\frac{k!}{2\pi}\frac{M}{\hat{t}^{k+1}}\cdot 2\pi\hat{t}=\frac{Mk!}{\hat{t}^k}$$

Theorem 1 (Cauchy-Kowalevskaya). Suppose $\phi(t)$ is the exact solution of Eq.(2), $\mathrm{m}<+\infty$, then there exists a $\tau>0$ such that $\mathrm{u}:[0,\tau]\to\mathrm{R}$ is also an analytic real function.

Proof. As $\mathrm{N}(\phi)$ is analytic near f, $\mathrm{m}<+\infty$, so $m^2N(\phi)$ is also analysis near f. By Cauchy's estimate, there exist $\mathrm{a},\mathrm{b}>0$ such that

$$m^2\frac{1}{k!}\left|\partial_\phi^k\phi(f)\right|\le\frac{b}{a^k},\forall k\ge 0, \tag{4}$$

where $\partial_\phi^k N(\phi)$ make the sense of the Fréchet derivative, it means that $\partial_\phi N(\phi)=N'(\phi)$, $\partial_\phi^2 N(\phi)=N''(\phi)$, ... The taylor series of $\mathrm{N}(\phi)$ at f converges when $|\phi - f|<\mathrm{a}$. What's more, if $\mathrm{r}=|\phi - f|$,we have

$$m\left|N(\phi)\right|\le m\sum_{\mathrm{k}=0}^{\infty}\frac{1}{\mathrm{k}!}\left|\partial_\phi^k N(f)\right|\left|(\phi-f)^k\right|\le\frac{b}{m}\sum_{\mathrm{k}=0}^{\infty}\frac{\left|\phi-f\right|^k}{\mathrm{a}^\mathrm{k}}=\frac{a\frac{b}{m}}{a-r}=:mg(r).$$

Consider the majorant function $\mathrm{g(r)}$, it is obviously that

$$m\frac{1}{k!}\left|\partial_\phi^k N(f)\right|\le m\frac{1}{k!}\partial_r^k g(0),\forall k\ge 0. \tag{5}$$

We get the initial-problem (2)' majorant problem which is given by

$$\begin{cases} \dot{r}(t) = mg(r), & t > 0, \\ r(0) = 0 \end{cases} \tag{6}$$

where $r > 0$. The majorant problem (6) is a simple ODE with the exact solution

$$r(t) = a - \sqrt{a^2 - 2\frac{ab}{m}t}.$$

It is clear that $r(t)$ is analytic on $(-\infty, \frac{a}{2b}m)$. According to the comparison principle, if $\phi(t)$ suit Eq. (3), then

$$|\phi(t) - f| \le m\int_0^t |N(\phi(s))| ds \le m\int_0^t |g(r(s))| ds = r(t) = \sum_{k=1}^{\infty} \frac{t^k}{k!}\partial_t^k r(0).$$

The taylor series of the majorant problem's solution $r(t)$ absolutely converges for all $|t| < \frac{a}{2b}m$. In order to prove $\phi(t)$ is also analytic in $t \in [0, \tau)$, where $\tau = \frac{a}{2b}m = \frac{a}{2b}\frac{hq}{hq - q + 1}$, we must to prove

$$\left|\partial_t^k \phi(0)\right| \le \partial_t^k r(0), \quad k \ge 1.$$

Suppose that's right, then the taylor series for $\phi(t)$ has the majorant series. According to Weierstrass M-test, consequently, it converges. First we prove the bound above by computing the following formulas for k = 1,2,3,

$$\partial_t \phi(t) = mN(\phi(t)),$$
$$\partial_t^2 \phi(t) = mN'(\phi(t))N(\phi(t)),$$
$$\partial_t^3 \phi(t) = mN''(\phi(t))N(\phi(t))N(\phi(t)) + mN'(\phi(t))N'(\phi(t))N(\phi(t)).$$

As a result,

$$\left|\partial_t \phi(0)\right| \le m\left|N(\phi(0))\right| \le mg(r(0)) = \partial_t r(0),$$
$$\left|\partial_t^2 \phi(0)\right| \le m\left|N'(\phi(0))\right|\left|N(\phi(0))\right| \le mg'(r(0))g(r(0)) = \partial_t^2 r(0),$$
$$\left|\partial_t^3 \phi(0)\right| \le \left|\partial_t^3 \phi(0)\right| m\left|N''(\phi(0))\right|\left|N(\phi(0))\right|\left|N(\phi(0))\right| + m\left|N'(\phi(0))\right|\left|N'(\phi(0))\right|\left|N(\phi(0))\right|$$
$$\le mg''(r(0))g(r(0))g(r(0)) + mg'(r(0))g'(r(0))g(r(0)) = \partial_t^3 r(0),$$

Generally speaking, for all $k \ge 0$,

$$\phi^{(k+1)}(t) = mP_k(N(\phi(t))),$$

where $P_k(N(\phi(t)))$ is a polynomial of N, and its Fréchet derivatives up to the kth-order with positive coefficients. In consequence, we have

$$\left|\partial_t^{(k+1)} \phi(0)\right| \le m\left|P_k(N(\phi(0)))\right| \le mP_k(\left|N(\phi(0))\right|) = \partial_t^{(k+1)} r(0),$$

So far, this bound have already concluded the proof.

Theorem 2. For the formula

$$\tau(h) = \frac{a}{2b}\frac{hq}{hq - q + 1}, \tag{7}$$

the closer the value of h is to zero, the larger the convergence region $\tau(h)$ is to ∞.

Proof. As $\tau(h) > 0$, we have

$$h \in (-\infty, 1 - \frac{1}{q}) \cup (0, +\infty).$$

Eq. (7) equivalent to

$$\tau(h) = \frac{a}{2b} \frac{1}{1 - \frac{1}{h}(1 - \frac{1}{q})}, \tag{8}$$

From Eq. (8), we have the conclusion for the convergence region τ.

(i) the closer the value of h is to ∞, the closer the convergence region τ is to $\frac{a}{2b}$.

(ii) the closer the value of h is to $1 - \frac{1}{q}$, the larger the convergence region τ. Eespecially, when $q \to 1^-$, that is, $h \to 0^-$ and $\tau \to +\infty$.

The proof has been finished in (ii) and the conclusion can be showed in Figure 1.

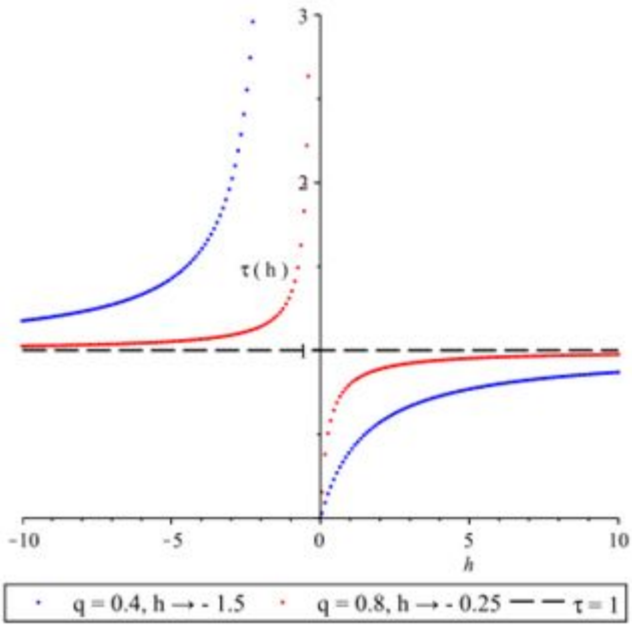


Figure 1. Take different value of $q = 0.4, 0.8$ with $a = 2, b = 1$, we find the closer thecorresponding value of $1 - \frac{1}{q}$ is to $-1.5, -0.25$, the closer τ is to $+\infty$.

HAM for NLS Equation

Some nonlinear equations are often presented as

$$\begin{cases} L(u) = N(u), & t > 0, \\ u(0) = f. \end{cases} \tag{9}$$

For example, consider the following continuous NLS equation

$$iu(t) = -\frac{1}{2}u_{xx} - |u|^2 u, \qquad t > 0, \tag{10}$$

where $u(x,t)$ is an amplitude function with the property of $|u|^2 = \bar{u}\, u$, this equation usually called NLS equation with cubic nonlinearity. Obviously, the equation fits to the above abstract formulation (9) with

$$N(u) = \frac{1}{2} i\partial_x^2 u + i|u|^2 u \; .$$

Do the transformation $u(x,t) = F(x)e^{it}$, then use $u(t)$ instead of $F(x)$. Moreover, take the initial data $u(0)=1, \dot{u}(0)=0$, we obtain

$$\begin{cases} \ddot{u}(t) = u - 2u^3, \qquad t > 0, \\ \dot{u}(0) = 0, \\ u(0) = 1, \end{cases} \tag{11}$$

The exact solutionof (11) is $\operatorname{sech}(t)$. It is straightforward represent

$$u(t) = \sum_{k=0}^{\infty} a_k t^k \, , \tag{12}$$

under the set of base functions

$$\{t^k, k = 0,2,4,...\}.$$

With the aid of the Eq. (11) and under the rule of solution expression,we choose the initial approximation

$$u_0(t) = 1,$$

and the auxiliary linear operator

$$L = \frac{\partial^2}{\partial t^2},$$

and

$$\mathrm{N}(\phi(t,q)) = \frac{\partial^2}{\partial t^2}\phi(t,q) - \phi(t,q) + 2\phi^3(t,q) \, ,$$

with $\phi(0,q) = 1$. According to the above conditions, we have the zeroth-order deformation equation

$$(1-q)L[\phi(t,q) - u_0(t)] = hqH(t)\mathrm{N}(\phi(t,q)) \tag{13}$$

From Eq. (13), we know $\phi(t,0) = u_0(t)$ and $\phi(t,1) = u(t)$ when q = 0 and q = 1. With respect to the embedding parameter q, we define

$$u_k(t) = \frac{1}{k!}\frac{\partial^k}{\partial q^k}\phi(t,q),$$

and $\phi(t,q)$ can be expanded in Taylor series

$$\phi(t,q) = \phi(0,q) + \sum_{k=0}^{\infty} u_k(t) q^k \, ,$$

Define the following vector

$$\vec{u}_k(t) = \{u_0(t), u_1(t), ..., u_k(t)\}.$$

Differentiating Eq. (13) k times with respect to q, then take q = 0 and dividing by k!. At last, we get the kth-order deformation equation

$$L[u_k - \chi_k u_{k-1}] = hH(t)R_k[\vec{u}_{k-1}] \, ,$$

with the initial condition $u_0(0) = 1, u_k(0) = 0 \; (k > 0),$ where

$$R_k[\vec{u}_{k-1}] = \ddot{u}_{k-1} - u_{k-1} + 2\sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i} u_i u_j u_{k-1-i-j} - (1-\chi_k)$$

and

$$\chi_k = \begin{cases} 0, & k \le 1, \\ 1, & k > 1. \end{cases}$$

According to the both of the rule of solution expression and the coefficient ergodicity, the corresponding auxiliary function should be determined uniquely

$$\mathrm{H(t)} = 1.$$

Then we successively have

$$u_1(t) = \frac{1}{2}ht^2,$$

$$u_2(t) = \frac{1}{2}ht^2 + \frac{1}{2}h^2t^2 + \frac{5}{24}h^2t^4,$$

$$u_3(t) = \frac{1}{2}ht^2 + h^2t^2 + \frac{1}{2}h^3t^2 + \frac{5}{12}h^2t^4 + \frac{5}{12}h^3t^4 + \frac{61}{720}h^3t^6,$$

......

According to the formula $u(t) = \sum_{k=0}^{\infty} u_k(t)$, $u(t)$ could be obtained and as is shown in Figure 2.

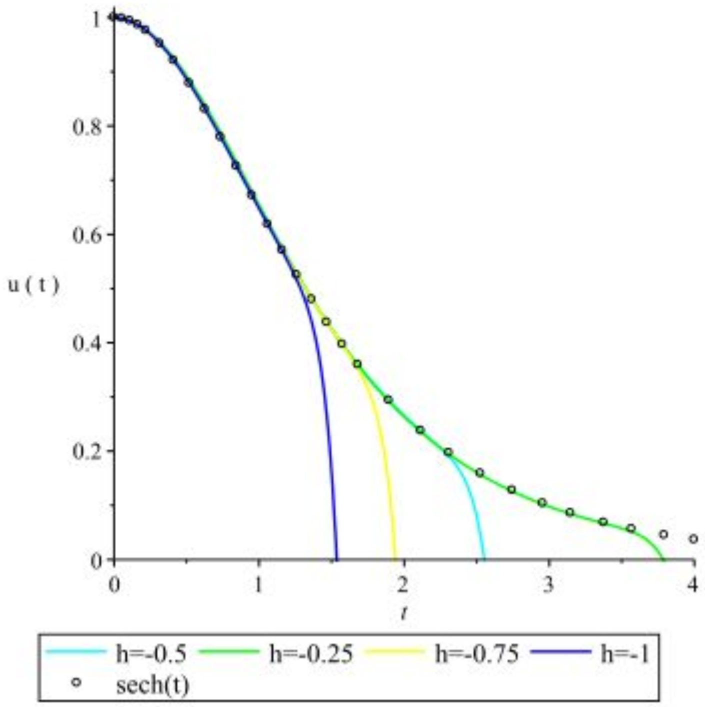


Figure 2. Take different value of $h = -1, -0.75, -0.5, -0.25$, we find the value of h is closer to 0, the corresponding graph is more fit with the graph of the exact solution.

Piecewise Homotopy Analysis Method

Considered the graph of h = −1 from Figure 2, we find the homotopy analysis solution are very closed to the exact solution in [0,1] but away from the exact solution in [1,4]. In generally,compare the approximate solution with the exact solution for any given value of h, we can choose a t0, such that the homotopy analysis solution and the exact solution be overlap for any $t \in [0, t_0]$ in an negligible error, but be away for $t \in [t_0, \infty]$. Now, conclude the idear of Piecewise homotopy analysis method: First, we can get the homotopy analysis solution of one nonlinear equation with the initial guess solution $u_0(t)$, here we assume the approximate solution is $a_0(t)$ with the starting point $(0, u_0(t))$, then we can obtain the graphs of the homotopy analysis solution $a_0(t)$ and the exact solution. Second, we can choose a point t_0 on the overlap section, the part $t \in [0, t_0]$ is to be preserved because the homotopy analysis solution $a_0(t)$ is absolutely equal to the exact for any $t \in [0, t_0]$ in an negligible error. Third, we choose the point $(t_0, a_0(t_0))$ as the starting point and use HAM to the nonlinear equation again to get a new analysis approximate solution $a_1(t)$ and its graph. We can find a point $t_1(> t_0)$ on the overlap section between the new approximate $a_1(t)$ and the exact solution's graphs, then keep the part $t \in (t_0, t_1]$ for the same reason. $t_2(> t_1), t_3(> t_2), t_4(> t_3), \ldots$ and $a_2(t), a_3(t), a_4(t), \ldots$would be find after repeat the above step again and again, where $\{t_k \mid k = 0,1,2,\ldots\}$ is not unique. At last, the Piecewise homotopy analysis solution $a(t)$ could be obtained as the following form.

$$\begin{cases} a_0(t), \quad 0 \le t \le t_0 \\ a_1(t) = a_0(t - t_0)\big|_{u_0 = a_0(t_0)}, \quad t_0 \le t \le t_1 \\ a_2(t) = a_1(t - t_1)\big|_{u_0 = a_0(t_1)}, \quad t_1 \le t \le t_2 \\ \ldots\ldots \\ a_k(t) = a_{k-1}(t - t_{k-1})\big|_{u_0 = a_0(t_{k-1})}, \quad t_{k-1} \le t \le t_k \\ \ldots\ldots \end{cases}$$

To calculate conveniently, we often choose the distance of t_k and t_{k+1} as a small and constant real number, and this number is called the tep size, d. That is, $d = t_{k+1} - t_k, k = 0,1,2,\ldots$. The detail answer is that, $t_0 = d, t_1 = 2d, \ldots, t_k = (k+1)d ,\ldots$. In this case, the Piecewise homotopy analysis solution a(t) could be represented as

$$\begin{cases} a_0(t), \quad 0 \le t \le d \\ a_1(t) = a_0(t - d)\big|_{u_0 = a_0(d)}, \quad d \le t \le 2d \\ a_2(t) = a_1(t - 2d)\big|_{u_0 = a_0(2d)}, \quad 2d \le t \le 3d \\ \ldots\ldots \\ a_k(t) = a_{k-1}(t - kd)\big|_{u_0 = a_0(kd)}, \quad kd \le t \le (k+1)d \\ \ldots\ldots \end{cases}$$

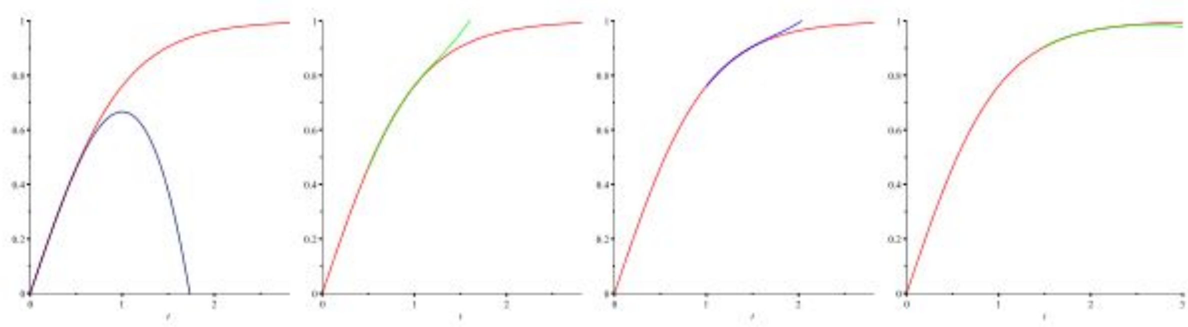
Piecewise Homotopy Analysis Method for Ricatti Equation

Consider the Ricatti equation,

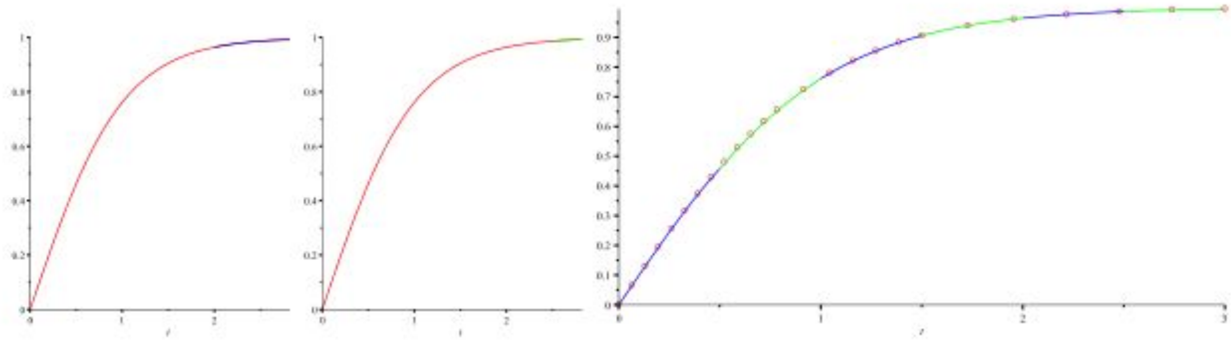
$$\dot{u}(t) = 1 - u^2(t), \qquad t > 0, \tag{14}$$

with initial condition u(0) = 0. The exact solution of (14) is u(t) = tanh(t).

For simplicity, we let h = −1 and just take 4th-order homotopy analysis solution $u(t) \approx t - \frac{1}{3}t^3$ after using the HAM to Eq.(14) with $u_0 = t$, because the graph of the approximate solution $t - \frac{1}{3}t^3$ and the exact solution tanh(t) has already overlap in [0,0.5] , that is to say, [0,0.5] is a valid interval for Ricatti equation. As is showed on Figure 3(a).



(a) Blue: $a_0, 0 \le t \le 3.0$ (b) Green: $a_1, 0.5 < t \le 3.0$ (c) Blue: $a_2, 1.0 < t \le 3.0$ (d) Green: $a_3, 1.5 < t \le 3.0$



(e) Blue: $a_4, 2.0 < t \le 3.0$ (f) Green: $a_5, 2.5 < t \le 3.0$ (g) solid: $a(t), 0 \le t \le 3$

Figure 3. (a-f): the red line is the figure of tanh(t), $0 \le t \le 3$. (g):the point of circle is the exact solution tanh(t), the blue and green line is the 4-th order approximate solution a(t) on divided sections with $t_0 = 0.5, t_1 = 1.0, t_2 = 1.5, t_3 = 2.0, t_4 = 2.5, t_5 = 3.0$.

In generally, use the HAM, from Liao's book [2], we know the kth-order deformation equation ,

$$\dot{u}_k - \chi_k \dot{u}_{k-1} = hH(t)[\dot{u}_{k-1} + \sum_{j=0}^{k-1} u_j u_{k-1-j} - (1-\chi_k)]$$

we can get u(t)'s 4th-order homotopy analysis solution under the initial data u_0.

$$a_0(t) = u_0 + (1 - u_0^2)t + (u_0^3 - u_0)t^2 + (-u_0^4 + \frac{4}{3}u_0^2 - \frac{1}{3})t^3 + (u_0^5 - \frac{5}{3}u_0^3 + \frac{2}{3}u_0)t^4 \quad (15)$$

with h = −1, H(t) = 1 near t = 0. Note that u0is an undetermined initial data. It can be easily find that $a_0(t) = t - \frac{1}{3}t^3$ when $u_0 = 0$. Then we use the Piecewise homotopy analysis method.

Take $t_0 = 0.5$, we know the 4th-order Piecewise homotopy analysis solution is $a_0(t)$ on [0,0.5]; take $t_1 = 1.0$, the 4th-order Piecewise homotopy analysis solution is $a_1(t) = a_0(t - 0.5)|_{u_0 = a_0(0.5)}$ on (0.5,1]. In generally, take $t_k = 0.5(k+1)$, we can get the 4th-order Piecewise homotopy analysis solution $a_k(t)$ on (0.5k,0.5(k + 1)].

By the Piecewise homotopy analysis method, u(t) ′ s 4th-order Piecewise homotopy analysis solution a(t) is

$$\begin{cases} a_0(t), \quad 0 \le t \le 0.5 \\ a_1(t) = a_0(t-0.5)\big|_{u_0=a_0(0.5)}, \quad 0.5 \le t \le 1.0 \\ a_2(t) = a_1(t-1.0)\big|_{u_0=a_0(1.0)}, \quad 1.0 \le t \le 1.5 \\ \ldots\ldots \\ a_k(t) = a_{k-1}(t-0.5k)\big|_{u_0=a_0(0.5k)}, \quad 0.5k \le t \le 0.5(k+1) \\ \ldots\ldots \end{cases}$$

The figure could be obtained in $t \in [0,\infty]$ with the values of $t_0 = 0.5, t_1 = 1.0, \ldots, t_k = 0.5(k+1), \ldots$ as showed on Figure 3(g).

Table 1. Comparisons of a(t)/tanh(t), a'(t)/tanh'(t) and a''(t)/tanh''(t) with different values of t.

t	$a(t)$	$tanh(t)$	$a'(t)$	$tanh'(t)$	$a''(t)$	$tanh''(t)$
0.0	0.000000	0.000000	1.000000	1.000000	0.000000	0.000000
0.5^+	0.458333	0.462117	0.789931	0.786448	-0.724103	-0.726862
0.5^-	0.458333	0.462117	0.750000	0.786448	-1.000000	-0.726862
1.0+	0.760947	0.761594	0.420960	0.419974	-0.640656	-0.639700
1.0^-	0.760947	0.761594	0.437507	0.419974	-0.520278	-0.639700
1.5^+	0.906028	0.905148	0.179113	0.180707	-0.324564	-0.327133
1.5^-	0.906028	0.905148	0.192241	0.180707	-0.246151	-0.327133
2.0^+	0.964366	0.964028	0.069998	0.070651	-0.135008	-0.136219
2.0^-	0.964366	0.964028	0.069814	0.070651	-0.137663	-0.136219
2.5^+	0.986599	0.986614	0.026623	0.026592	-0.052532	-0.052472
2.5^-	0.986599	0.986614	0.024931	0.026592	-0.063039	-0.052472
3.0^+	0.994970	0.995055	0.010034	0.009866	-0.019968	-0.019634
3.0^-	0.994970	0.995055	0.009109	0.009866	-0.025581	-0.019634

Error analysis: We know that the solution of HAM is analytic and the Piecewise homotopy analysis solution a(t)'s accurate is $O(0.5^4)$ when $u_0 = 0$, or $O(0.5^3)$ when $u_0 = 0$. The first-order derivative of the approximate solution a(t)'s accurate is $O(0.5^3)$. when $u_0 \neq 0$, or $O(0.5^2)$ when

$u_0 = 0$. The second-order derivative of the approximate solution $a(t)'$ s accurate is $O(0.5^2)$ when $u_0 \neq 0$, or $O(0.5^1)$ when $u_0 = 0$. All of those is showed on Figure 4, and the graphs of $a(t), a'(t), a''(t), t \in (0.5,1.0]$ are showed on Figure 4.

Table I shows that the error of a(t) is satified with the conjecture. For instance, when $t = 0.5^-$, the error between a(t) and tanh(t) is $0.003784(\leq 0.5^3)$, the error between a(t) and tanh(t) is $0.036448(\leq 0.5^2)$, the error between a(t) and tanh(t) is $0.273138(\leq 0.5^1)$; when $t = 1.0^-$, the error between a(t) and tanh(t) is $0.0006476(\leq 0.5^4)$, the error between a(t) and tanh(t) is $0.017533(\leq 0.5^3)$, the error between a(t) and tanh(t) is $0.119422(\leq 0.5^2)$, and so do the others value of t. Moreover, the errors of $a^{(n)}(t^+), n = 0,1,2.$ is smaller than the errors of $a^{(n)}(t^-), n = 0,1,2.$

Figure 4 shows the errors in (0.5,1.0]. From the graph, we find that the larger value of t, the bigger error of every $a(t), a'(t), a''(t)$. And of course, to any $t \in (0.5,1.0]$, the error of a(t) is smaller than the error of $a'(t)$ and the error of $a'(t)$ is smaller than the error of $a''(t)$.

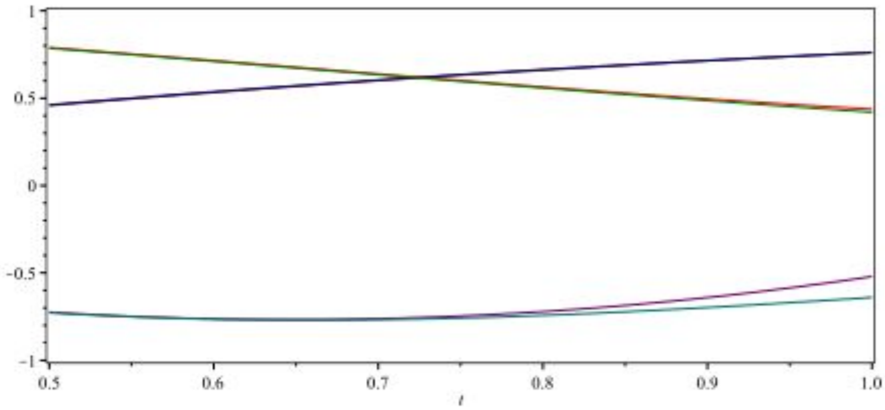


Figure 4. From left top to bottom, the middle two lines is the exact solution and the approximate solution, whose precision is 0.5^4; the first two lines is the first-order derivative of the exact solution and the approximate solution, whose precision is 0.5^3; the last two lines is the second-order derivative of the exact solution and the approximate solution, whose precision is 0.5^2.

Piecewise Homotopy Analysis Method for Duffing Equation

Let's consider the following Duffing equation as an example,

$$\ddot{u} + \frac{u^3}{1+u^2} = 0, \qquad t > 0, \tag{16}$$

with initial conditions $\dot{u}(0) = 0, u(0) = A$. Clearly, Eq.(16) can be deformed into

$$\ddot{u} + \ddot{u}u^2 + u^3 = 0, \tag{17}$$

In order to use the Piecewise homotopy analysis method, we should apply the HAM to Eq.(17) in the first place.

The zeroth-order homotopy equation is

$$(1 - q)L[\phi(t,q) - u_0(t)] = hqH(t)N[\phi(t,q)] \tag{18}$$

with $\phi(0,q) = 0, \phi(0,q) = A$,where

$$N[\phi(t,q)] = \ddot{\phi}(t,q) + \ddot{\phi}(t,q)\phi^2(t,q) + \phi^3(t,q).$$

Take

$$\phi(t,q) = u_0(t) + \sum_{k=0}^{\infty} u_k(t)q^k, L = \frac{\partial^2}{\partial t^2}, h = -1, H(t) = 1$$

into Eq. (18), then differentiating the Eq. (18) k times with respect to q and dividing by k! at last, we obtain the kth-order deformation equation

$$\ddot{u}_k - \chi_k \ddot{u}_{k-1} = -\ddot{u}_{k-1} - \sum_{j=0}^{k-1}\sum_{i=0}^{j} (\ddot{u}_i + u_i)u_{j-i}u_{k-1-j} + (1-\chi_k).$$

u(t)'s mth-order homotopy analysis solution is

$$\tilde{a}_0(t) = \sum_{k=0}^{m} u_k(t).$$

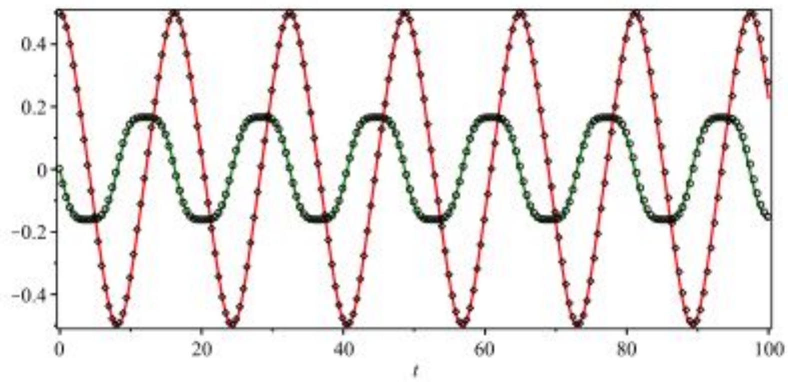


Figure 5. The rhombus points are Eq.(16)'s 4th-order Piecewise homotopy analysis solution $\tilde{a}_0(t)$, the red line is the 4th-order homotopy analysis solution $\tilde{a}_0(t)$); the cycle points are the first-order derivative of $\tilde{a}_0(t)$, the green line is the first-order derivative of $\tilde{a}_0(t)$. From the above equations, we know $\tilde{a}_0(t)$ is only based on $u_0(t)$ and m. Of course, we always want to take a big value of m to get a better $\tilde{a}_0(t)$. Due to the limitations of the paper, we get the 4th-order $\tilde{a}_0(t)$ with initial data $u_0 = a + bt$.

4th-order $\tilde{a}_0(t)$ homotopy analysis solution with the initial data $u_0 = a + bt$ is

$$\tilde{a}_0(t) = a + bt + (\frac{1}{2}a^5 - \frac{1}{2}a^3 - \frac{1}{2}a^7 + \frac{1}{2}a^9)t^2 + (\frac{3}{2}a^8 b - \frac{1}{2}a^2 b + \frac{5}{6}a^4 b - \frac{7}{6}a^6 b)t3$$
$$+ ......$$
$$+ (\frac{211}{416000}b^7 a^2 + \frac{271}{1108800}b^9)t^{15} + \frac{211}{3328000}b^8 a t^{16} + (\frac{211}{56576000})b^9 t^{17}$$

The first initial data $\dot{u}(0) = 0, u(0) = A$ can be obtained when a = 0,b = A, and in this situation,

$$\begin{aligned}\tilde{a}_0(t) = At - \frac{1}{20}A^3 t^5 + \frac{1}{42}A^5 t^7 + (\frac{1}{480}A^5 - \frac{1}{72}A^7)t^9 + (-\frac{9}{3080}A^7 + \frac{1}{110}A^9)t^{11} \\ + (\frac{11}{3360}A^9 - \frac{11}{124800}A^7)t^{13} + \frac{271}{108800}A^9 t^{15} + \frac{211}{56576000}A^9 t^{17}\end{aligned} \quad (19)$$

According to the above 4th-order homotopy analysis solution $\tilde{a}_0(t)$, we can use the Piecewise homotopy analysis method to solve Eq. (16). Here we also take $t_0 = 0.5, t_1 = 1.0, ..., t_k = 0.5(k+1)$,...

to get the different initial data of the corresponding intervals, then we can obtain the 4th-order Piecewise homotopy analysis solution $\tilde{a}_0(t)$ of the corresponding intervals. As is showed in figure 5.

Figure 5 shows the comparison of Eq.(16)'s 4th-order Piecewise homotopy analysis solution $\tilde{a}_0(t)$ and 4th-order homotopy analysis solution $a_0(t)$ on [0,100], and the comparison of the two solution's first-order derivative on [0,100] when A=0.5.

Conclusions

In this paper, we first prove that the closer the value of $h(<0)$ is to zero, the larger the convergence region is to $+\infty$ for initial-value problems. Based on the convergence, another analysis method called Piecewise homotopy analysis method is proposed to solve nonlinear equations.

Compared HAM with Piecewise homotopy analysis method through two examples, we find the latter has the following characteristics: First of all, Piecewise homotopy analysis method has all the characteristics that HAM has. Second, Piecewise homotopy analysis method could get a better solution with a relatively small value of m than HAM for large t, it over comes a shortcoming in HAM that the farther point' s approximation effect is not as good as the point near the initial point, and for this reason the calculations are reduced to get the same accuracy of large t. Moreover, Piecewise homotopy analysis method not only has a higher accuracy than HAM, but its derivatives have an acceptable accuracy. In addition, Piecewise homotopy analysis method can solve equations with arbitrary initial data, which is depend on the step size. At last, a higher accuracy can get by shortening the tep size.

References

[1]. S.J. Liao. The proposed homotopy analysis technique for the solution of nonlinear problems. Ph.D. Thesis, 1992.
[2]. S.J. Liao. Beyond perturbation: Introduction to the homotopy analysis method. Chapman and Hall/CRC Press, Boca Raton, 2003.
[3]. Y. Tan and S. Abbasbandy. Homotopy analysis method for quadratic riccati differential equation. Communications in Nonlinear Science and Numerical Simulation,13(3):539 – 546, 2008.
[4]. H. Xu and J.. Cang. Analysis of a time fractional wave-like equation with the homotopy analysis method. Physics Letters A, 372(8):1250 – 1255, 2008.
[5]. S.J. Liao. An analytic approximate approach for free oscillations of self-excited systems. International Journal of Non-Linear Mechanics, 39(2):271 – 280, 2004.
[6]. S.J. Liao. Homotopy analysis of nonlinear progressive waves in deep water. Journal of engineering mathematics, 2003.
[7]. S.j. Liao. A kind of approximate solution technique which does not depend upon small parameters ii. an application in fluid mechanics. International Journal of NonLinear Mechanics, 32(5):815 – 822, 1997.
[8]. C. S. Liu. The essence of the generalized taylor theorem as the foundation of the homotopy analysis method. Communications in Nonlinear Science and Numerical Simulation, 16(3):1254 – 1262, 2011.
[9]. A. Abdelrazec and D. Pelinovsky. Convergence of the adomian decomposition method for initial-value problems. Numerical Methods for Partial Differential Equations, 27(4):749–766, 2011.