Improved edge-based smoothed finite element method (IES-FEM) for Mid-frequency acoustic analysis

***Z.C. He[1,2], G.Y.Li[1], Eric. Li[1], G.Y. Zhang[2], Xin.Nie[1]**

[1] State Key Laboratory of Advanced Design and Manufacturing for Vehicle Body, Hunan University, Changsha, 410082 P. R. China

[2] State Key Laboratory of Structural Analysis for Industrial Equipment, School of Naval Architecture, Dalian University of Technology, Dalian, 116024, P. R. China

*Corresponding author: hezhicheng815@163.com

Abstract

The 3D ES-FEM using tetrahedral mesh proposed recently has showed many great features in mechanics, and mid-frequency acoustics analysis, such as $kh \approx 1$. When it comes to higher frequencies, the 3D ES-FEM also encounters the dispersion error, which is related to "slightly overly-soft stiffness" induced by the excessive edge-based smoothing operations compared to the continua system. In this paper, an improved 3D edge-based smoothed finite element method (IES-FEM) is proposed by introducing a parameter controlling the ratio of "slightly over-softness" of the ES-FEM and "over-stiffness" of the FEM, and the balance of the discretized system can then be tuned to reduce the dispersion error in higher frequency range, i.e. $1 \le kh \le 2$. Numerical results demonstrate the advantages of IES-FEM for acoustic problems, in comparison with the ES-FEM using the same mesh.

Keywords: Numerical method; Edge-based smoothed finite element method (ES-FEM); Acoustic; Dispersion error;

1. Introduction

Acoustic analysis has been a hot topic due to the increasing customer demands on the comfort of automobiles or aircraft, etc. As a traditional numerical modal is used, especially for a linear finite element system, the mid-frequency acoustic analysis often encounters computational difficulties. Researches in [1] reveal that the standard FEM can provide proper results with the restriction of $kh<1$, which is related to the "rule of thumb". In the numerical analysis, the linear FEM cannot give accurate prediction even if the $kh<1$ is satisfied.

In order to explain the root cause of the acoustic error and predict the acoustic field in the mid-frequency range, Ihlenburg [1] found that the acoustic error bounds contain a pollution term that are related to the loss of stability with large wave numbers, and he firstly showed that FEM with higher-order polynomial approximations (the *hp* version of the FEM) work well in reduce the acoustic error. The higher order methods such as *p*-FEM [2], and the discontinuous enrichment method (DEM) [3] have also been studied for acoustic computations in the mid-frequency regime. Babuška [4] attempted to correct the loss of stability in the Helmholtz operator and designed a Generalized Finite Element Method (GFEM) for the Helmholtz equation such that the pollution effect is minimal. The Galerkin/least-squares finite element method (GLS)[5], the quasi-stabilized finite element method (QSFEM) [4] were also proposed to improve the accuracy of acoustic analysis.

In a standard Galerkin FEM formulation, the discretized system behaves "overly-stiff", which leads to inaccurate results in acoustic simulation, especially for the linear FEM in the mid-frequency regime. Due to the adaptively of linear elements (such as triangular and tetrahedral elements) in generating mesh in complex domain, the research on the use of low order mesh with high accuracy is of practical significance. Recently, Liu's group conducted intensive studies on the softness of low-order elements and proposed smoothed finite element methods (S-FEM) by using the gradient smoothing technique together with the finite element methods. Using the node-based strain smoothing operation, a node-based smoothed point interpolation method (NS-PIM or LC-PIM [6, 7]) and a node-based smoothed finite element method (NS-FEM) [8] have been formulated in the frame of meshfree and FEM, respectively. The node-based smoothed technique can provide a much better gradient solution than the standard FEM, while the smoothing domain contains too many elements leading to "overly-soft" of NS-FEM and instability in solving dynamic and acoustic problems [9,10]. The edge-based smoothed finite element methods (ES-FEM) [10-13] are thus proposed for 2D and 3D mechanics and acoustic problems, respectively. The ES-FEM showed super convergence properties, ultra accuracy and high computational efficiency compared to the traditional FEM using the same set of triangular and tetrahedral meshes, and is very suitable for dynamic and acoustic problems.

In studying the 3D ES-FEM for acoustic problems[13], the ES-FEM provides better results than improved FEM using hexahedral mesh at $kh \approx 1$. When it comes to higher frequencies, the ES-FEM also encounters the dispersion error [14], which is rooted at the "slightly over-softness" of stiffness matrix in ES-FEM. In this paper, an improved 3D edge-based smoothed finite element method (IES-FEM) is proposed by introducing a parameter controlling the ratio of "slightly over-softness" of the ES-FEM and "over-stiffness" of the FEM, and the balance of the discretized system can then be tuned to reduce the dispersion error in higher frequency range, *i.e.* $1 \le kh \le 2$. Numerical results demonstrate the advantages of IES-FEM for acoustic problems, in comparison with the standard ES-FEM using the same elements.

In this work, we mainly focus on the acoustic analysis in mid-frequency regime, and "the $1<kh<2$" is adopted as a reference of mid-frequency. The paper is organized as follows: Section 2 briefly describes the mathematical model of the acoustic problems. The idea of IES-FEM method is formulated detailed in Section 3. The numerical example is used to evaluate the performance of the proposed method in Section 4. Finally the conclusions from the numerical results are made in Section 5.

2. Acoustic problems and its standard Galerkin discretization

The acoustic pressure p in a bounded domain Ω is governed by the following well-known Helmholtz equation:

$$\Delta p + k^2 p = 0 \tag{1}$$

where Δ is the Laplace operator, k is the wavenumber defined by

$$k = \frac{\omega}{c} \tag{2}$$

where c and ω denote the speed of sound traveling in the homogeneous media and angular frequency, respectively. For general interior acoustic problems with boundary Γ, there are three types of boundary conditions prescribed on the boundary of Γ_D, Γ_N and Γ_A, where $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_A$, and the three sets of boundary conditions are expressed

$$p = p_D \qquad \text{on } \Gamma_D \tag{3}$$

$$\nabla p \cdot n = -j\rho\omega v_n \qquad \text{on } \Gamma_N \tag{4}$$

$$\nabla p \cdot n = -j\rho\omega A_n p \qquad \text{on } \Gamma_A \tag{5}$$

where p_D is the prescribed acoustic pressure on the boundary Γ_D , v_n is the normal velocity on boundary Γ_N, ρ is the medium density and A_n represents the admittance coefficient on boundary Γ_A. In the standard FEM, the discrete of acoustic pressure p can be expressed in the following form:

$$p = \sum_{i=1}^{m} N_i p_i = \mathbf{Np} \tag{6}$$

where N_i are the nodal shape functions obtained using standard finite element procedure and p_i are the unknown nodal pressures. In standard Galerkin weak form, the shape function $\mathbf{N}$ is also chosen as the weight function w and the discretized form for acoustic problems can be obtained as:

$$-\int_{\Omega}\nabla\mathbf{N}\cdot\nabla\mathbf{NP}\mathrm{d}\Omega+k^2\int_{\Omega}\mathbf{N}\cdot\mathbf{NP}\mathrm{d}\Omega-j\rho\omega\int_{\Gamma_N}\mathbf{N}\cdot v_n\mathrm{d}\Gamma-j\rho\omega A_n\int_{\Gamma_A}\mathbf{N}\cdot\mathbf{NP}\mathrm{d}\Gamma=0 \tag{7}$$

The discretized system equations can be finally obtained and written in the following matrix form:

$$[\mathbf{K}^{\mathrm{FEM}}-k^2\mathbf{M}^{\mathrm{FEM}}+j\rho\omega\mathbf{C}]\{\mathbf{P}\}=-j\rho\omega\{\mathbf{F}\} \tag{8}$$

where $\mathbf{K}^{\mathrm{FEM}}$ is the acoustical stiffness matrix, $\mathbf{M}^{\mathrm{FEM}}$ is the acoustical mass matrix, $\mathbf{C}$ is the acoustical damping matrix, $\{\mathbf{P}\}^{\mathrm{T}}$ is nodal acoustic pressure vector, $\mathbf{F}$ is the nodal acoustic forces vector, and all of them are described as follows

$$\mathbf{K}^{\mathrm{FEM}}=\int_{\Omega}(\nabla\mathbf{N})^{\mathrm{T}}(\nabla\mathbf{N})\mathrm{d}\Omega \tag{9}$$

$$\mathbf{M}^{\mathrm{FEM}}=\int_{\Omega}\mathbf{N}^{\mathrm{T}}\mathbf{N}\mathrm{d}\Omega \tag{10}$$

$$\mathbf{C}=\int_{\Gamma_A}\mathbf{N}^{\mathrm{T}}\mathbf{N}A_n\mathrm{d}\Gamma \tag{11}$$

$$\mathbf{F}=\int_{\Gamma_N}\mathbf{N}^{\mathrm{T}}v_n\mathrm{d}\Gamma \tag{12}$$

3. The idea of Improved edge-based smoothed finite element method (IES-FEM)

3.1 Brief the Smoothing domain in ES-FEM for acoustic problems

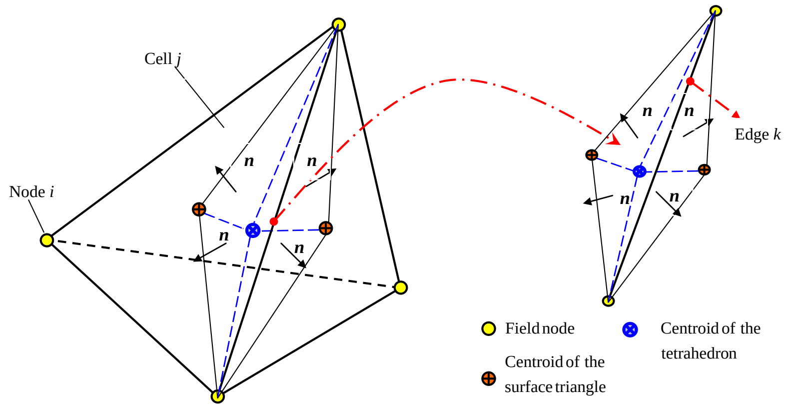

Figure 1 The sub-smoothing-domain of edge k in cell j

In the scheme of the ES-FEM for 3D problems, the numerical integration of Eq. (9) is not performed based on the tetrahedral elements but the edge-based smoothing domains. In the construction of local smoothing domains, the sub-domain of the smoothing domain Ω_k^s for edge k located in the particular cell j can be obtained by connecting two end nodes of the edge to the centroids of the surface triangles and the centroid of cell j, as shown in Fig. 1. The sub-smoothing-domain for edge k is one sixth region of this tetrahedral element. Finding out other sub-domains located in other elements containing edge k and the smoothing domain for edge k can be constructed by uniting all the sub-domains. The number of local smoothing domain is equals to the number of elemental edges. Using edge-based smoothing operation, the gradient component $\nabla\mathbf{N}$ is replaced by the smoothed item $\overline{\nabla\mathbf{N}}$, and the global smoothed acoustic stiffness matrix can be written as:

$$\overline{\mathbf{K}}^{\text{ES-FEM}}=\int_{\Omega}\left(\overline{\nabla\mathbf{N}}\right)^{\mathrm{T}}\left(\overline{\nabla\mathbf{N}}\right)\mathrm{d}\Omega \qquad \text{The smoothed acoustical stiffness matrix} \tag{13}$$

3.2 Improved ES-FEM (IES-FEM) for acoustic problems

Recently, it is found that the ES-FEM also encounters the dispersion error[14] in large wave number, i.e.$1\leq kh\leq 2$, which is related to "slightly overly-soft stiffness" induced by the excessive edge-based smoothing operations compared to the continua system. The numerical method can be tuned to the exact or nearly exact stiffness using appropriate domains as well. Similar as α-FEM by combining the FEM and NS-FEM, an IES-FEM is also proposed by introducing a parameter alpha making the best use of "over-stiffness" of the FEM model and "slightly over-softness" of the ES-FEM model to achieve the ultimate performance.

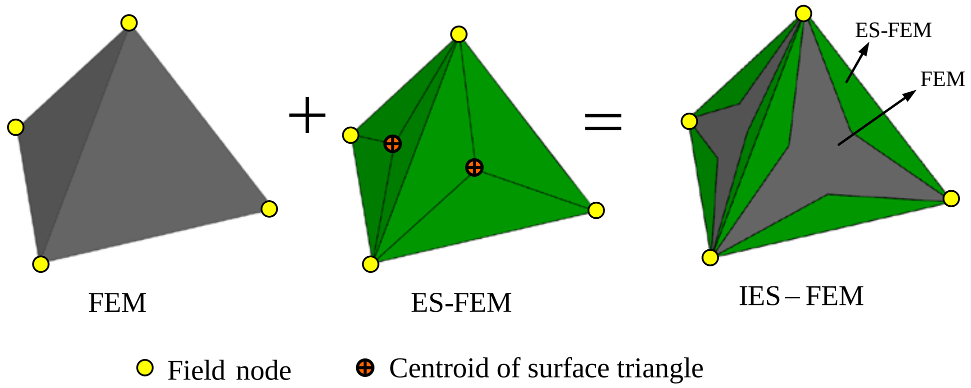

Figure 2 The IES-FEM is formulated by combining of the FEM and ES-FEM

In the 3D problem of IES-FEM, each tetrahedral element is divided into seven portions, as shown in Fig.2: six volumes containing tetrahedral edges have an equal volume of $\frac{1}{6}\alpha V_{\mathrm{e}}$, and the remaining part in the middle of the element has a volume of $(1-\alpha)V_{\mathrm{e}}$, where the V_{e} is the volume of the tetrahedral element. Six volumes containing tetrahedral edges compose a part of six edge-based smoothing domain of ES-FEM, and the middle part is used to calculate the contribution to the local stiffness matrix of IES-FEM using FEM, thus the stiffness of IES-FEM can be formulated as follows:

$$\mathbf{K}^{\text{IES-FEM}}=(1-\alpha)\sum_{i}^{N^e}\mathbf{K}^{\text{FEM}}+\alpha\sum_{i}^{N^n}\mathbf{K}^{\text{ES-FEM}} \tag{14}$$

where N^{e} is the number of total elements in the entire problem domain and N^n is the number of total edges in the entire problem domain. The stiffness matrix $\mathbf{K}^{\text{FEM}}$ and $\mathbf{K}^{\text{ES-FEM}}$ can be calculated by Eq. (9) and Eq. (13). Note that the parameter alpha which controls the contribution of ES-FEM and FEM can be selected as constant to improve the results of acoustic simulation [14]. In the 3D IES-FEM for acoustic problems, numerical analysis indicates that when alpha equals to 0.82, it can always provide very good results.

4. Numerical study

4.1Numerical error for acoustic problems

The pollution is mainly a consequence of the dispersion effect, meaning that the wave number of the numerical solution and the wave number of the exact solution disagree. The dispersion effect was first examined for the wave equation [1]

In the mid-frequency analysis, the standard FEM cannot provide proper results under the condition of $kh>1$ due to dispersion error. In this work, we mainly focus on the acoustic analysis in mid-frequency regime, and "the $1<kh<2$" is adopted as a reference of mid-frequency. In the following computations, the numerical global error indicator is given in terms of velocity and can be described as follows:

$$\left|p^e - p^h\right|_1^2 = \int_\Omega \left(\tilde{v}^e - \tilde{v}^h\right)^{\mathrm{T}} \left(v^e - v^h\right) \mathrm{d}\Omega \tag{15}$$

where $\tilde{v}$ is complex conjugate of the velocity v, the superscript e denotes the exact solutions and h denotes the numerical solutions obtained from numerical methods including the present IES-FEM, ES-FEM using low-order elements.

4.2 3D tube with Neumann boundary condition

The numerical example adopted in 3D interior acoustic problem is a cylinder tube filled with air as shown in Fig. 3. The air density ρ is 1.225kg/m^3 and the speed of sound in the air is 340m/s. The dimension of this cylinder tube with length l=1m, diameter d=0.3m is considered. The boundary conditions of this problem are that: the left end of tube is specified normal velocity boundary condition with v_n=10m/s, and the right end of tube is a rigid wall with zero velocity. The analytical solutions for this problem can be easily derived and the acoustic pressure and velocity are given by

$$p = -j\rho c v_n \frac{\cos\left(k\left(1-\xi\right)\right)}{\sin(k)} \tag{16}$$

$$v = \frac{v_n \sin\left(k\left(1-\xi\right)\right)}{\sin(k)} \tag{17}$$

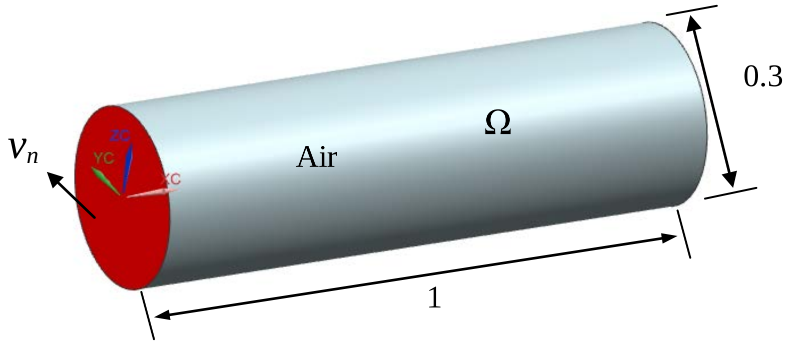

Figure 3 3D acoustic tube with the Neumann boundary condition.

4.2.1 The verification of parameter alpha

In 3D acoustic analysis, the parameter alpha is firstly investigated. The frequency value of 1352Hz is used. Fig. 4 plots the numerical error of IES-FEM against the parameter alpha from 0 to 1 with a step of 0.01 by the use of four models with nodal spacing of 0.06m, 0.05m, 0.04m and 0.03m. It can be seen from this figure that: (1) at frequency of 1352Hz , the monotonic convergence can be obtained for the IES-FEM with alpha varies from 0 to 1 with the refinement of mesh; and the IES-FEM can always provide very good results when α equals to about 0.82.

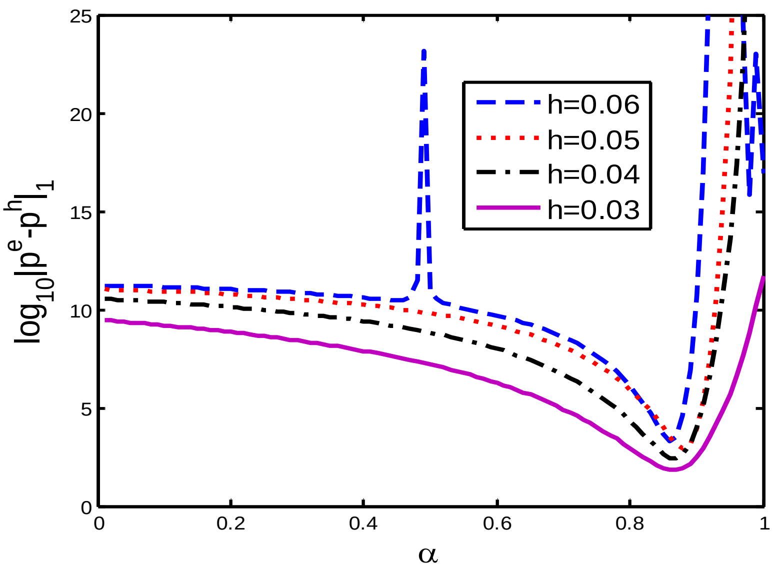

Figure 4 Numerical error for different mesh size by varing the parameter alpha at different frequencies

4.2.2 Acoustic accuracy and convergence

The tube is discretized by unstructured mesh with an average mesh size of 0.06m. The frequency value of 1352Hz (kh=1.5) has been studied using present IES-FEM. For the purpose of comparison, ES-FEM[13] solutions are also computed using this set of tetrahedral mesh. The results obtained from these two numerical methods have been plotted in Fig. 5, together with the exact solutions. It can be seen from the plots that: the IES-FEM can give much better results than the ES-FEM, that gives solution departing a lot from the exact one. The convergence property is also investigated by using four models with 588, 990, 1753 and 3894 uniformly distributed nodes with nodal spacing of 0.06m, 0.05m, 0.04m and 0.03m. Fig. 6 presents the convergence curves in terms of global error against the average nodal spacing h at the frequency of 1352Hz for ES-FEM and IES-FEM. From the figure, it can be observed that the present IES-FEM can give much more accurate gradient results than that of ES-FEM.

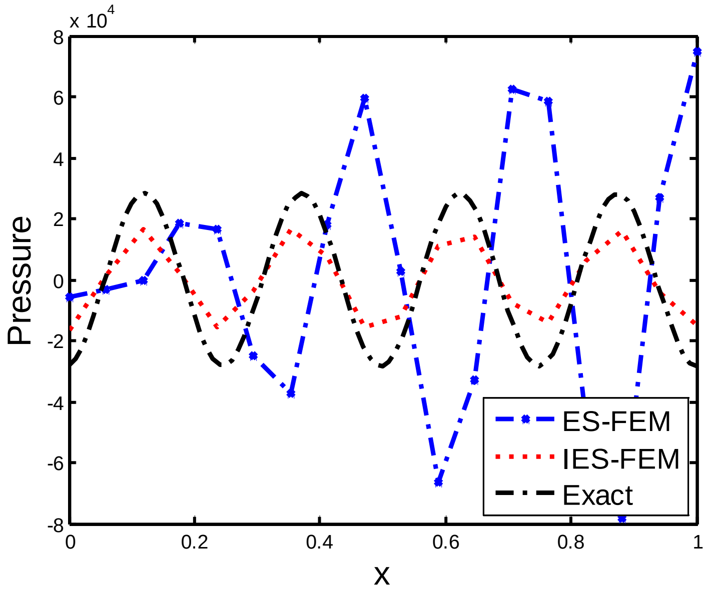

Figure 5 Exact and numerical solutions of acoustic pressure for the 3D acoustic tube at kh=1.5

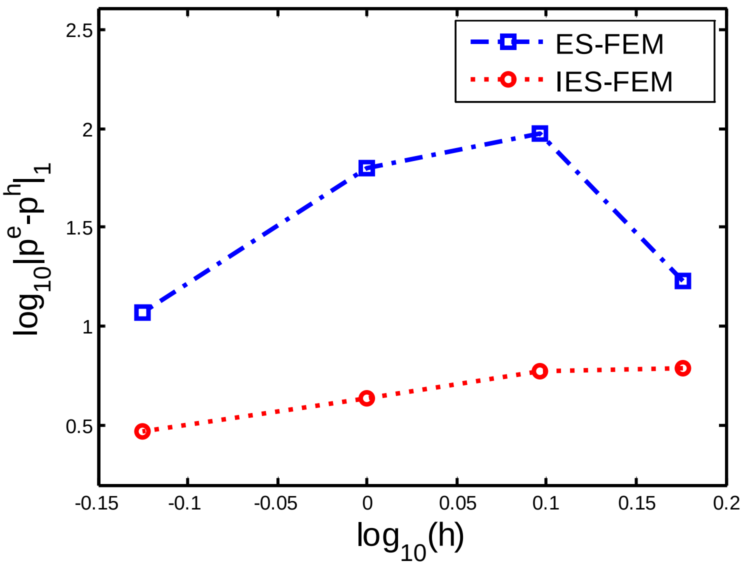

Figure 6 Comparison of accuracy and convergence for 3D acoustic tube

4.2.3 Computational time and efficiency study

Fig. 7 compares the computational time between IES-FEM and ES-FEM. Note the bandwidth of IES-FEM is the same as ES-FEM, and the computational time using IES-FEM is almost the same as ES-FEM. However, when it comes to the computational efficiency in terms of CPU time for the same accuracy, the IES-FEM is found to much more efficient than the ES-FEM as shown in Fig. 8.

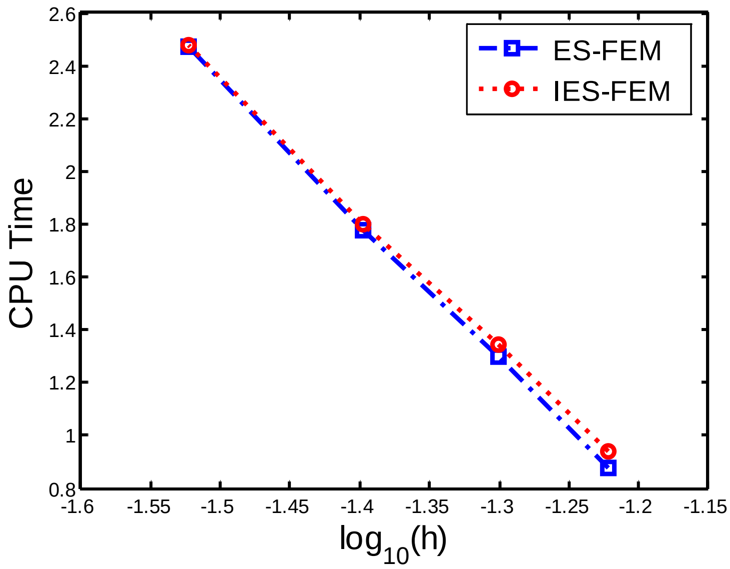

Figure 7 Comparison of the computational time for the IES-FEM and ES-FEM

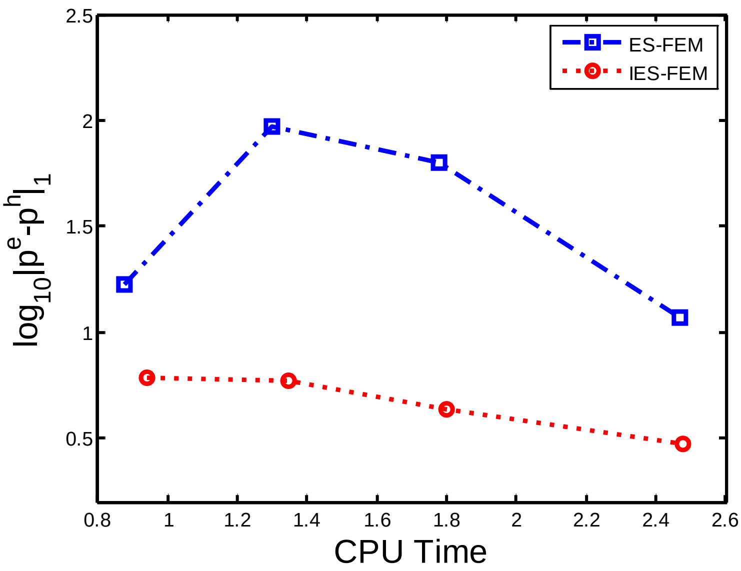

Figure 8 Comparison of the efficiency of numerical results in terms of global error

5. Conclusions

In this paper, an improved 3D edge-based smoothed finite element method (IES-FEM) is proposed by introducing a parameter controlling the ratio of "slightly over-softness" of the ES-FEM and "over-stiffness" of the FEM, and the balance of the discretized system can then be tuned to reduce the dispersion error in higher frequency range, i.e. $1 \le kh \le 2$. The determined parameter alpha controls a proper gradient smoothing operation in the IES-FEM, and provides a perfect balance between stiffness and mass in the discrete system matrix, which dramatically reduces the dispersion error. Numerical results demonstrate the advantages of IES-FEM for acoustic problems, in comparison with the ES-FEM using the same mesh.

a) The IES-FEM uses the simplest linear tetrahedral mesh, which can be easily generated for any complicated domains of acoustic media, and hence is ideal for automated modeling and simulation.
b) In the IES-FEM, no additional parameters or degrees of freedom are introduced, and the present method can be implemented in a straightforward way with little change to the standard FEM code.
c) In IES-FEM for acoustic problems, when the parameter alpha equals to 0.82, the numerical studies show that it can always provide much better results and is more efficient than ES-FEM.

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