Development of a Discontinuous Galerkin Method for Supersonic Flow Simulations on Hybrid Mesh

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Abstract

A two dimensional discontinuous Galerkin finite element method for supersonic flow field simulation on hybrid meshes is proposed. The governing equations are Euler equations, and the 3rd order explicit Runge–Kutta method is used for temporal discretization. The Hermit WENO limiter is introduced to increase the stability of this scheme when it is applied to supersonic flow fields. Two dimensional hybrid unstructured meshes are used in spatial discretizations, which contain both triangle elements and quadrangle elements. This method is validated with supersonic test problems, the results show that this method can solve supersonic flow fields using hybrid unstructured meshes.

Keywords: Supersonic, Euler Equations, Discontinuous Galerkin, Hybrid Mesh

Introduction

Supersonic flow field simulation plays an important role in flight aerodynamic predictions and space craft designs. There are many computational fluid dynamic methods for solving supersonic flow problems, these methods can be put into three categories: finite difference method (FDM), finite volume method (FVM) and finite element method (FEM). The FDM is suitable for constructing high order numerical schemes, and widely used in academic researching, but this method is mainly developed under Cartesian grids, it is difficult to extend this method to unstructured or hybrid meshes, which are common when dealing with real world complex geometries. The FVM, on the other hand, has no limitation on mesh types or geometry complexities, but it is difficult for FVM to achieve a scheme higher than second order on an unstructured mesh, mainly due to the difficulties to implement a compact reconstruction stencil for high order FVM.

The discontinuous Galerkin method (DGM) is a special kind of FEM, this method was first proposed by Reed and Hill [Reed and Hill (1973)], in the 1990s, Cockburn and Shu proposed the Runge-Kutta Discontinuous Galerkin method [Cockburn and Shu (1998)]. After that, the DGM is widely used in many areas, such as aerodynamics, hydrodynamics, wave propagations and computational acoustics. The DGM has both the advantages of FDM and FVM, it is suitable for constructing high order numerical schemes by using high order basis functions, the computational mesh and element shape has no limitations. The stencil in DGM is compact with any order of basis functions, that means to get the solutions of unknown variables in one element, only the unknown variables in its neighbor elements are needed. All these characteristics of the DGM make it a promising method for solving real world engineering flow problems.

The present authors have developed a two dimensional discontinuous Galerkin method for compressible Euler equations on unstructured and hybrid meshes. In order to suppress the non-physical oscillation, a Hermit WENO limiter [Hong et al. (2007)] is introduced. The numerical tests show that this scheme provides an attractive way for solving supersonic flow problems with complex geometries.

Governing equations

The governing equations are two dimensional inviscid Euler equations, which can be expressed in the form as:

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=0 \tag{1}$$

The variable U is conservative state vector, F and G are the inviscid flux vectors:

$$U=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho E\end{bmatrix};\ F=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ (\rho E+p)u\end{bmatrix};\ G=\begin{bmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ (\rho E+p)v\end{bmatrix} \tag{2}$$

The equation of state for perfect gas is used:

$$p=\rho RT \tag{3}$$

These form the complete set of equations, ready to be solved with proper numerical methods.

Discontinuous Galerkin method

Spatial discretization

Assuming that the computational domain is divided into a set of non-overlapping elements K_j, the governing equations are solved in a weak form, we introduce test functions v, multiply test functions with the governing equations, then integrate over element K_j, we have:

$$\int_{K_j}\frac{\partial U}{\partial t}v\mathrm{d}V+\int_{K_j}\left(\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}\right)v\mathrm{d}V=0 \tag{4}$$

After integrating by parts, we obtain the final form:

$$\int_{K_j}\frac{\partial U}{\partial t}v\mathrm{d}V+\oint_{\partial K_j}v\hat{f}\bullet n\mathrm{d}\Omega-\int_{K_j}\nabla v\bullet f\mathrm{d}V=0 \tag{5}$$

Where the $\hat{f}$ is numerical flux between the two adjacent elements, any numerical flux calculation schemes can be used, here the Van Leer scheme [Toro (2009)] is chosen to calculate $\hat{f}$.

The approximate solution is defined in each element as a polynomial:

$$u_h=\sum_{i=1}^{N}u_i\phi_i(x,t) \tag{6}$$

Where $\phi_i(x,t)$ is the basis function, u_i is the solution coefficient, the order of discontinuous Galerkin scheme is defined as the maximum order of the basis functions.

Time discretization

Replace the solution vectors and test functions in Eq.(5) by their approximation polynomials, a system of ordinary differential equations is obtained:

$$M\frac{dU}{dt} = R(U) \tag{7}$$

$R(U)$ is the residual vector, this ODE system can be solved step forward in time using explicit Runge-Kutta scheme, a third order scheme is used:

$$\begin{aligned}
U^{(1)} &= U^n + \Delta t M^{-1} R(U^n), \\
U^{(2)} &= \frac{3}{4}U^n + \frac{1}{4}\left[U^{(1)} + \Delta t M^{-1} R(U^{(1)})\right] \\
U^{(3)} &= \frac{1}{3}U^n + \frac{2}{3}\left[U^{(2)} + \Delta t M^{-1} R(U^{(2)})\right] \\
U^{n+1} &= U^{(3)}
\end{aligned} \tag{8}$$

Where $U^n = U(t)$ and $U^{n+1} = U(t+\Delta t)$.

Slope limiter

When there are strong discontinuous in the flow field, the discontinuous Galerkin solving procedure may fail due to the severe oscillations near strong discontinuity regions, these oscillations will cause non-physical solutions such as negative pressure or negative density. When shock waves exist in flow fields, certain amount of numerical dissipation is crucial for the successful solving. The DGM with piecewise constant basis functions could offer enough dissipations by itself, but when the order of basis functions is equal or greater than unity, some additional dissipations is often needed, limiters are most commonly used tools to do this.

There are many kinds of limiters proposed by researchers, the Hermit WENO limiter proposed by Luo and Shu is becoming popular among them. The Hermit WENO limiter is based on the idea of Hermit polynomial reconstruction and WENO reconstruction. The major advantage of this limiter is the compactness of its stencils, this makes it suitable for hybrid unstructured mesh, detailed implementations of Hermit WENO limiter on unstructured mesh can be found in [Hong et al. (2007; 2010)].

Numerical tests

The following numerical tests is obtained by the discontinuous Galerkin method using piecewise linear basis functions in two dimensional space.

Riemann problem

The two dimensional Riemann problem is designed to test the performance of numerical methods when there are shock waves and contact discontinuities in supersonic flow fields. The computational domain is [0,1]×[0,1], the domain is divided into four parts, they are (a): [0,0.5]×[0,0.5], (b): [0.5,1]×[0,0.5], (c): [0,0.5]×[0.5,1] and (d): [0.5,1]×[0.5,1], each part has its own initial condition, the initial conditions are:

$$\text{(a)}: \begin{pmatrix}\rho\\u\\v\\p\end{pmatrix}=\begin{pmatrix}0.8\\0.1\\0.1\\1\end{pmatrix};\ \text{(b)}: \begin{pmatrix}\rho\\u\\v\\p\end{pmatrix}=\begin{pmatrix}1\\0.1\\0.8276\\1\end{pmatrix},\ \text{(c)}: \begin{pmatrix}\rho\\u\\v\\p\end{pmatrix}=\begin{pmatrix}1.0222\\-0.6179\\0.1\\1\end{pmatrix},\ \text{(d)}: \begin{pmatrix}\rho\\u\\v\\p\end{pmatrix}=\begin{pmatrix}0.5313\\0.1\\0.1\\0.4\end{pmatrix}$$

Two kinds of Meshes are used in this numerical test case, Fig.1 shows the elements distributions of these meshes, the first mesh is a Cartesian grid which contains only uniform distributed quadrangle elements, the second mesh is a hybrid mesh with two blocks of Cartesian grids and two blocks of unstructured grids, this mesh contains both quadrangle and triangle elements. The solutions of the flow field are calculated until the time t=0.23.

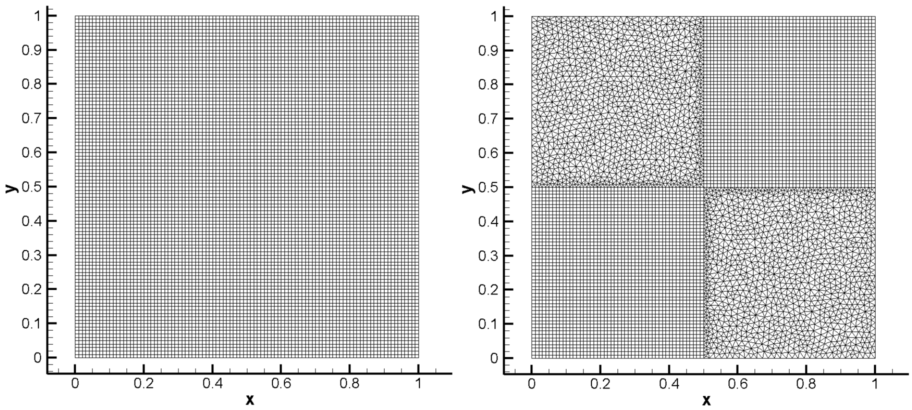

Figure 1. Uniform cartesian grid (left) and hybrid mesh (right) used in the numerical simulation

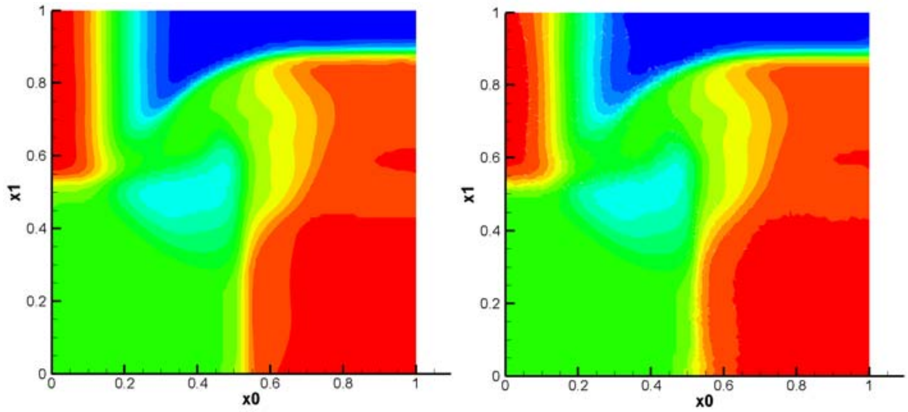

Figure 2. Comparison of density distributions at t=0.23 obtained by cartesian grid (left) and hybrid mesh (right)

Fig. 2 shows the flow field density distribution when t=0.23, the Cartesian grid and hybrid mesh give similar density profiles, which means that the discontinuous Galerkin method has the capable of solving Euler equations on unstructured hybrid meshes, this characteristic of DGM gives a lot flexibilities in modeling complex flow geometries.

Supersonic cylinder

Supersonic cylinder flow is a common test case in supersonic flow field simulations, the flow field contains a strong bow shock wave in front of the cylinder, the radius of cylinder is chosen as 0.01, inflow Mach number M=3, and the non-dimensional inflow parameters are chosen as : $\rho = 1$, $u = 1$, $v = 0$, $p = 1/\left(\gamma M^2\right)$, Fig.3 shows the computational mesh, which contains both quadrangle elements and triangle elements. There is a quadrangle element block near the cylinder surface, and a quadrangle element block designed to capture the bow shock wave, between these two quadrangle element blocks are triangle elements.

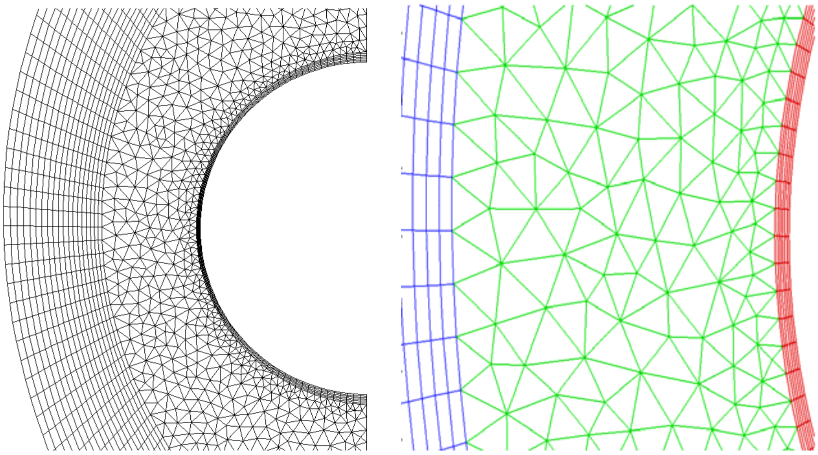

Figure 3. Computational mesh (left) and its local details (right)

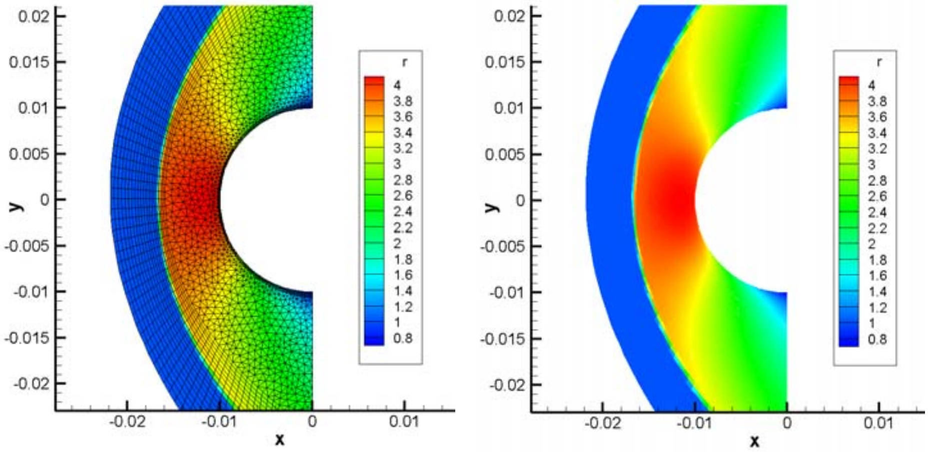

Figure 4. Solution of density distribution compared with mesh (left) and the density contour (right)

Fig. 4 shows the solution of density distribution using this hybrid mesh, the shock wave is distributed in the quadrangle element region and has a sharp resolution with these quadrangle elements, The density profile in the flow field between shock wave and cylinder gets a smooth distribution with triangle elements. The result gives a demonstration of the advantage to use discontinuous Galerkin method combined with hybrid meshes in supersonic flow simulations, the parallel distributed quadrangle elements are suitable for capturing shock waves, mean while for smooth flow regions, the usage of triangle elements can offer more geometrical flexibilities.

Conclusions

A two dimensional discontinuous Galerkin finite element method for supersonic flow field simulations is proposed. The governing equations are Euler equations, and the computational mesh is unstructured hybrid mesh. Numerical tests show that the discontinuous Galerkin mehod provides an effective way of solving engineering supersonic problems on hybrid meshes.

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