Parametric reliability sensitivity analysis using failure probability ratio function

†Wei Pengfei
School of Mechanics, Civil Engineering and Architecture, Northwestern Polytechnical University, Xi'an, China

†Corresponding author: wpf0414@163.com; pengfeiwei@nwpu.edu.cn

Abstract
Reducing the failure probability is an important task in the design of engineering structures. In this paper, a reliability sensitivity analysis technique, called failure probability ratio function, is firstly developed for providing the analysts quantitative information on failure probability reduction while one or a set of distribution parameters of model inputs are changed. The proposed failure probability ratio function can be especially useful for failure probability reduction, reliability-based optimization and reduction of the epistemic uncertainty of parameters. The Monte Carlo simulation (MCS), Importance sampling (IS) and Truncated Importance Sampling (TIS) procedures, which need only a set of samples for implementing them, are introduced for efficiently computing the proposed sensitivity indices. A numerical example is introduced for illustrating the engineering significance of the proposed sensitivity indices and verifying the efficiency and accuracy of the MCS, IS and TIS procedures.
Keywords: Sensitivity analysis; Failure probability ratio function; Importance sampling

1. Introduction

In design of engineering structure, the analysts often build a lot of computational models (e.g. finite element model, FEM) for simulating the behavior of real structures. However, due the extensively existing uncertainty (aleatory or epistemic), the performance of the structure turn out to be unsteady, which often prevents the analysts from understanding the behavior of structure. In the probabilistic framework, the model inputs are often treated as random variables and represented by probability density function (PDF). The distribution parameters are either fixed at constant value (only aleatory uncertainty is presented) or characterized by confidence interval (due to epistemic uncertainty). Under these assumptions, the analysts' two main concerns are reliability analysis and safety improvement.

The reliability analysis aims at assessing the failure probability of existing structure. During the past several decades, many methods have been developed by researchers for this purpose such as the Monte Carlo Simulation (MCS), First-order Second-moment (FOSM) method [Hasofer and Lind (1974)], Importance Sampling (IS) [Au and Beck (2002); Harbitz(1986); Melchers(1989); Melchers(1990);], Subset Simulation (SS) [Au and Beck (2001)], Line Sampling (LS) [Schuëller et al. (2004)], directional sampling [Ditlevsen et al. (1990)] response surface method [Faravelli (1989)]. However, analysts still find it difficult to employ these methods especially when the epistemic uncertainty is presented in model inputs.

Safety improvement focuses on reducing the failure probability of existing structure via selecting optimal values for distribution parameters (if possible). This is often dealt with in two ways: reliability-based optimization (RBO) and sensitivity analysis. Up to now, three groups of methods are available for RBO: double-loop method [Enevoldsen and Sorensen (1994); Tu et al. (1999)], single-loop method [Chen et al. (1997); Kuschel and Rackwitz (1997)], decoupling method [Au (2005); Royset et al.(2001); Zou and Mahadevan (2006)]. Compared with the methods for reliability analysis, these RBO methods are generally more time-consuming. One can refer to Schueller and Jensen (2008) and Valdebenito and Schuëller (2010) for overviews of the RBO methods. Sensitivity analysis techniques for safety improvement can be divided into three groups: local, global and regional sensitivity analysis.

Local reliability sensitivity analysis indices are generally defined as the partial derivatives of failure probability or reliability index with respect to distribution parameters [Bjerager and Krenk (1989); Lu et al. (2008); Melchers and Ahammed (2004); Wu and Mohanty (2006)]. These indices measure the change on failure probability while the parameters are perturbed at one given point. If the sensitivity index of one parameter is positive, then the failure probability tends to decrease if one reduces this parameter at the given point, otherwise, if the index is negative, then the failure probability tends to increase. The higher the absolute value of the local sensitivity index is, the more dramatically the failure probability will change. Theoretically, a parameter has significant effect on failure probability at one point doesn't mean that it is influential at each point, similarly, although the index of one parameter is close to zero at one point, one cannot think that this parameter is non-influential anywhere. Therefore, the local reliability sensitivity indices cannot tell the analysts the global sensitivity information of the distribution parameters of model inputs to the failure probability.

The global reliability sensitivity indices, which aims at measuring the contribution of individual or a set of inputs to the failure probability by investigating their full distribution ranges, are developed by Wei et al. (2012) based on Sobol's indices [Homma and Saltelli (1996) and Sobol’ (1993)]. The higher the main effect index of one inputs is, the more reduction of failure probability can be obtained while one reduce the uncertainty of this input, otherwise, as the total effect index is close to zero, the failure probability will not change obviously while one reducing the uncertainty of this input. The global reliability sensitivity indices can be estimated by the methods developed for Sobol's indices such as MCS [Saltelli (2002); Saltelli et al. (2010); Sobol’ (2001)], Fourier Amplitude sensitivity Test (FAST) [Xu and Gertner (2008)] and meta-modelling method [Ratto (2007)], thus can be easily implemented. The global reliability sensitivity indices can only tell the analysts which inputs to focus on so that the failure probability can be reduced efficiently and cheaply, but cannot tell the analysts the amount of failure probability reduction due to specific reduced uncertainty of model inputs.

In this paper, the failure probability ratio function is introduced for measuring the change on failure probability while the distribution parameters of model inputs vary in intervals. The proposed sensitivity index is similar to the function of failure probability developed by Au (2005) to some extent. The failure probability ratio function has important significance in many engineering application. Firstly, it can tell the analysts the amount of failure probability reduction while one change the distribution parameters of inputs to any specific ones, thus can help the analysts reducing the failure probability efficiently and quantitatively. Second, it can provide plenty of information for RBO. After the failure probability ratio function been obtained, the RBO problem can be transformed to a deterministic one. Third, for model with epistemic uncertainty (due to lack of information (data), the distribution parameters of model inputs are represented by an confidence interval), it is helpful for selecting the inputs distribution parameters which are most valuable for collecting more information.

For numerically estimating the failure probability ratio function, the MCS procedure, which needs only a set of samples for implementing it, is firstly introduced. For problem with relatively large failure probability, the MCS procedure is accurate and efficient. For problem with small failure probability, we suggest using the Importance Sampling (IS) and Truncated Importance Sampling (TIS) procedures for reducing the computational burden.

2. Failure probability ratio function

Let $Y = g(\boldsymbol{X})$ denote the limit state function of the structure under investigation, where $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ is the n-dimensional inputs vector. The joint PDF of the input vector $\boldsymbol{X}$ is given as $f_{\boldsymbol{X}}(\boldsymbol{x})$, and the marginal PDF of the input X_i is denoted by $f_i(x_i)$. In this paper, we assume that the failure of structure happens when the model output Y is less than zero, thus the failure domain F is defined as:

$$F = \{\boldsymbol{x} : g(\boldsymbol{x}) < 0\} \tag{1}$$

and the failure probability P_f of the structure can be derived as:

$$P_f = P(F) = \int_F f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int I_F(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = E_f\left(I_F(\boldsymbol{x})\right) \tag{2}$$

where $P(\cdot)$ denotes the probability measure, $E_f(\cdot)$ indicates the expectation with respect to the joint PDF $f_{\boldsymbol{X}}(\boldsymbol{x})$ and $I_F(\boldsymbol{x})$ stands for the indictor function of the failure domain, which is given as:

$$I_F(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in F \\ 0 & \boldsymbol{x} \notin F \end{cases} \tag{3}$$

Generally, the failure probability is related with the distribution parameters of model inputs such as the mean, variance and correlation of multivariate normal distribution. If one of these distribution parameters is changed, then the failure probability will also be changed. Let p denote one of these parameters, e.g., variance. Suppose now the parameter p varies in the interval $\left[p^{(l)}, p^{(u)}\right]$, then let $p = \left(p^{(u)} - p^{(l)}\right)q + p^{(l)}$, where q is a variable. While the parameter p varies in the interval $\left[p^{(l)}, p^{(u)}\right]$, q varies in the unit interval $[0,1]$. If we keeps all the other parameters of model inputs constant, then denote the updated joint PDF of model inputs due to changed p as $f_{\boldsymbol{X}}^*(\boldsymbol{x};q)$. For example, suppose the n-dimensional inputs vector follows independent normal distribution, i.e., $f_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{j=1}^{n} f_j(x_j)$, where $f_j(x_j) = e^{-\frac{(x-\mu_j)^2}{2\sigma_j^2}} \Big/ \sqrt{2\pi\sigma_j^2}$. Let p denote the variance σ_i^2 of X_i and p varies in the interval $\left[0, \sigma_i^2\right]$, then the joint PDF $f_{\boldsymbol{X}}(\boldsymbol{x})$ can be updated as:

$$f_X^*(\boldsymbol{x};q)=\frac{1}{\sqrt{2\pi q\sigma_i^2}}e^{-\frac{(x-\mu_i)^2}{2\sigma_i^2}}\prod_{j=1,j\neq i}^{n}\frac{1}{\sqrt{2\pi\sigma_j^2}}e^{-\frac{(x-\mu_j)^2}{2\sigma_j^2}}, \qquad q\in[0,1] \tag{4}$$

The failure probability with respect to the updated joint PDF $f_X^*(x;q)$ is computed as:

$$P_f^*(q)=\int I_F(\boldsymbol{x})\,f_X^*(\boldsymbol{x};q)\,\mathrm{d}\boldsymbol{x}, \qquad q\in[0,1] \tag{5}$$

The univariate failure probability ratio function $RPF_p(q)$ is defined as:

$$RPF_p(q)=\frac{P_f^*(q)}{P_f} \tag{6}$$

By definition, $RPF_p(q)$ measures the ratio between residual and total failure probabilities while the parameter p is changed to $\left(p^{(u)}-p^{(l)}\right)q+p^{(l)}$, where $q\in[0,1]$.

Similarly, we can develop the multivariate failure probability ratio function for measuring the reduction of failure probability while multiple distribution parameters of model inputs are changed. Suppose now we have m distribution parameters p_k ($k=1,2,\ldots,m$) of model inputs, each of which varies in a interval $\left[p_k^{(l)},p_k^{(u)}\right]$. Let $p_k=\left(p_k^{(u)}-p_k^{(l)}\right)q_k+p_k^{(l)}$, where $q_k\in[0,1]$. Then, similarly, we can obtain the updated joint PDF $f_X^*(\boldsymbol{x};q_1,q_2,\ldots,q_m)$, and define the failure probability $P_f^*(q_1,q_2,\ldots,q_m)$ with respect to $f_X^*(\boldsymbol{x};q_1,q_2,\ldots,q_m)$ as:

$$P_f^*(q_1,q_2,\ldots,q_m)=\int I_F(\boldsymbol{x})\,f_X^*(\boldsymbol{x};q_1,q_2,\ldots,q_m)\,\mathrm{d}\boldsymbol{x}, \qquad q_1,q_2,\ldots,q_m\in[0,1] \tag{7}$$

The m-dimensional failure probability ratio function $RPF_{p_1,p_2,\ldots,p_m}(q_1,q_2,\ldots,q_m)$ is defined as

$$RPF_{p_1,p_2,\ldots,p_m}(q_1,q_2,\ldots,q_m)=\frac{P_f^*(q_1,q_2,\ldots,q_m)}{P_f} \tag{8}$$

The local reliability sensitivity index S_p of the parameter p is defined as the derivative of the failure probability with respect to p [Wu and Mohanty (2006)]:

$$S_p=\left.\frac{\partial P_f/P_f}{\partial p/p}\right|_{p=p^*}=\frac{p}{P_f}\bullet\left.\frac{\partial P_f}{\partial p}\right|_{p=p^*} \tag{9}$$

where p^* is a constant value (often chosen as the true value of the parameter p). The partial derivative $\left.\partial P_f/\partial p\right|_{p=p^*}$ indicates the change of failure probability while one perturbing the parameter p at point p^*. p/P_f is a normalization factor that makes the sensitivity index dimensionless.

Then it can be proved that (see Appendix for proof):

$$\left.\frac{\partial RPF_p(q)}{\partial q}\right|_{q=q^*}=\frac{p^{(u)}-p^{(l)}}{p^*}S_p \tag{10}$$

where $p^*=\left(p^{(u)}-p^{(l)}\right)q^*+p^{(l)}$ is a constant. Eq. (10) indicates that the derivative of the probability ratio function with respect to the parameter p is proportional to the local reliability sensitivity index of p. The local sensitivity index S_p only reflect the sensitivity information of the parameter p at one given point p^*, whereas, the failure probability ratio function $RPF_p(q)$ measures the reduction of failure probability when the parameter p is fixed at any point, thus it provides much more information on failure probability reduction than the local sensitivity index.

3. Estimators for Failure probability ratio function

3.1 Monte Carlo simulation

By definition

$$P_f^*(q_1,q_2,\ldots,q_m)=\int I_F(\boldsymbol{x})\frac{f_{\boldsymbol{X}}^*(\boldsymbol{x};q_1,q_2,\ldots,q_m)}{f_{\boldsymbol{X}}(\boldsymbol{x})}f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=E_f\left(I_F(\boldsymbol{x})\frac{f_{\boldsymbol{X}}^*(\boldsymbol{x};q_1,q_2,\ldots,q_m)}{f_{\boldsymbol{X}}(\boldsymbol{x})}\right) \quad (11)$$

Eq. (13) indicates that $P_f^*(q_1,q_2,\ldots,q_m)$ can be expressed as the expectation with respect the original joint PDF $f_{\boldsymbol{X}}(\boldsymbol{x})$, thus can be estimated by MCS procedure using one set of samples. The Monte Carlo estimator is given as:

$$\hat{P}_f^*(q_1,q_2,\ldots,q_m)=\frac{1}{N}\sum_{j=1}^{N}I_F\left(\boldsymbol{x}^{(j)}\right)\frac{f_{\boldsymbol{X}}^*\left(\boldsymbol{x}^{(j)};q_1,q_2,\ldots,q_m\right)}{f_{\boldsymbol{X}}\left(\boldsymbol{x}^{(j)}\right)} \quad (12)$$

where $\boldsymbol{x}^{(j)}$ ($j=1,2,\ldots,N$) stands for the jth sample of model inputs generated by using the original joint PDF $f_{\boldsymbol{X}}(\boldsymbol{x})$.

Apparently, $\hat{P}_f^*(q_1,q_2,\ldots,q_m)$ is an unbiased estimator of $P_f^*(q_1,q_2,\ldots,q_m)$. In this paper, we use the mean square error (MSE) for quantifying the error of the estimate. Take the univariate failure probability ratio function $RPF_p(q)$ as an example, the MSE of the estimate $\widehat{RPF}_p(q)$ is given as:

$$MSE_p=\sqrt{\int_0^1\left(\widehat{RPF}_p(q)-RPF_p(q)\right)^2\mathrm{d}q} \quad (13)$$

where $RPF_p(q)$ is the reference result. In this paper, the MSE is estimated via bootstrap, thus the reference result $RPF_p(q)$ is computed by averaging the repeated estimates.

The samples can be generated using many methods such as simple random sampling, Latin-hypercube sampling [Helton and Davis (2003); Loh (1996)] and Sobol's sequence [Sobol' (1976)]. In this paper, the Sobol's sequence is recommended since it leads to better convergence rate and lower discrepancy of estimates especially for input dimension less than a few hundred [Sobol' (1976); Varet et al. (2012)].

The above MCS procedure needs only a set of sample for estimating the failure probability ratio function. For structure with relative large failure probability, it is efficient and accurate. However, for small failure probability ($<10^{-3}$), the MCS procedure need more samples for promising some of them dropping in the failure domain so that the failure probability can be correctly estimated, thus the computational cost increases heavily.

3.2 Importance Sampling

To reduce the computational burden of simulation method for computing the small failure probability, many researchers have suggested using the IS procedure [Harbitz (1986); Melchers (1989); Melchers (1990)]. The basic idea of the IS procedure is choosing an importance sampling density (ISD) for generating samples so that more samples drop into the failure domain.

Denote the ISD as $h_X(x)$, then Eq. (13) can be written as:

$$P_f^*(q_1,q_2,\dots,q_m)=\int I_F(x)\frac{f_X^*(x;q_1,q_2,\dots,q_m)}{h_X(x)}h_X(x)\mathrm{d}x=E_h\left(I_F(x)\frac{f_X^*(x;q_1,q_2,\dots,q_m)}{h_X(x)}\right) \quad (14)$$

where $E_h(\bullet)$ is the expectation operator taken with respect to the ISD $h_X(x)$. Eq. (16) indicates that $P_f^*(q_1,q_2,\dots,q_m)$ can be expressed as an expectation with respect to $h_X(x)$, thus can be estimated by using the sample mean, i.e.,

$$\hat{P}_f^*(q_1,q_2,\dots,q_m)=\frac{1}{N}\sum_{k=1}^{N}\left(I_F\left(x^{(k)}\right)\frac{f_X^*\left(x^{(k)};q_1,q_2,\dots,q_m\right)}{h_X\left(x^{(k)}\right)}\right) \quad (15)$$

Similarly, one can verify that Eq. (17) provides an unbiased estimator for the failure probability ratio function $P_f^*(q_1,q_2,\dots,q_m)$. The efficiency of the IS procedure is greatly affected by the choice of the ISD $h_X(x)$. Theoretically, the optimal ISD can be identified by minimizing the variance of the estimator in Eq. (17), i.e.,

$$\min_h V_h\left(I_F(x)\frac{f_X^*(x;q_1,q_2,\dots,q_m)}{h_X(x)}\right) \quad (16)$$

where the subscript in $V_h(\bullet)$ indicates that the variance is computed with respect to $h_X(x)$. It can be proved that the solution of the optimization problem is Eq. (18) is [Au and Beck (2002)]:

$$h_{X,opt}(x)=\frac{I_F(x)f_X^*(x;q_1,q_2,\dots,q_m)}{\int I_F(x)f_X^*(x;q_1,q_2,\dots,q_m)\mathrm{d}x}=\frac{I_F(x)f_X^*(x;q_1,q_2,\dots,q_m)}{P_f^*(q_1,q_2,\dots,q_m)} \quad (17)$$

By using the optimal ISD in Eq. (19), the variance of the estimates in Eq. (17) can be derived as:

$$V_{h_{opt}}\left(\hat{P}_f^*(q_1,q_2,\dots,q_m)\right)=\frac{1}{N}V_{h_{opt}}\left(I_F(x)\frac{f_X^*(x;q_1,q_2,\dots,q_m)}{h_{X,opt}(x)}\right)=\frac{1}{N}V_{h_{opt}}\left(P_f^*(q_1,q_2,\dots,q_m)\right)=0 \quad (18)$$

Although the optimal ISD $h_{X,opt}(x)$ can be derived by Eqs. (18) and (19), it is not available in practical application. One reason is that the identification of $h_{X,opt}(x)$ involves the information of the failure probability $P_f^*(q_1,q_2,\dots,q_m)$, which is what to be estimated from the simulation. Another reason is due to the fact that the indicator function $I_F(x)$ is unknown in advance. Even though the optimal ISD $h_{X,opt}(x)$ can be constructed via some numerical method, it is often a complicated and time-consuming task to generate samples from the constructed ISD especially for high dimensional ISD [Au and Beck (2002)].

In practical application, researchers often attempt to construct the approximate optimal ISD under some assumptions. One of the most common used methods is assuming the ISD belongs to one family of distribution, and then choosing the optimal ISD via optimizing the distribution parameters. Au and Beck (2002) use the cross-entropy for searching the optimal distribution parameters of ISD belongs to assumed distribution family. Others suggested generating the ISD by shifting the center of the original joint PDF $f_X(x)$ to the design point x^* with the highest probability density in the failure region [Harbitz (1986); Melchers (1989)]. In the examples of this paper, for simplicity, the later method is used. For problem with multiple failure modes, the ISD can be constructed by mixing multiple PDFs, each of which is centered at one design point. The design points can be identified by using many optimization algorithms such as FOSM [Hasofer and Lind (1974)] and genetic algorithm [Obadage and Harnpornchai (2006)].

Compared with the MCS procedure, the above IS procedure is more suitable for problem with small failure probability since the ISD $h_{\boldsymbol{X}}(\boldsymbol{x})$ allows more sample drop into the failure domain. One note that, as the distribution parameters vary, the failure domain will not change since the limit state function remains unchanged. Those samples within the failure domain will always stay in the failure domain no matter how the distribution parameters change. Therefore, one need only one set of samples generated by one pre-identified ISD $h_{\boldsymbol{X}}(\boldsymbol{x})$ for computing the failure probability ratio function $RPF_{p_1,p_2,\ldots,p_m}(q_1,q_2,\ldots,q_m)$ at any points.

The main drawback of the above IS procedure is that it may not always be suitable for high-dimensional (up to a few hundred) nonlinear problems since that the identification of an approximate fixed ISD is practically impossible [Katafygiotis and Zuev (2008)].

3.3 Truncated Importance Sampling

In subsection 3.2, the introduction of the ISD constructed using the design point has substantially reduced the computational burden for computing the failure probability ratio function. In fact, the computational cost can be further reduced by using the TIS procedure [Grooteman (2008)].

In a standard Gaussian space, the reliability index β is in fact the distance from the from the design point $\boldsymbol{x}^*$ to the origin of coordinate, as shown by Figure 1. Then a hypersphere with radius β can be obtained. We denote this hypersphere as β -sphere. Define the indictor function $I_\beta(\boldsymbol{x})$ of the β-sphere as:

$$I_\beta(\boldsymbol{x}) = \begin{cases} 1 & \|\boldsymbol{x}\| \geq \beta^2 \\ 0 & \|\boldsymbol{x}\| < \beta^2 \end{cases} \tag{19}$$

It is shown in Fig. 1 that the failure region outsides the β-sphere, and there is no failure point drop in the region of β-sphere, then the failure probability $P_f^*(q_1,q_2,\ldots,q_m)$ can be further derived as:

$$\begin{aligned} P_f^*(q_1,q_2,\ldots,q_m) &= \int I_F(\boldsymbol{x}) I_\beta(\boldsymbol{x}) \frac{f_{\boldsymbol{X}}^*(\boldsymbol{x};q_1,q_2,\ldots,q_m)}{h_{\boldsymbol{X}}(\boldsymbol{x})} h_{\boldsymbol{X}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \\ &= E_h\left(I_F(\boldsymbol{x}) I_\beta(\boldsymbol{x}) \frac{f_{\boldsymbol{X}}^*(\boldsymbol{x};q_1,q_2,\ldots,q_m)}{h_{\boldsymbol{X}}(\boldsymbol{x})} \right) \end{aligned} \tag{20}$$

Then $P_f^*(q_1,q_2,\ldots,q_m)$ can be estimated by:

$$\hat{P}_f^*(q_1,q_2,\ldots,q_m) = \frac{1}{N}\sum_{j=1}^N \left(I_F\left(\boldsymbol{x}^{(j)}\right) I_\beta\left(\boldsymbol{x}^{(j)}\right) \frac{f_{\boldsymbol{X}}^*\left(\boldsymbol{x}^{(j)};q_1,q_2,\ldots,q_m\right)}{h_{\boldsymbol{X}}\left(\boldsymbol{x}^{(j)}\right)} \right) \tag{21}$$

In Eq. (23), if the sample point $\boldsymbol{x}^{(j)}$ drop into the β-sphere, then $I_\beta\left(\boldsymbol{x}^{(j)}\right)=0$, further $I_F\left(\boldsymbol{x}^{(j)}\right) I_\beta\left(\boldsymbol{x}^{(j)}\right) f_{\boldsymbol{X}}^*\left(\boldsymbol{x}^{(j)};q_1,q_2,\ldots,q_m\right) \big/ h_{\boldsymbol{X}}\left(\boldsymbol{x}^{(j)}\right)=0$, thus one needs not to compute the value of limit state function at the point $\boldsymbol{x}^{(j)}$. By this way, the computational burden is further reduced.

The above TIS procedure can further reduce the computational cost by introducing the β-sphere. However, since the design point $\boldsymbol{x}^*$ is computed numerically, as some computational error exists, the β-sphere may include some non-negligible part of the failure region. This will further leads to computational error of the estimate of $P_f^*(q_1, q_2, \ldots, q_m)$ [Wei et al. (2012)].

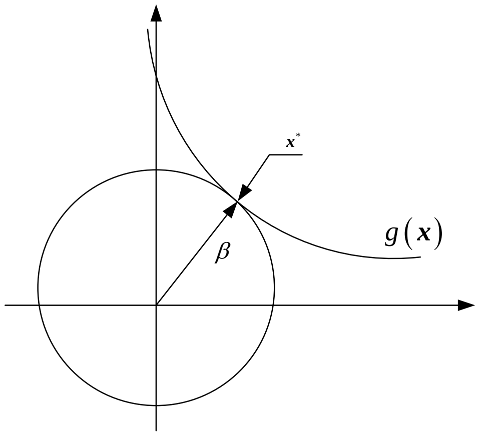


Figure 1. Schematic illustration of the β-sphere.

4. Numerical test case

In this section, we use a numerical example for illustrating the engineering significance of the failure probability ratio function, and verifying the efficiency and accuracy of the proposed numerical methods. The limit state function of the structure is represented by

$$Y = g(\boldsymbol{X}) = 4 - X_1 - X_2 - X_1 X_3 \tag{22}$$

where $\boldsymbol{X} = (X_1, X_2, X_3)$ is a vector including three inputs, which are assumed to follow standard normal distribution, i.e., $X_i \sim N(0,1)$ for $i = 1,2,3$.

In this test case, we consider the sensitivity of the mean and variance of the model inputs to the failure probability. We assume that the means and variances of the three inputs vary in the interval $[-0.1, 0.1]$ and $[0,1]$ respectively. Then the mappings from q to the mean μ and variance σ^2 are given as $\mu = 0.2q + 0.1$ and $\sigma_i^2 = q$, where $q \in [0,1]$.

The univariate failure probability ratio functions with respect to the means and variances of the three inputs are computed by MCS, IS and TIS procedures using one set of sample, and the estimates are plotted in Figures 2-4. The total number of function evaluations of the MCS, IS and TIS procedure are 10^6, 1022 and 719, thus the estimates of the MCS procedure can be regarded as the exact solution. It is shown that the estimates of the IS and TIS procedures are in good agreement with their exact solutions. Compared with the MCS procedure, the computation burden of the IS procedure is sufficiently reduced, and due to the introduction of the β-sphere, the computational cost is further reduced without affecting the accuracy of estimates. For further investigating the convergence of the IS procedure, we plot the MSEs of the estimates with respect to the sample size in Figure.5. It can be seen that the MSE of each estimate tends to zero as the sample size increases.

It is shown by Fig.2(a) that, the failure probability increases monotonically and linearly with respect to the means of the three inputs. As one decrease the means of the model inputs, the failure probability tends to reduce linearly, and the slopes of the failure probability ratio functions in Fig.2(a) indicate the rate of reduction. As can been, as reducing the same amount of means, X_1 leads to the most reduction of failure probability, followed by X_3, and then X_2.

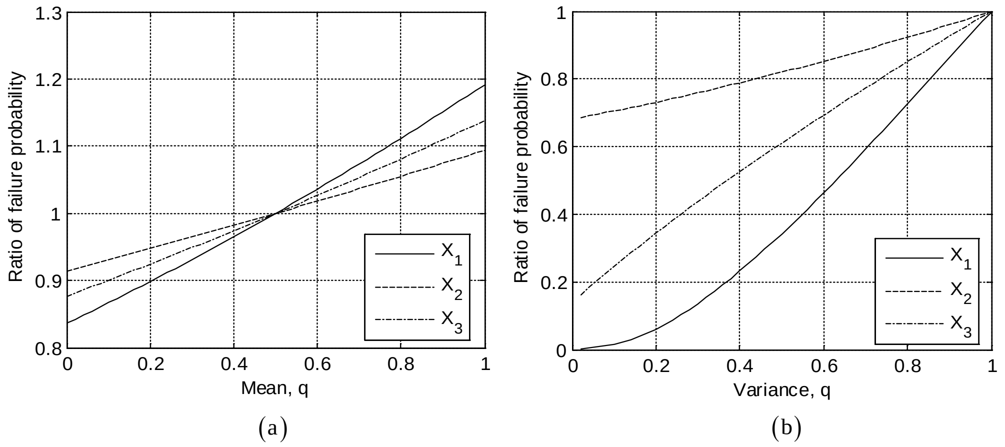


Figure 2. Univariate failure probability ratio functions computed by MCS procedure: (a) with respect to means of inputs; (b) with respect to variances of inputs.

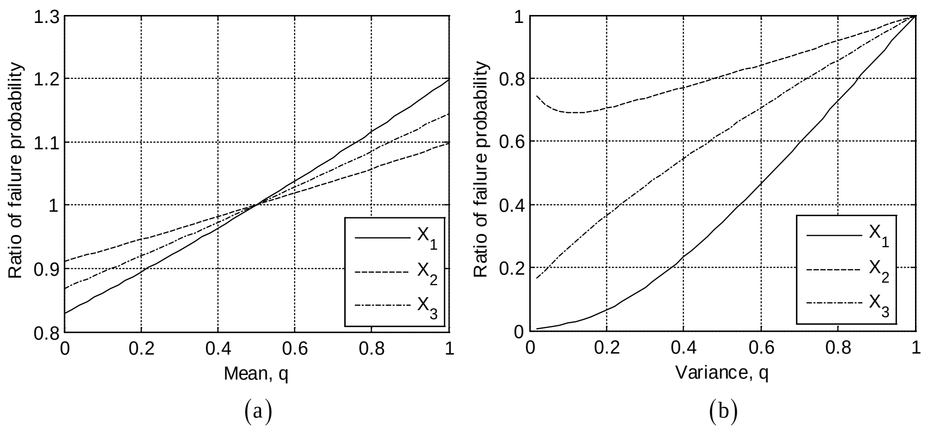
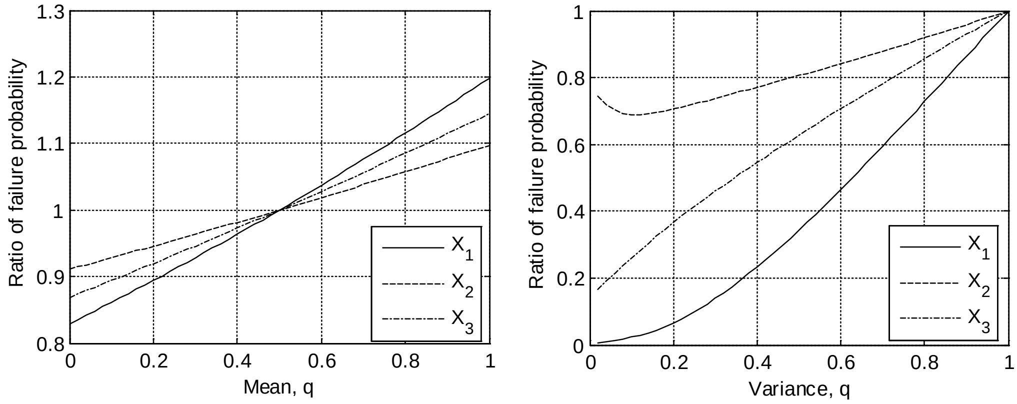
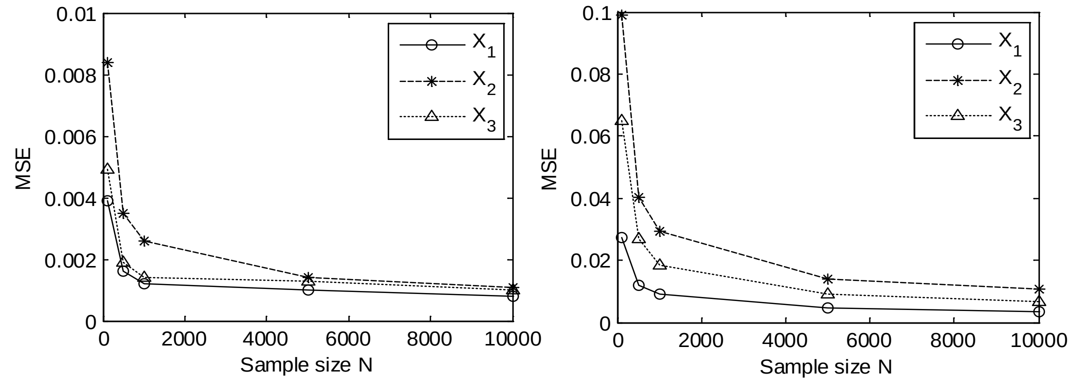


Figure 3. Univariate failure probability ratio functions computed by IS procedure: (a) with respect to means of inputs; (b) with respect to variances of inputs.



(a) (b)

Figure 4. Univariate failure probability ratio functions computed by TIS procedure: (a) with respect to means of inputs; (b) with respect to variances of inputs.



(a) (b)

Figure 5. Convergence plots of the univariate failure probability ratio functions with respect to (a): mean of inputs; (b) variance of inputs.

In many engineering application, the failure probability is reduced via decreasing the dispersion (variance) of model inputs. Suppose now our target is to reduce the failure probability by 80%. As one can see in Figure.2(b), the failure probability tends to decrease as the variances of inputs are reduced. It can be read from Figure.2(b) that, to achieve our targeted reduction of failure probability, one need to reduce the variance of X_1 by 63% or that of X_3 by 94% individually. It is possible to achieve our target by reducing the variance of X_2.

In most cases, it is difficult to reduce the variance of one input by 63%. For reducing the failure probability by 80%, one needs to use the bivariate failure probability ratio function. Since that reducing the variances of X_1 and X_3 leads to more failure probability reduction that reducing the variance of X_2, we plot the bivariate failure probability ratio function (computed by MCS procedure) with respect to the variances of X_1 and X_3 in Figure.6. The diagonal line of this bivariate failure probability ratio function measures the ratio between the residual and total failure probability while both the variances of X_1 and X_3 are reduced to q, thus we plot it in Figure 7. As can be seen that, by reducing the variances of X_1 and X_3 by 48% simultaneously, we can achieve our targeted reduction of failure probability.

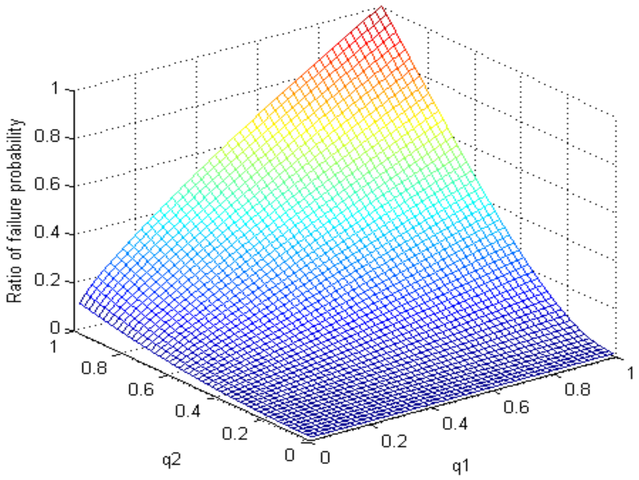


Figure 6. 3D plots of the bivariate failure probability ratio function with respect to the variances of the pair (X_1, X_3), where q_1 and q_2 indicate the variance reduction of X_1 and X_3 respectively

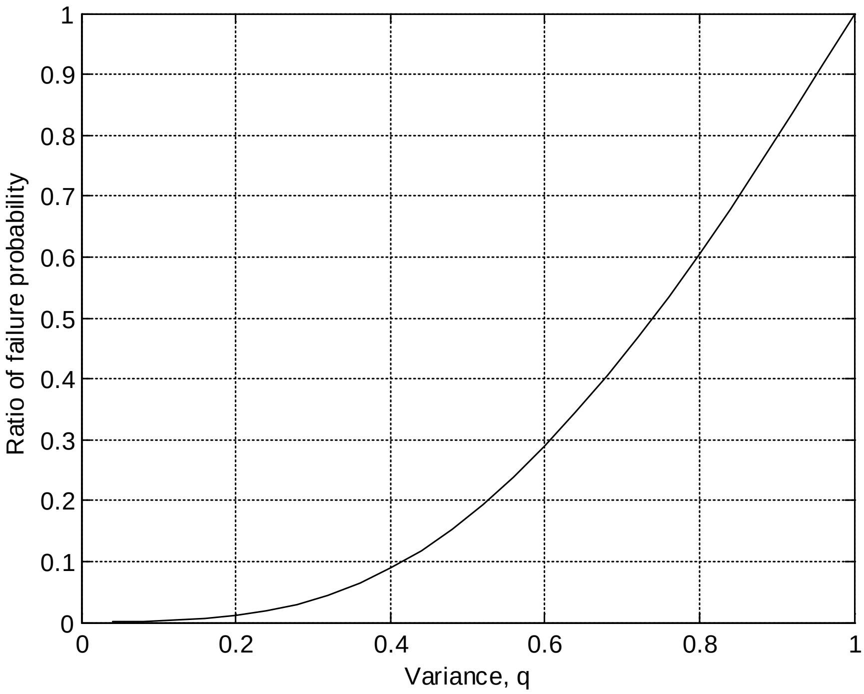


Figure 7. Diagonal line of the failure probability ratio function with respect to the variance of the input pair (X_1, X_3)

Acknowledgement

This work is supported by National Natural Science Foundation of China (NSFC 51475370)

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