On the solution of sine-Gordon solitons via localized kernel-based Method

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Abstract

In this work, a local kernel based numerical scheme is constructed for numerical solution of sine-Gordon equation in circular domain. The global kernel method resulted the dense differentiation matrices and hence difficult to apply for problem with large amount of data points. The present numerical scheme is local with sparse differentiation matrices, consequently capable of removing the deficiency of ill-conditioning in the global kernel method.

Keywords: Local kernel based scheme; two-dimensional sine-Gordon equation

1. Introduction

The kernels (RBFs) was first used for solving partial differential equations by Kansa in the year 1990 [Kansa (1990), fluid dynamics-I; Kansa (1990), fluid dynamics-II]. In this original work the fluid mechanics problems were solved by approximating the derivatives by the derivative of MQ kernel functions directly. The differentiation matrices obtained in this method are unsymmetric as well as dense. The dense linear system in the global kernel method solved by Gaussian elimination methods needed $O(N^3)$ floating point operations. Due to the high resolution for large amount of data points it becomes difficult to solve the problem with global kernel based method. Many robust numerical approximation methods have been developed to overcome this difficulty some of them are the transforms based methods and the multi-pole approaches [Greengard and Strain (1991); Cherrie et al. (2002); Gumerov and Duraiswami (2007)], the domain decomposition methods [Beatson et al. (2001); Kansa and Hon (2000); Li, and Hon (2004)], the partition of unity methods [Wendland (2002)], the greedy algorithms [Hon et al. (2003); Schaback and Wendland (2000); Ling and Schaback (2008)], the multilevel methods [Fasshauer (1999)], and the use of locally supported kernel functions [Wendland (1995); Floater and Iske (1996)]. An other alternative approach to overcome this difficulty was developed by Tolstykh [Tolstykh (2000)], here local kernel interpolants in small domains centered around each node is used to form differentiation weights. This idea has been used to construct various types of local kernel based approximate methods and has been applied successfully to a wide range of problems. These include convection-diffusion [Chandhini and Sanyasiraju (2007); Stevens et al (2009); Sarler and Vertnik (2006); Sarra (2012)], incompressible NavierStokes [Chinchapatnam et al (2009); Shan et al

(2008); Shu et al (2003)], elliptic equations [Tolstykh and Shirobokov (2003); Wright and Fornberg (2006)] and [Wong et al (1999); Xiao and McCarthy (2003); Brown et al (2005)]. In the present work we used the same idea to construct local kernel based numerical scheme for simulating two-dimensional sine-Gordon equation.

The sine-Gordon equation in two space dimension is

$$s_{\tau\tau} = s_{\xi\xi} + s_{\eta\eta} - \mu(\xi,\eta)\sin s, \text{ where } \tau > 0 \text{ and } (\xi,\eta) \in \Omega, \tag{1}$$

with associated initial conditions

$$s(\xi,\eta,0) = h_1(\xi,\eta),\ s_\tau(\xi,\eta,0) = h_2(\xi,\eta), \tag{2}$$

and with boundary condition

$$\beta s = (\xi,\eta,\tau), \tag{3}$$

In science and engineering we always need some robust numerical scheme to solve soliton type equations for large scaled data points in irregular domain for example the sine-Gordon type solitons. Many robust numerical scheme have been developed by many researchers over the years to approximate the sine-Gordon equation. For example the finite difference scheme [Guo et al (1986)], The leapfrog scheme [Christiansen and Lomdahl (1981)], the finite-elements approach [Argyris et al (1991)]. A predictor-corrector scheme [Khaliq, A. Q. M. et al. (2000)], and a split cosine scheme [Sheng, Q. et al. (2005)]. Bratsos [Bratsos (2007)] used a three-time level fourth-order explicit finite-difference scheme for solving sine-Gordon equation. In this work we used local kernel based numerical scheme to approximate the solution of 2d sine-Gordon equation.

2. Description of the method

In multivariate scattered data interpolation, we always need to recover an unknown function $s : R^d \to R$ from a given set of N function values $\{s(\xi_1), s(\xi_2),..., s(\xi_N)\} \subset R.$ Where the scattered centers $\xi_1, \xi_2,...,\xi_N \in \Omega$ and $\Omega \subset R^d$ is arbitrary shaped domain and the centers can be chosen anywhere in the domain. In the local kernel based approximation method, at each center $\xi_i \in \Omega$, the local interpolant takes the form

$$s(\xi_i,\tau) = \sum_{\xi_j \in \Omega_i} a_j(\tau)\kappa\left(\left\|\xi_i - \xi_j\right\|\right), \tag{4}$$

where $a^i = [a_1, ..., a_n]$ is a vector of expansion coefficients, $\kappa : \Omega \times \Omega \to R$ is a radial kernel defined by $\kappa(\xi, \xi_j) = \kappa(r_j)$ with $r_j = \|\xi - \xi_j\|$ and $\Omega_j \subset \Omega$ is a local domain corresponding to center ξ_i contains $n < N$ centers. The corresponding N number of $n \times n$ linear systems are given as,

$$s^i = \Lambda^i a^i, i = 1, 2, ..., N, \tag{5}$$

Where the entries of the matrix Λ^i are $\{\kappa(\|\xi_k - \xi_j\|)\}^i$, $k, j \in \Omega_i$, the matrix Λ^i is called the interpolation matrix, and each system have to be solved for the expansion coefficients. Now to approximate the differential operator $Ls(\xi, \tau)$, we have

$$Ls(\xi_i, \tau) = \sum_{\xi_j \in \Omega_i} a_j(\tau) L\kappa(\|\xi_i - \xi_j\|), \tag{6}$$

The expression in (6) may be given in matrix form,

$$Ls(\xi_i, \tau) = \delta^i \bullet a^i, \tag{7}$$

Where a^i is the $n \times 1$ vector of expansion coefficients, and δ^i is the $1 \times n$ vector with entries

$$\delta^i = L\kappa(\|\xi_i - \xi_j\|),\ \xi_j \in \Omega_i. \tag{8}$$

To eliminate the expansion coefficients, we have from equation (5)

$$a^i = (\Lambda^i)^{-1} s^i, \tag{9}$$

we substitute the values of a^i from (9) in (7) to get,

$$Ls(\xi_i, \tau) = \delta^i (\Lambda^i)^{-1} s^i = \sigma^i s^i, \tag{10}$$

where,

$$\sigma^i = \delta^i (\Lambda^i)^{-1}, \tag{11}$$

is the weight corresponding to center ξ_i. Hence for all centers locations, we have

$$Ls = \sum s, \tag{12}$$

where, $\sum$ is $N \times N$ sparse differentiation matrix, each row of the matrix $\sum$ contains n non-zeros elements. After spatial local RBF approximation, we obtained the following system of ODEs

$$\frac{\partial s}{\partial \tau} = F(s). \tag{13}$$

Time integration can be carried out using any ODE solver like ode15s, ode113, ode45 etc from Matlab. In general, ode45 is the best function to apply as a first try for most problems. A good ODE solver will automatically select a reasonable time step $\delta\tau$ and detect stiffness of the ODE system. For this ODE computation we have used Runge-Kutta method of order four.

3. Stability of the local meshless numerical scheme:

In the present local meshless method of lines our numerical scheme is given by

$$s_\tau = \sum s, \tag{14}$$

here the time-dependent PDE is transformed into a system of ODEs in time. The method of lines refers to the idea of solving the coupled system of ODEs by a finite difference method in τ (e.g. Runge-Kutta, etc.) The numerical stability of the method of lines is investigated by a rule of thumb. *The method of lines is stable if the eigenvalues of the (linearized) spatial discretization operator, scaled by $\delta\tau$, lie in the stability region of the time-discretization operator* [Trefethen and Bau (1997)]. The stability region is a part of a complex plane consisting of those eigenvalues for which the technique produces a bounded solution. In the present meshless method of lines our numerical scheme is given in (13). We can investigate the stable and unstable eigenvalue spectrum for the given model by computing the eigenvalues of the matrix $\sum$, scaled by $\delta\tau$.

4. Choosing a good value of shape parameter:

A variety of kernel functions are available in the literature. In our computation we used the multiquadrics kernel fuctions, $\phi(r) = \sqrt{1+\varepsilon^2 r^2}$. As usual these RBFs contain a shape parameter and the solution accuracy greatly depends on this parameter. There exist some strategies for the optimization of the shape parameter [Hardy (1971); Franke (1982); Carlson and Foley (1991); Foley (1994); Rippa (1999); Trahan and Wyatt (2003); Fasshauer and Zhang (2007); Scheuerer (2011)]. A condition number may be used to quantify the sensitivity to perturbations of a linear system, and to estimate the accuracy of a computed solution. The conditioning results require that in order for the system matrix to be well conditioned that the shape parameter and minimum separation distance be large. Obviously, both situations cannot occur at the same time. This observation has been referred to as the uncertainty principle [Schaback (1995)]. Incorporating this idea the smallest errors occur when the condition number v of the system matrix is approximately kept in the range $10^{13} < v < 10^{15}$ in our computations. The system matrix is decomposed as **A**, **E, B** = $svd(\Lambda^i)$. Here svd is the singular value decomposition of the interpolation matrix Λ^i. **A**, **B** are $n \times n$ orthogonal matrices and **E** is $n \times n$ diagonal matrix contains the n singular values of Λ^i, and $v = \|\Lambda^i\| \|(\Lambda^i)^{-1}\| = \max(E)/\min(E)$ is the condition number of the matrix Λ^i. When an acceptable value of shape parameter is returned by the above algorithm, then the svd is used to compute $(\Lambda^i)^{-1} = (AEB^T)^{-1} = BE^{-1}A^T$ (see [Trefethen and Bau (1997)]). Note that for orthogonal matrices the inverse of the matrix is equal to its transpose. Consequently, we can compute the weights σ^i in (11).

5. Application of the method

In this section we apply the method described above to solve the two-dimensional sine-Gordon equation. We considered various types of initial solutions in the form of circular, ring solitons, interaction of two and four circular ring solitons. The two-dimensional sine-Gordon equation has been transformed into a system of two partial differential equations given by

$$s_\tau = p, \; p_\tau = s_{\xi\xi} + s_{\eta\eta} - \mu(\xi,\eta)\sin s, \tau > 0,$$

with the boundary condition as $\nabla s \bullet q = 0$, and $\nabla p \bullet q = 0$, and with the initial conditions
$s(\xi,\eta,0) = h_1(\xi,\eta),\; p(\xi,\eta,0) = h_2(\xi,\eta),$ respectively and where q is a unit normal vector.

5.1. *Circular solitons*

We apply the proposed method for the case when $\mu(\xi,\eta) = 1$, the initial solution is taken as circular solitons [Argyris et al (1991)] given by

$$h_1(\xi,\eta) = 4\tan^{-1}\exp[3 - \sqrt{\xi^2 + \eta^2}], \tag{16}$$

$$h_2(\xi,\eta) = 0, \tag{17}$$

the problem is solved in the circular domain of radius r = 8 with N = 3000 uniformly distributed interpolation nodes. We select $n = 200$ points in each local domain Ω_i corresponding to each node $i = 1,2,3,...,N \in \Omega$.The time integration is carried out with Runge-Kutta method of order 4 with time step $\delta t = 0.005$. The results are obtained by the present numerical method in terms of sin(s/2), where s is the approximate solution of the given model obtained with present local method. The obtained results at different times are shown in Figure 1.

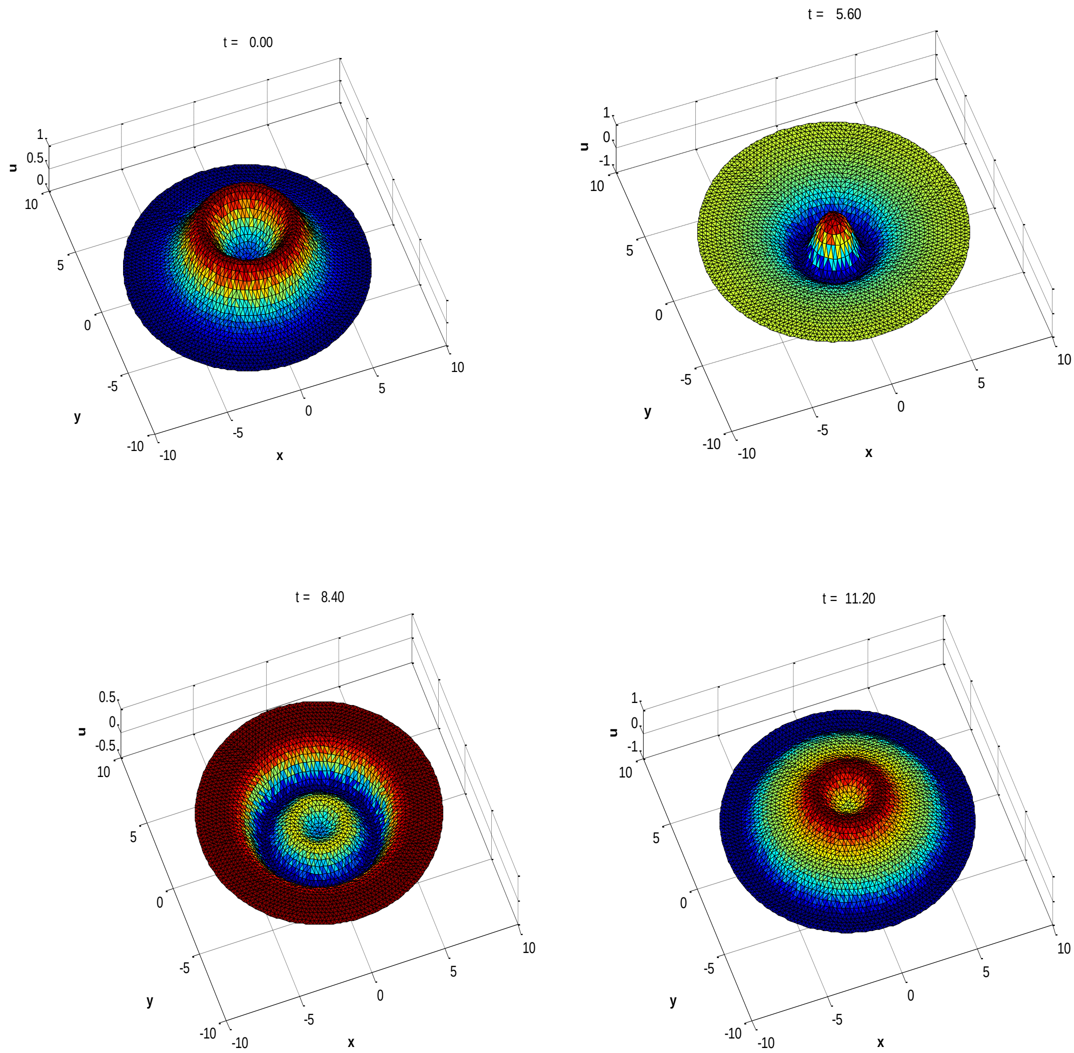

Figure 1: Circular soliton: approximate solution in the form of sin(*s*/2) at *τ* = 0, 5.6, 8.5, and 11.2, in the domain Ω with radius *r* = 8, *N* = 3000.

5.2. *Two solitons collision*

Here we consider the interaction of two expanding solitons for the choice $\mu(\xi,\eta)=1$ and with the initial solutions

$$h_1(\xi,\eta)=\sum_{i=1}^{2} f_i(\xi,\eta)\,,\ h_2(\xi,\eta)=\sum_{i=1}^{2} g_i(\xi,\eta) \tag{18}$$

$$f_i(\xi,\eta)=4\tan^{-1}\exp[(4-\sqrt{(\xi\pm3)^2+(\eta\pm7)^2}\,)/0.436\,], \tag{19}$$

and

$$g_i(\xi,\eta)=4.13\sec h[(4-\sqrt{(\xi\pm3)^2+(\eta\pm7)^2}\,)/0.436\,]. \tag{20}$$

We select $N=5000$ number of uniformly distributed interpolation points in the circular domain Ω of radius $r=25$. We solved the problem without using the symmetry features that was used in the earlier work [Argyris et al (1991), Sheng, Q.et al. (2005); Dehghan, M. and Shokri, Ali. (2008)] for simulating the collision of two circular solitons. We take the interpolation points in the whole computation domain to demonstrate the robustness of local radial kernel method. This demonstrates the capability and efficiency of the present method for solving large scale problem in circular domain. The results of the present method are shown in Figure 2.

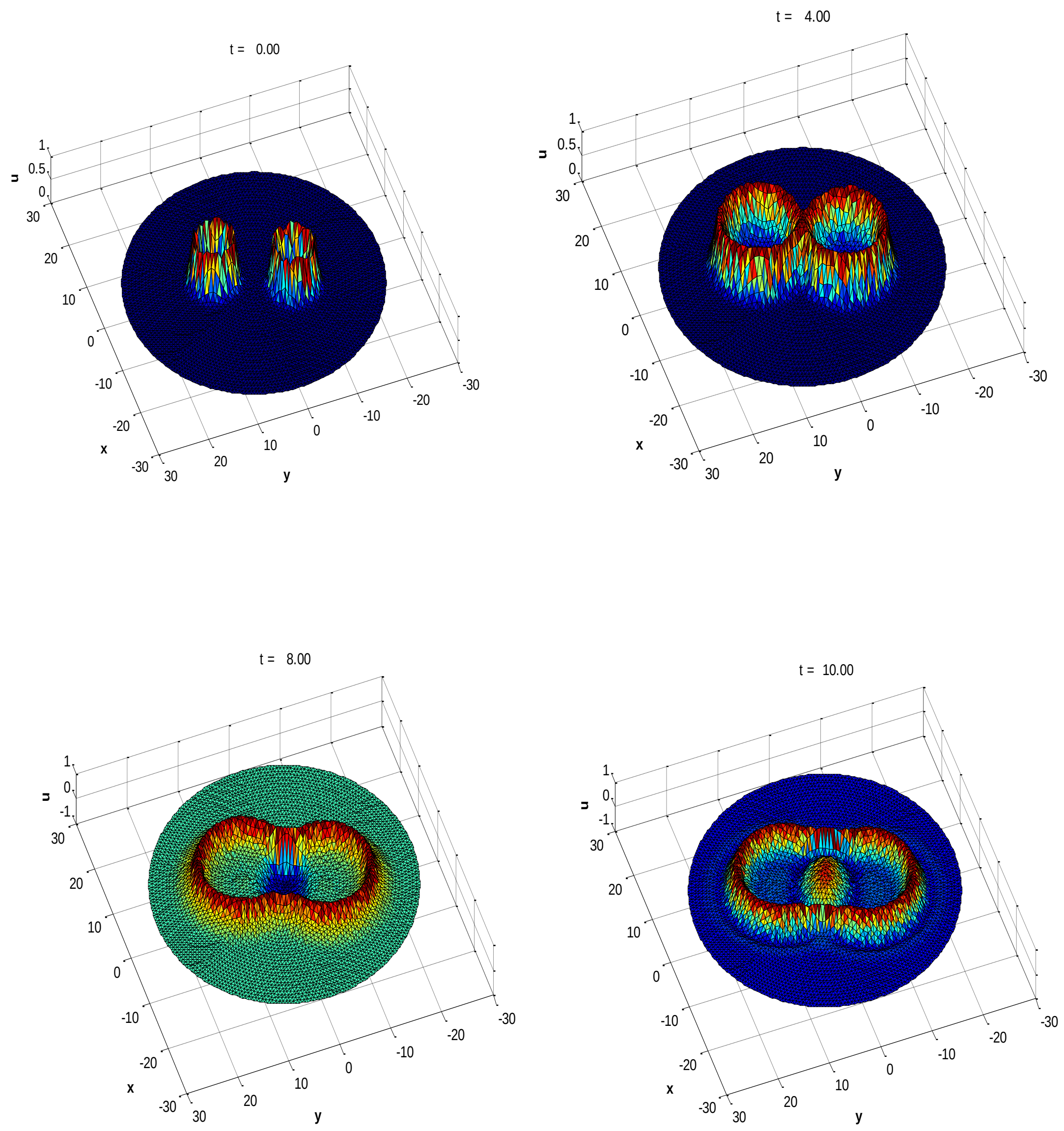

Figure 2: Two solitons collision: the function sin(*s*/2), at *t* = 0, 4, 8 and 10, in the domain Ω with radius *r* = 25, *N* = 5000.

5.3. ***Four expanding solitons collision***

The collision of four expanding circular solitons are considered for the choice $\mu(\xi,\eta)=1$, and with the initial solutions

$$h_1(\xi,\eta)=\sum_{i=1}^{4} f_i(\xi,\eta),\, h_2(\xi,\eta)=\sum_{i=1}^{4} g_i(\xi,\eta) \tag{21}$$

$$f_i(\xi,\eta)=4\tan^{-1}\exp[(4-\sqrt{(\xi\pm 7)^2+(\eta\pm 7)^2}\,)/0.436\,], \tag{22}$$

and

$$g_i(\xi,\eta)=4.13/\cosh[(4-\sqrt{(\xi\pm 7)^2+(\eta\pm 7)^2}\,)/0.436\,]. \tag{23}$$

This problem is solved in the circular domain of radius $r=25$ with $N=5000$ uniformly distributed interpolation points. Again we are not using the symmetry features used in the earlier work [Argyris et al (1991), Sheng, Q. et al. (2005); Dehghan, M. and Shokri, Ali. (2008)]. The evolution of the four expanding solitons in times are shown in Figure 3.

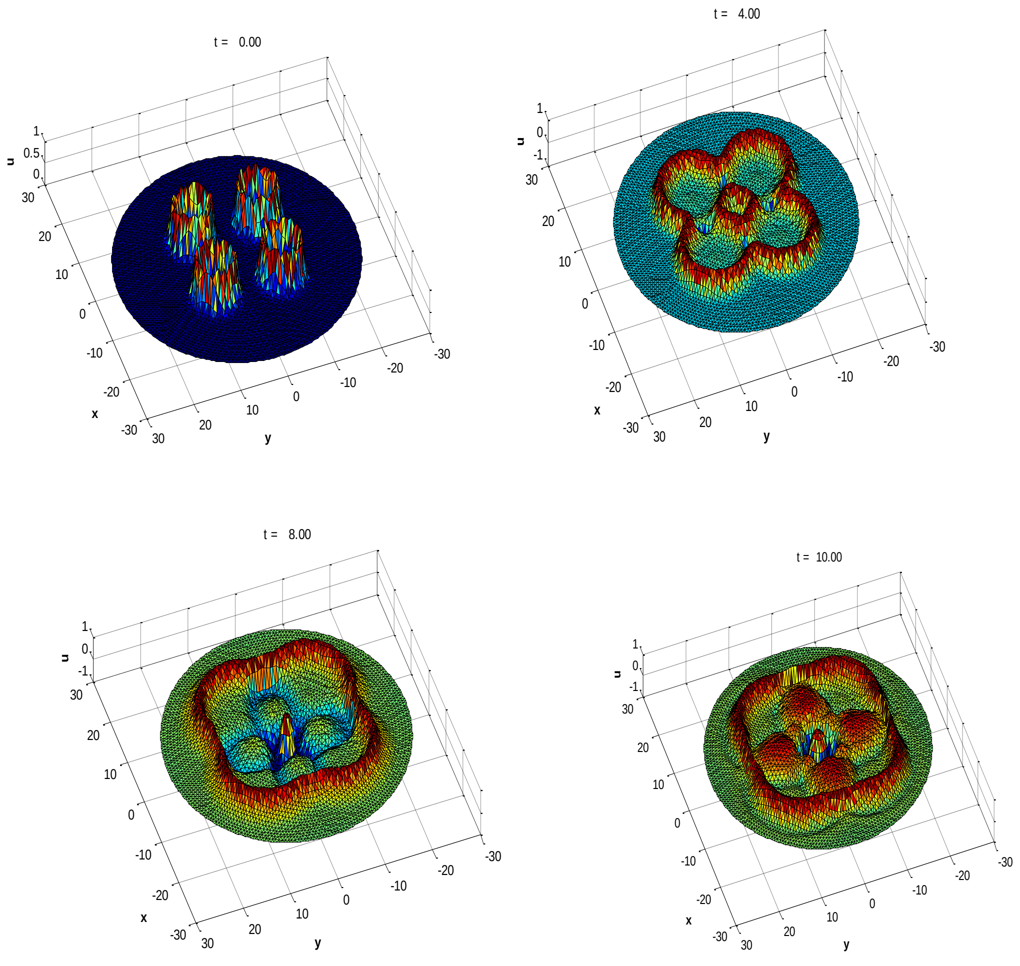

Figure 3: Four expanding solitons collision: the function sin(*s*/2), at *t* = 0, 4, 8 and 10, in the domain Ω with radius *r* = 25, and *N* = 5000.

6. Conclusions

In this work we have constructed local kernel based numerical scheme for simulating the two dimensional sine-Gordon equation. As contrary to the global based kernel based methods [Dehghan, M. and Shokri, Ali. (2008)], the present local scheme performed efficiently for large data points in complex shaped domain. The present local method may be used to similar types of time-dependent partial differential equations in irregular shaped domain.

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