Solution of computational acoustics and wave propagation problems using a high order, high resolution coupled compact difference scheme

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Abstract

Here, we report a high order, high resolution coupled compact difference scheme for solving computational acoustics problems. Proposed coupled compact difference scheme displays significant spectral resolution while estimating spatial derivatives and has a physical dispersion relation preserving (DRP) ability over a wide range of wave number when a fourth order four stage Runge-Kutta scheme is used for time integration. Proposed scheme simultaneously computes the first, the second and the fourth derivative in a coupled manner at all the grid points in the domain. We have purposefully evaluated the fourth derivative term using coupled compact difference scheme to add numerical diffusion for the attenuation of unphysical spurious waves in the computed solution.

Keywords: Computational Acoustics, Coupled Compact Difference Scheme, DRP Property, High resolution scheme

Introduction

Propagation of an acoustic wave over a small distance inside a homogeneous medium displays non-dissipative, non-dispersive characteristics [Kinsler et al. (1999)]. Simulation of acoustic wave propagation problem involves computation of acoustic wave field either directly from the linearized compressible flow equations [Tam & Webb (1993)] or by solving hyperbolic partial differential equation for wave propagation [Sengupta (2013)]. The numerical scheme used for solving acoustic problems must have a significant spectral resolution to effectively resolve all the scales present in the acoustic field. Compact schemes provide higher spectral resolution as compared to the explicit difference schemes for same stencil size [Lele (1992), Fung et al. (1995), Chu & Fan (1998), Sengupta et al. (2003), Zhou et al. (2007), Bhumkar et al. (2014)] and are preferred for obtaining highly accurate solutions. As propagation of an acoustic wave displays non-dispersive, non-dissipative and isotropic nature, numerical schemes used for simulating computational acoustic problems must be neutrally stable and preserve the physical dispersion relation numerically [Tam & Webb (1993), Sengupta (2013)]. For an adopted numerical scheme, it is not only important to resolve all physical spatial and temporal scales but also display neutrally stable, DRP nature [Sengupta (2013)]. This has prompted researchers to search for a high resolution, dispersion relation preserving schemes which are useful for computing wave propagation problems [Chiu & Sheu (2009), Tam & Webb (1993), Hu et al. (1996)].

Here, we are proposing a new high order, high resolution coupled compact difference scheme to compute the spatial derivative terms while a fourth order four stage Runge-Kutta scheme has been used for time integration. The derived coupled compact difference scheme evaluates the first derivative, the second derivative and the fourth derivative simultaneously at all the grid points in the domain.

Methodology

Stencil for the coupled compact difference scheme:

Almost all discrete difference computations involve implicit filtering and corresponding solution components in high wavenumber range show spurious nature which are often responsible for numerical instabilities [Sengupta (2013)]. Unphysical amplification of high wavenumber components can be controlled by either using upwind scheme or by using explicit filters [Yu et al. (2015), Visbal & Gaitonde (2002)]. A general stencil for the upwind scheme is given as [Sengupta (2013)],

$$\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x}\right)_{CD} + \alpha\left(\frac{\partial^{2n} f}{\partial x^{2n}}\right) \tag{1}$$

In upwind schemes, one adds explicit numerical diffusion through the even order derivative term in Eq. 1. For adding controlled amount of numerical diffusion, diffusion coefficient α has been used. In Eq. 1, n is an integer and the subscript CD shows derivative has been obtained using a central difference scheme. Diffusion coefficient α can be either positive or negative based on the direction of propagation of information at a particular point in the domain.

A high accuracy, spectrally optimized upwind CCD scheme has been proposed in [Chiu et al. (2009)], to evaluate the first and second derivative terms together. In the present work, we propose following coupled compact difference scheme with a central stencil to evaluate the first, the second and the fourth derivative terms together. Information associated with the fourth derivative term has been used here to attenuate unphysical spurious waves.

Consider a domain discretized using equi-spaced grid points with a grid spacing h. The coupled compact difference scheme for simultaneous evaluation of the first, the second and the fourth derivative term is given as,

$$\frac{87}{176}\left(u^{I}_{j+1}+u^{I}_{j-1}\right)+u^{I}_{j}+\left(\frac{-13}{176}\right)h\left(u^{II}_{j+1}-u^{II}_{j-1}\right)+\frac{1}{528}h^{3}\left(u^{IV}_{j+1}-u^{IV}_{j-1}\right)=\frac{1}{h}\left(\frac{47}{48}\left(u_{j+1}-u_{j-1}\right)+\frac{1}{132}\left(u_{j+2}-u_{j-2}\right)\right) \tag{2}$$

$$\frac{1805}{1224h}\left(u^{I}_{j+1}-u^{I}_{j-1}\right)+\left(\frac{-79}{408}\right)\left(u^{II}_{j+1}+u^{II}_{j-1}\right)+u^{II}_{j}+\frac{1}{408}h^{2}\left(u^{IV}_{j+1}+u^{IV}_{j-1}\right)=\frac{1}{h^{2}}\left(\frac{1624}{459}\left(u_{j+1}+u_{j-1}\right)+\left(\frac{-241}{34}\right)u_{j}+\frac{11}{1836}\left(u_{j+2}+u_{j-2}\right)\right) \tag{3}$$

$$\left(\frac{-13945}{408h^{3}}\right)\left(u^{I}_{j+1}-u^{I}_{j-1}\right)+\frac{755}{136h^{2}}\left(u^{II}_{j+1}+u^{II}_{j-1}\right)+\left(\frac{-13}{136}\right)\left(u^{IV}_{j+1}+u^{IV}_{j-1}\right)+u^{IV}_{j}=\frac{1}{h^{4}}\left(\frac{-8600}{153}\left(u_{j+1}+u_{j-1}\right)+\frac{1920}{17}u_{j}+\frac{-40}{153}\left(u_{j+2}+u_{j-2}\right)\right) \tag{4}$$

In order to obtain the first, the second and the fourth derivative terms, equations (2)-(4) are solved in an iterative and coupled manner. For iterative approach, we propose to evaluate the various derivative terms using explicit central difference schemes as an initial guess. This will reduce the computational cost required for iterative approach as the iterations are performed with reasonable initial guess which is more close to final solution as compared to some random initial guess. Equations (2)-(4) are solved in an iterative manner till the maximum residue becomes less than the prescribed tolerance value which is chosen as 10^{-6} in present study. Here, a simple traditional Gauss-Seidel iterative algorithm has been used.

One can use equations (2)-(4) at all the grid points for the periodic problem. However for the non-periodic problems, different stencils for the boundary and the near boundary nodes are required. In this regard, we propose to use following stencils for the respective derivative terms at the inlet boundary [Sengupta (2013)]. The second and the fourth derivative terms are usually not required at the boundary nodes where one usually prescribe a Dirichlet boundary condition and hence are not evaluated at the boundary nodes while the fourth derivative term is not evaluated at the second and second last node. Same stencils can be used for the boundary and the near boundary points on the other end by reverting the stencils and adding a minus sign [Sengupta (2013)]. Thus, we have used explicit stencils for the boundary and the near boundary nodes as,

$u^I_1 = (-1.5u_1+2u_2-0.5u_3)/h$; $u^{II}_1 = 0$; $u^{IV}_1=0$ (5)

$u^I_2 = (u_3-u_1)/(2h)$; $u^{II}_2 = (u_1-2u_2+u_3)/h^2$; $u^{IV}_2=0$ (6)

$u^I_3 = (-u_5+u_1+8(u_4-u_2))/(12h)$; $u^{II}_3 = (u_1-2u_2+u_3)/h^2$; $u^{IV}_3=(u_1+u_5-4(u_2+u_4)+6u_3)/h^4$ (7)

Finite difference schemes depend on information available at the nearby points to estimate derivative values. Taylor series approximation is used to derive stencil for difference scheme in such a way that the lower order derivative terms are matched accurately while the higher order derivatives terms are truncated. Thus the numerically estimated derivative and the exact derivative values differ due to the truncation error. This is also known as implicit filtering associated with the difference schemes [Sengupta (2013)]. Different finite difference schemes can have same order of accuracy however different spectral resolution while evaluating derivative terms. In such case, the scheme having higher spectral resolution will produce more accurate results as compared to low resolution schemes. Thus, it is imperative to evaluate performance of numerical scheme based on its spectral resolution and not on its order of accuracy [Sengupta (2013)]. Here, we have estimated spectral resolution of the proposed scheme using the full domain matrix global spectral analysis (GSA) technique in [Sengupta (2013)]. Details about this technique are not provided here to avoid repetition.

Following the work in [Sengupta (2013)], if one denotes K and K_{eq} as the exact and the numerically obtained wavenumber in a difference computation then the discretization effectiveness for the first, the second and the fourth derivative can be obtained as shown in Figure 1(a)-(c), respectively. Figure 1(a) compares the effectiveness of spectral resolution (K_{eq}/K) for the proposed coupled compact difference scheme and a 12th order compact difference scheme. Figure shows proposed scheme has even better spectral resolution as compared to the 12th order compact difference scheme. Discretization effectiveness for the second and the fourth derivative terms show near spectral resolutions. Thus the proposed coupled compact difference scheme has significantly improved spectral resolution as compared to the existing difference schemes.

Apart from estimation of effectiveness for spatial derivative terms, one needs to estimate combined effects of spatial and temporal discretization terms for solving unsteady problems. Consider a 1D wave equation which also serves as a model equation for computational acoustics problems as,

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \quad (8)$$

Simulation of the computational acoustics problems involves estimation of both space and time derivative terms. For such unsteady problems, one needs to first estimate important numerical properties such as the numerical amplification factor |G| and the numerical group velocity V_{gN}/c. Variation of these important numerical properties with respect to non-dimensional wavenumber Kh and CFL number Nc has been discussed in [Sengupta (2013)] for the solution of 1D wave equation using different discretization schemes. Using the same methodology, we obtained the contours of the numerical amplification factor |G| and the numerical group velocity V_{gN}/c for the solution of 1D wave equation when coupled compact difference scheme has been used for spatial discretization while RK4 scheme is used for time integration.

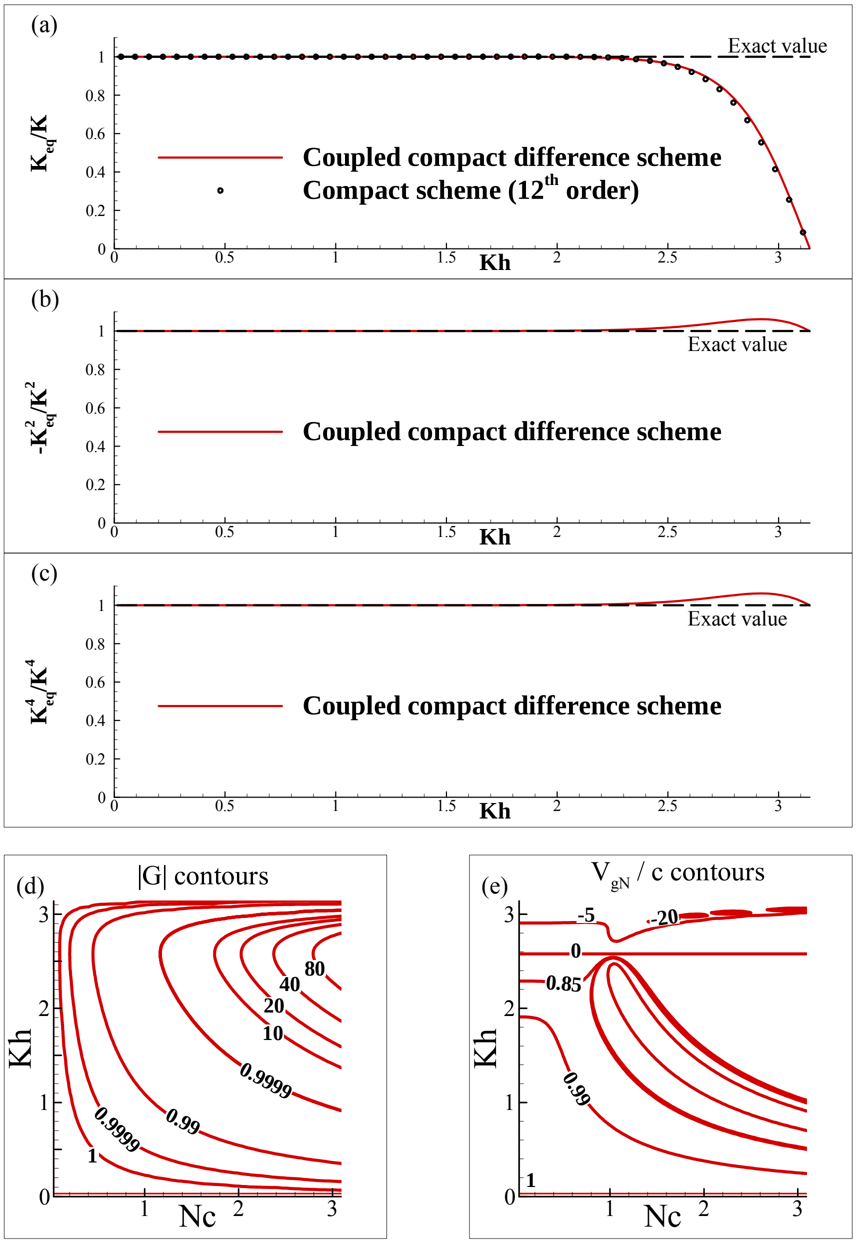

Figure 1: (a) Comparison of the effectiveness of spectral resolution (K_{eq}/K) for different spatial discretization schemes to evaluate the first derivative term at the central node; Discretization effectiveness of the second and the fourth derivative terms is shown in (b) and (c) respectively; Contours for the variation of the numerical amplification factor |G| and the numerical group velocity V_{gN}/c are shown in (d) and (e), respectively when the coupled compact difference scheme is used with the fourth order RK4 scheme for time integration to solve Eq. (8).

Figure 1 (d) and (e) shows numerical properties corresponding to the central node. Numerical amplification factor contours in Fig. 1(d) show that for a small CFL number Nc, one observes a neutrally stable region across a complete wavenumber range. Non-dimensionalized numerical group velocity contours in Fig. 1(e) show that for a small CFL number, physical dispersion relation has been preserved accurately up to non-dimensional wavenumber Kh = 1.9. Thus present scheme has significant DRP ability. One observes presence of negative group velocity above Kh=2.6. This region has been identified as a q-wave region [Sengupta (2013)] in which solution components not only travel with wrong velocity but also in wrong direction. Such waves are often responsible for numerical instabilities.

Spurious waves are often triggered due to presence of sharp discontinuities, irregularly spaced grid points, discontinuities in the initial and boundary conditions [Sengupta (2013)]. One needs to attenuate these spurious waves by addition of numerical diffusion to avoid numerical instabilities. One can add numerical diffusion using the information associated with the fourth derivative evaluated at each grid point as shown in Eq. (1). Amount of added numerical diffusion directly depends on diffusion coefficient α. Figure 2 (a)-(b) show variation of numerical amplification factor and numerical group velocity contours for the solution of 1D wave equation when the indicated diffusion coefficient is used to obtain the upwind coupled compact difference scheme. Figure 2 shows that with increase in α, scheme displays more and more stable behavior for a small CFL number across a complete wavenumber range Kh. Thus one can construct upwind coupled compact difference scheme to damp out unphysical, spurious components from the solution.

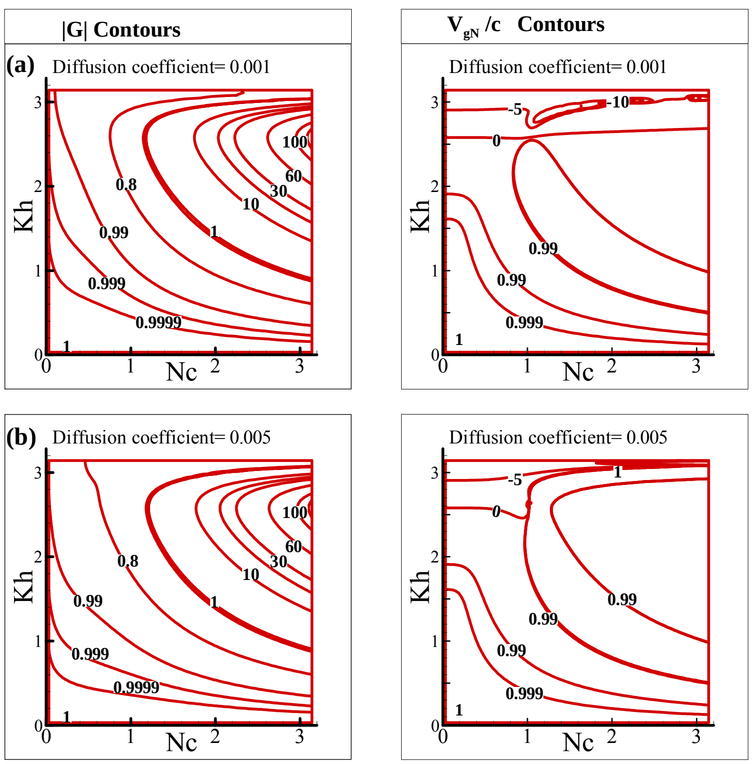

Figure 2: Comparison of the numerical amplification factor |G| contours and the numerical group velocity V_{gN}/c contours for the central node is shown for different diffusion coefficients. Contours are obtained for the solution of Eq. (8) when the coupled compact difference scheme has been used with the fourth order RK4 scheme for time integration following work in [Sengupta (2013)]. Note that with addition of numerical diffusion, stability is achieved (|G|<1) for a small CFL number.

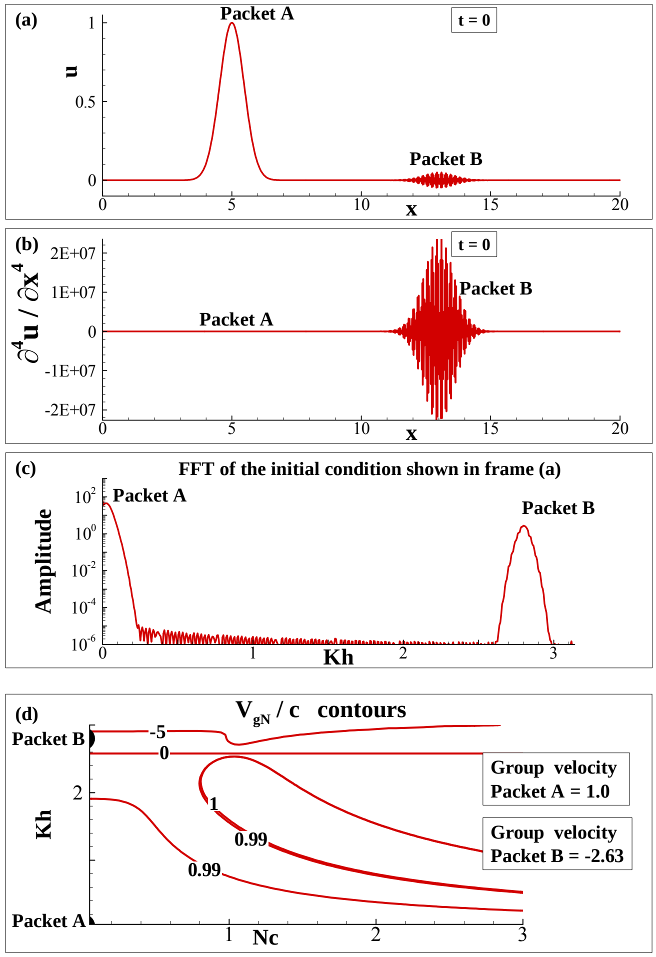

Figure 3: Initial condition of the wave packet along with its fourth derivative and FFT of the initial condition have been shown in (a), (b) and (c), respectively. Numerical group velocity contours for the solution of Eq. (8) have been shown in (d).

Results and Discussion

In this section, we use the coupled compact difference scheme to solve the model wave equation problems as well as for solving computational acoustics problems.

1. *Solution of 1-D wave equation for wave propagation problem.*

We have chosen this problem to verify the efficacy as well as advantages of the proposed scheme while solving unsteady problems. Here, we have obtained solution of 1D wave equation Eq. (8) subjected to the initial condition as shown in Fig. 3(a). We have purposefully designed the initial condition as a combination of two different wave packets, packet A and packet B. Figure 3(b) shows the fourth derivative of the initial condition and it indicates large values corresponding to packet B due to rapid variation of amplitude associated with packet B. The FFT of the initial condition has been shown in Fig. 1(c). Central wavenumber of packet A is very small while that for packet B is close to the Nyquist limit. In figure 1(d), we have shown the numerical group velocity contours for the solution of Eq. (8) when the spatial discretization has been obtained using coupled compact difference scheme while RK4 scheme has been applied for time integration. We have also marked the central wavenumbers corresponding to packet A and packet B.

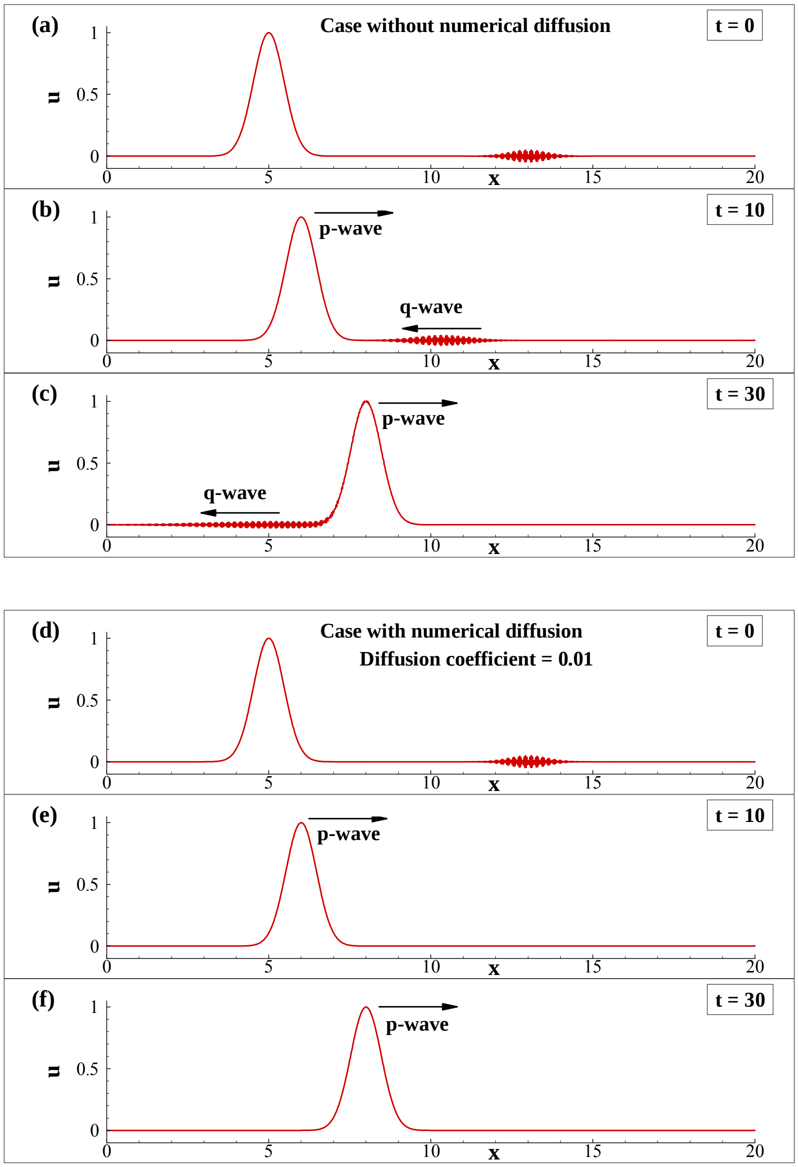

Figure 4: Solutions of 1D wave equation Eq. (8) without and with addition of numerical diffusion have been shown in (a)-(c) and (d)-(f), respectively. Note, in frames (d)-(f) addition of numerical diffusion attenuates spurious packet B which is present in frames (a)-(c) when no diffusion has been added.

We have considered a domain $0 \leq x \leq 20$ with *101* equi-spaced grid points. Phase speed *c* is kept as *0.10*. Computations are performed using the coupled compact difference scheme for the spatial discretization and RK4 scheme for the time integration by keeping CFL number as *0.01*. Corresponding V_{gN}/c contours show group velocity for the packet A as 1.0 while that for packet B as -2.63. Figures 4(a) –(c) show the solution of Eq. (8) subjected to the initial condition in Fig. 3(a). Due to positive group velocity packet A propagates towards right hand side while packet B displays spurious nature and propagates in completely opposite direction towards left. In order to attenuate such spurious waves and prevent numerical instabilities, one can add numerical diffusion as shown in Eq. (1). Figures 4(d)-4(f) show propagation of wave packet when numerical diffusion has been added with a diffusion coefficient as 0.01. Due to addition of numerical diffusion, spurious packet B gets attenuated completely while packet A retain itself and travel towards correct direction with correct velocity. This shows the advantage of coupled contact difference scheme.

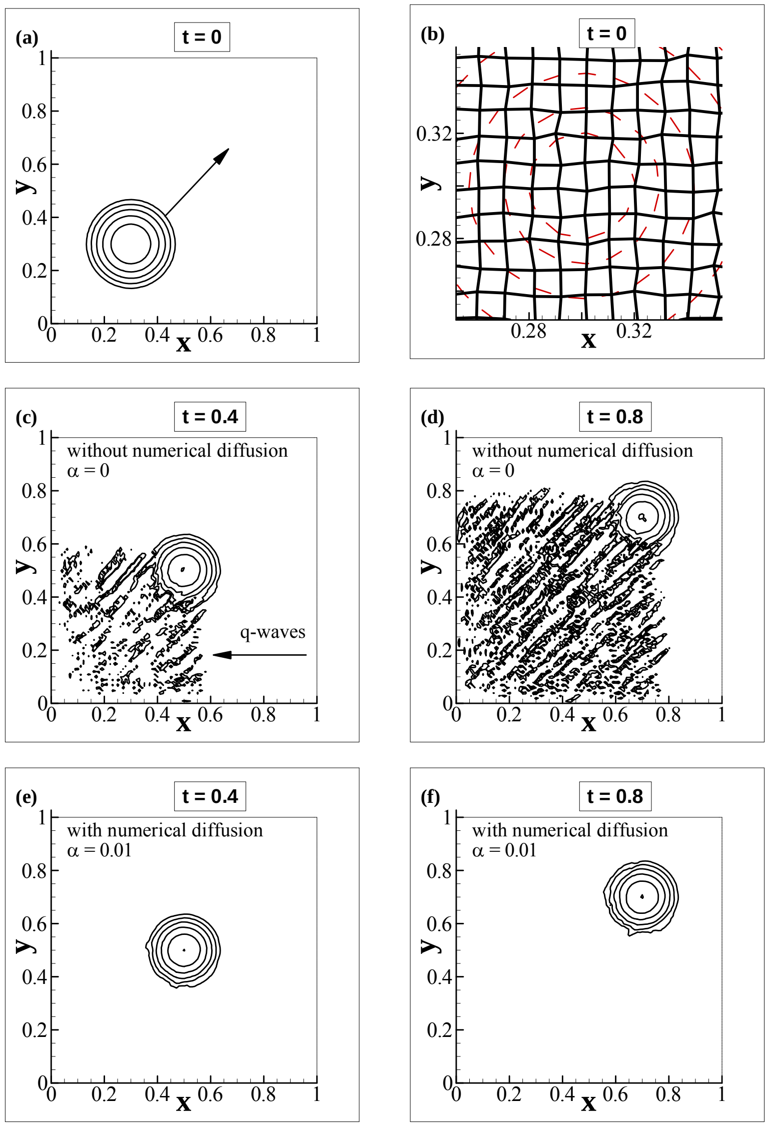

Figure 5: Initial condition of a 2D wave packet and zoomed view of a grid are shown in (a) and (b), respectively. Solutions of 2D wave equation Eq. (9), without and with addition of numerical diffusion are shown in (c)-(d) and (e)-(f), respectively. Note, in frames (e)-(f) addition of numerical diffusion attenuates spurious packets present in frames (c)-(d) when no diffusion has been added.

2. Propagation of a wave packet on a discontinuous grid

Next, we consider a propagation of a wave packet inside a *2D* domain following the *2D* wave equation given as,

$$\frac{\partial u}{\partial t} + c_x \frac{\partial u}{\partial x} + c_y \frac{\partial u}{\partial y} = 0 \tag{9}$$

In Eq. (9), c_x and c_y are phase speeds in x- and y-direction. In order to check the performance and applicability of the present scheme, we have purposefully considered wave propagation on a discontinuous grid as the discontinuous distribution of grid points triggers spurious high wavenumber oscillations. Figures 5 (a) and (b) show the initial condition of a *2D* wave packet and zoomed view of a grid, respectively. We have obtained solutions of *2D* wave equation following Eq. (9) using coupled compact difference scheme for the spatial discretization terms and RK4 scheme for time integration. We have constructed a domain $0 < x, y < 1$, with *101* points in either direction. We have purposefully assigned a random distribution to grid point spacing so as to test the efficacy of the coupled compact difference scheme. Figures 5(c) and (d) show propagation of *2D* wave packet at the indicated instants. Due to discontinuous distribution of grid points one observes large amount of spurious q-waves in the domain. However, when a fourth order numerical diffusion has been added to the solution, spurious waves are attenuated. This shows the advantage of proposed coupled compact difference scheme while working on a discontinuous grid.

3. *Propagation of the acoustic and the entropic disturbances*.

Next, we solve the computational acoustic wave propagation problem which consists of simultaneous propagation of acoustic and entropic disturbances. These disturbances propagate following the linearized compressible Navier-Stokes equations given by [Tam et al. (1995)],

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0 \tag{10}$$

Where,

$$U = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}, \quad E = \begin{bmatrix} M_x \rho + u \\ M_x u + p \\ M_x v \\ M_x p + u \end{bmatrix}, \; F = \begin{bmatrix} M_y \rho + u \\ M_y u \\ M_y v + p \\ M_y p + v \end{bmatrix} \text{ with } M_x = 0.5,\; M_y = 0$$

Initial condition for this problem is given as,

$$p = \exp\left[-(\ln 2)\left(\frac{x^2+y^2}{9}\right)\right];\; \rho = \exp\left[-(\ln 2)\left(\frac{x^2+y^2}{9}\right)\right] + 0.1\exp\left[-(\ln 2)\left(\frac{(x-67)^2+y^2}{25}\right)\right]$$

$$u = 0.04y\exp\left[-(\ln 2)\left(\frac{(x-67)^2+y^2}{25}\right)\right]$$

$$v = -0.04(x-67)\exp\left[-(\ln 2)\left(\frac{(x-67)^2+y^2}{25}\right)\right] \tag{11}$$

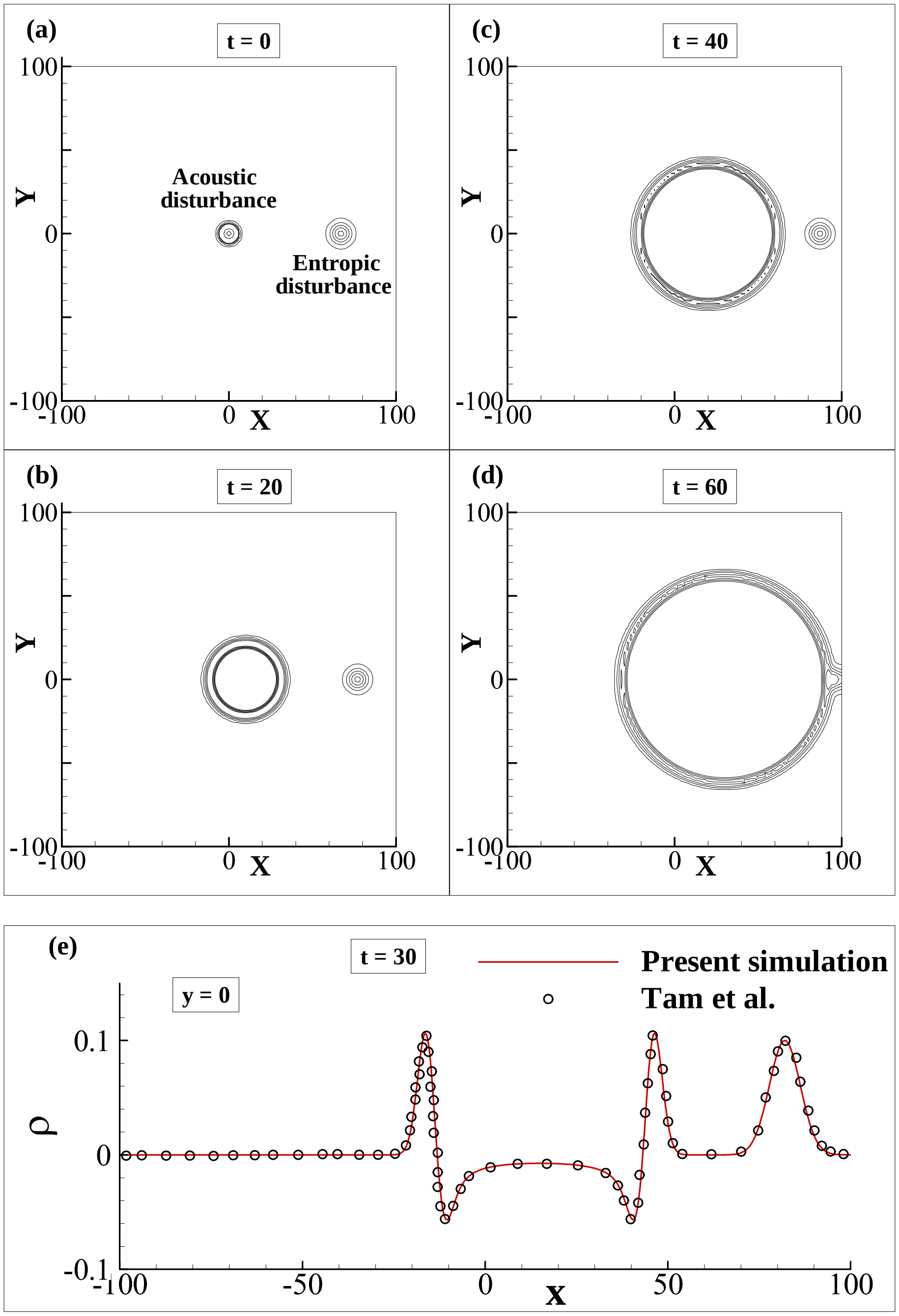

Figure 6: Initial condition and propagation of acoustic and the entropic disturbances following Eq. (10) are shown in frames (a)-(d). Comparison of density variation on the line y=0 obtained from present simulation with that of [Tam et al. (1995)] is shown in (e).

This case consists of an acoustic pulse generated by a Gaussian pressure distribution at the center of the computational domain as shown in the initial condition in Fig 6. The mean flow Mach number is 0.5. We have constructed the domain using 501 X 501 grid points. Downstream of the pressure pulse, at x = 0.67 an entropy pulse has also been superimposed. Acoustic pulse travels faster than entropy disturbances in the downstream direction as observed in Figs. 6(b) to 6(d) which show development and propagation of acoustic as well as entropic disturbances with time. We have compared the density variation on the line y=0 obtained from present simulation with that of [Tam et al. (1995)] in Fig. 6(e). Comparison shows a good match and justifies use coupled compact difference scheme for obtaining high accuracy solutions of computational acoustics problems.

Conclusions

Here, we have proposed a new coupled compact difference scheme to solve computational acoustics problems. Proposed scheme has significant resolution and physical dispersion relation preserving ability. In addition, one can add controlled amount of numerical diffusion to attenuate spurious waves in the solution. Solution of model wave propagation problems and computational acoustic problem highlights the advantages of the proposed coupled compact difference scheme.

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