

Viscoelastic Response of Composite Laminated Shells based on Efficient Higher-Order Theory

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Abstract

The Laplace transformation has been applied for analyzing viscoelastic behaviors of composite laminated shells. The constitutive equation in integral can be reduced to linear stress-strain relationship in Laplace domain. Therefore, the formulation such as displacement field, can be expressed similarly with elastic one. In order to describe accurately the viscoelastic behaviors of laminated shells, the efficient higher-order shell theory (EHOST) has employed. The general tensor formulation is used for accurate description and arbitrary shell. The time-dependent results of strain, stress distributions through the thickness of the shell are obtained by converting back to real time domain for creep and relaxation process.

Keywords: Laplace transform, viscoelastic, composite laminated shell, efficient higher-order shell theory

Introduction

Due to the time-dependent effects of matrix, viscoelastic behavior is one of important factors in the analysis of composite structures. These responses significantly depend on the applied loading and environmental conditions such as temperature or moisture. Nonetheless, the Boltzmann's superposition principle in integral form of viscoelastic analysis is much more complicate than Hook's law of elastic one [Yi and Hilton (1995); Venkat et al. (2012)]. Consequence, the computation costs extremely expensive for thick laminates. It needs to find a method which satisfies both efficiency and accuracy.

Some researcher have employed the Laplace or Fourier transforms to avoid above limitation [Lin and Hwang (1989); Chen (1995)]. The converted constitutive equation of viscoelastic materials in Laplace domain is very similar to the Hook's laws of elastic one. However, all of formulation procedures are done in Laplace domain. Finally, the results in real time domain are obtained by inversion Laplace transforms [Hassanzadeh and Pooladi-Darvish (2007); Dubner and Abate (1968)]. Hence, these methods are effective for long-term problems of viscoelastic analysis.

On the other hand, plates and shells more and more are popular structures, especially in aerospace engineering such as wings and fuselages. Because of potential application, various plate and shell theories have been developed to improve the accuracy as well as efficiency [Reddy (2004); Pagano (1969); Cho and Parmerter (1993; 1994); Kim and Cho (2003); Nguyen et al. (2015)]. To express adequately viscoelastic behaviors and the advance of present method, EHOST is good choice which can predict accurately and efficiently strain stress distribution through the thickness of composite laminated shell [Kim and Cho (2003)].

The present study aims to develop EHOST for viscoelastic composite laminates. Besides, the Laplace transform is employed to reduce the mortal formulation and save an expanse computation amount. Thus, this study provides a powerful tool for investigating the time-dependent response of composite laminated shell with the advantage of accuracy and efficiency.

Mathematical Formulation

1. Constitutive equation for viscoelastic material and Laplace transform

The constitutive equation for a linear viscoelastic materials can be expressed, as follows:

$$\sigma_i(t) = \int_0^t Q_{ij}(t-t') \frac{\partial \varepsilon_j(t')}{\partial t'} dt \quad (1)$$

where t denotes time, t' is a dummy variable for integration, $Q_{ijkl}(t)$ is relaxation modulus which can be well represented by a series of decaying exponentials, as Prony series.

By applying Laplace transform, the above complex equation can be reduced as linear relationship in Laplace domain as follows:

$$\sigma_i^*(s) = s Q_{ij}^*(s) \varepsilon_j^*(s) \quad (2)$$

where $()^*$ denotes parameters in the Laplace domain. Therefore, the procedures of viscoelastic composite laminated shell analysis in Laplace domain is the same to elastic counterpart in time domain.

Finally, the viscoelastic responses in real time domain can be obtained by using inverse Laplace techniques. In present study, the Fourier series algorithm is employed to convert the strain, stress values back into the real time domain as follows:

$$f(t) = \frac{e^{at}}{t} \left\{ \frac{1}{2} F(a) + \text{Re} \sum_{k=1}^{\infty} F\left(a + j \frac{k\pi}{t}\right) (-1)^k \right\} \quad (3)$$

where $F(s)$ is a function in the Laplace domain that need to converted, $f(t)$ is the converted function in real time domain, and a and n are the parameters that must be optimized for accuracy.

2. Efficient higher-order shell theory for viscoelastic composite laminates

The composited shells are considered, consisting of n orthotropic layers with uniform thickness h in a curvilinear orthogonal coordinate system x^α as shown in Fig. 1. $(x^1, x^2, x^3=0)$ represents curvature of the mid-surface which is chosen as the reference surface. The position vector $\mathbf{R}$ can be expressed by position vector $\mathbf{r}$ pointing to reference surface and unit vector $\mathbf{a}^3$, which is perpendicular to the surface at the point x^α , as follows:

$$\bar{\mathbf{R}}(x^\alpha, x^3) = \bar{\mathbf{r}}(x^\alpha) + x^3 \bar{\mathbf{a}}_3(x^\alpha) \quad (4)$$

where x^3 is the distance from the reference surface to the material point. The covariant base vector is defined as:

$$\bar{\mathbf{g}}_i = \frac{\partial \bar{\mathbf{R}}}{\partial x^i} = \frac{\partial \bar{\mathbf{r}}}{\partial x^i} + \frac{\partial}{\partial x^i} (x^3 \bar{\mathbf{a}}_3) \quad (5)$$

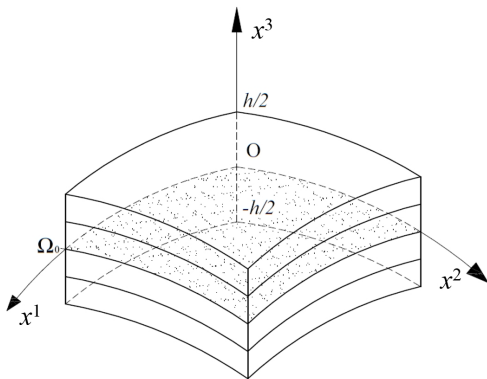


Fig. 1. Geometry and coordinates of the rectangular laminated shell.

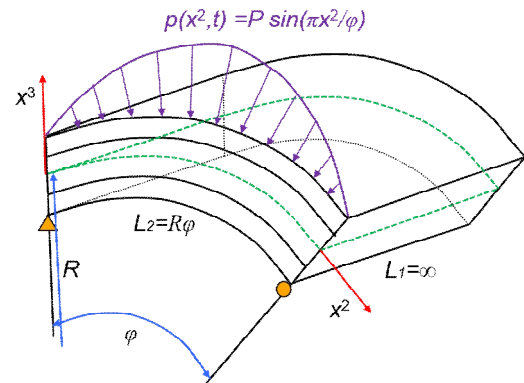


Fig. 2. Geometry and loading of cylindrical shell

where $\mathbf{g}^3 = \mathbf{a}^3$ and $\mathbf{a}_i = \mathbf{r}_{,i}$ are vector tangent to the surface coordinate curves. Hence, the displacement vector of the shell can be expressed as follows:

$$\begin{aligned} \vec{V}(x^i) &= V_\alpha \vec{g}^\alpha + V_3 \vec{g}^3 = U_\alpha \vec{a}^\alpha + U_3 \vec{a}^3 \\ V_\alpha &= \mu_\alpha^\beta U_\beta, \quad V_3 = U_3 \end{aligned} \quad (6)$$

where $\mathbf{a}_i, \mathbf{g}_i$ are covariant base vectors and $\mathbf{a}^i, \mathbf{g}^i$ are contravariant base vectors; μ_α^β denote the shifter tensor which is expressed as follows:

$$\vec{g}_\alpha = \mu_\alpha^\beta \vec{a}_\beta = (\delta_\alpha^\beta - x^3 b_\alpha^\beta) \vec{a}_\beta \quad (7)$$

The metric tensor $\mathbf{a}_{\alpha\beta}$ and surface tensor $\mathbf{b}_{\alpha\beta}$ are defined from first and second fundamental forms respectively, as follows:

$$\begin{aligned} ds^2 &= d\vec{r} \cdot d\vec{r} = a_{\alpha\beta} dx^\alpha dx^\beta \\ d\vec{r} \cdot d\vec{a}_3 &= -b_{\alpha\beta} dx^\alpha dx^\beta = -b_\beta^\alpha dx^\beta dx_\alpha \end{aligned} \quad (8)$$

The in-plane displacement fields of laminated shells are assumed as the one of the original EHOST, which superimposes linear zigzag displacements, with different slope in each layer, as follows:

$$\begin{cases} U_{\alpha(x^i, t)} = u_{\alpha(x^i, t)} + x^3 \psi_{\alpha(x^i, t)} + (x^3)^2 \xi_{\alpha(x^i, t)} + (x^3)^3 \phi_{\alpha(x^i, t)} + \sum_{k=1}^{n-1} S_\alpha^{(k)}(x^i, t) (x^3 - x_{(k)}^3) H(x^3 - x_{(k)}^3) \\ U_{3(x^i, t)} = w_{(x^i, t)} \end{cases} \quad (10)$$

where $\mathbf{u}_\alpha$ and $\mathbf{w}$ are displacements defined at the mid-surface, and $\mathbf{H}(x^3 - x_{(k)}^3)$ is the Heaviside unit step function. The requirement that the transverse stress should be vanished at the top and bottom surface, and continuous through the thickness of the shells are employed to reduce the number of unknown primary variables. Due to the advantage of Laplace transform, this work can be executed conveniently in Laplace domain, and the displacements can be obtained as follows:

$$\begin{cases} U_\alpha^*(x^i, s) = \mu_\alpha^\beta u_\beta^*(x^i, s) - x^3 u_{3||\alpha}^*(x^i, s) + f_\alpha^{\beta*}(x^3) \phi_\beta^*(x^i, s) \\ U_3^*(x^i, s) = w^*(x^i, s) \end{cases} \quad (11)$$

where

$$f_\alpha^{\beta*}(x^3) = \nu_\alpha^{\beta*} x^3 + \kappa_\alpha^{\beta*} (x^3)^2 + \delta_\alpha^\beta (x^3)^3 + \sum_{k=1}^{N-1} (\varpi^{(k)})_\alpha^{\beta*} (x^3 - x_{(k)}^3) H(x^3 - x_{(k)}^3) \quad (12)$$

where $\nu_\alpha^{\beta*}, \kappa_\alpha^{\beta*}$ and $(\varpi^{(k)})_\alpha^{\beta*}$ denote the matrices in Laplace domain which depend on the material properties and the thickness of each ply.

Then, the converted virtual work principle () in Laplace domain is employed and integrated by part to obtain the equilibrium (), as follows:

$$\int_v (\sigma^{\alpha\beta} \delta e_{\alpha\beta} + 2\sigma^{\alpha 3} \delta e_{\alpha 3}) - \int_\Omega p \delta u^3 d\Omega = 0 \quad (13)$$

$$\begin{aligned} \delta u_\omega : (N^{1*})^\omega - (N^{2*})^{\omega\gamma} - (M^{1*})^\omega + (M^{2*})^{\omega\nu} &= 0 \\ \delta \phi_\omega : (R^{1*})^\omega - (R^{2*})^{\omega\gamma} + (V^{1*})^{\omega 3} &= 0 \\ \delta u_3 : N^{3*} + (M^{3*})^\omega - (M^{4*})^{\omega\nu} &= p^* \end{aligned} \quad (14)$$

where the stress resultants will be shown in final version.

3. Cylindrical bending for cylindrical shell

For cylindrical bending problem for cylindrical shallow shell as shown in Fig. 2, the coordinates are specified as $(x^1, x^2, x^3) = (x, \theta, z)$, the curvature teams will be reduced as follows:

$$\mu_\alpha^\beta = \delta_\alpha^\beta, \quad g_{\alpha\beta} = a_{\alpha\beta} = \delta_{\alpha\beta}, \quad \mu = 1, \quad \bar{\Gamma}_{\alpha\beta}^v = 0, \quad b_{\alpha\beta} = b_\beta^\alpha = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{bmatrix} \quad (15)$$

The sinusoidal loading is applied on the upper surface of laminated shell, as follows:

$$p_z(x, \theta, t) = P_z(t) \sin\left(\frac{\pi\theta}{\varphi}\right) \quad (16)$$

The simply supported boundary conditions can be satisfied by the following forms of displacements:

$$u_2^0 = U(t) \cos\left(\frac{\pi\theta}{\varphi}\right), \quad \phi_2 = \Phi(t) \cos\left(\frac{\pi\theta}{\varphi}\right), \quad w = W(t) \sin\left(\frac{\pi\theta}{\varphi}\right) \quad (17)$$

where φ is chosen as $\pi/3$.

Substituting the form of displacement in Eq. (17) into Eq. (1) with the help of resultants-displacements relations, we can obtain the algebraic relations between the primary variables and the external force in the Laplace domain as follows:

$$\begin{bmatrix} K_{11}^* & K_{12}^* & K_{13}^* \\ K_{21}^* & K_{22}^* & K_{23}^* \\ K_{31}^* & K_{32}^* & K_{33}^* \end{bmatrix} \begin{Bmatrix} U^* \\ \Phi^* \\ W^* \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P^* \end{Bmatrix} \quad (18)$$

where K^* is the stiffness matrix in Laplace domain. The detail formulation of stiffness matrix will be given in the presentation. After solving the above equation, the primary variables as well as the deflection, in-plane displacements, stress distributions in real time domain can be obtained by applying the numerical inverse Laplace transform $\mathcal{L}^{-1}$.

Results and discussion

To compare with previous studies, the laminated shell (90/0/90), assumed to be made of Graphite/Epoxy GY70/339 at reference condition ($T=75^\circ\text{C}$, $M=0.1\%$), is chosen as numerical example with elastic properties are shown as follows:

$$\begin{aligned} E_L &= 2.89 \times 10^5 \text{ MPa} & E_T &= 6.063 \times 10^3 \text{ MPa} \\ G_{LT} &= 4.134 \times 10^3 \text{ MPa} & G_{TT} &= 2.067 \times 10^3 \text{ MPa} \\ \nu_{LT} &= \nu_{TT} = 0.31 \end{aligned} \quad (19)$$

where L denotes a fiber direction and T denotes a perpendicular direction to the fiber. The time-dependent function $\varphi(t)$ for GY70/339 determined by mastering curve Crossman's experimental data is omitted because of limited space [14]. In present study, for the static creep process, the applied force is kept as constant $p(t)=1.0$. For the relaxation process, after the force $p|_{t=0}=1.0$ is applied, the deflection will be kept as constant as the time go by. In this version, the in-plane displacement distribution U_2 is represented; the other stress-strain distributions through the thickness of the shell will be shown in the presentation.

The Fig. 3 shows the distribution of in-plane displacement U_2 through the thickness for creep process. The viscoelastic solution at initial time is the same to elastic solution. Then, it changes as the time goes by, but still have good agreement with elastic one. Due to the decay of the stiffness of composite matrix, the amplitude of U_2 increases respect to time. The increment at upper and lower

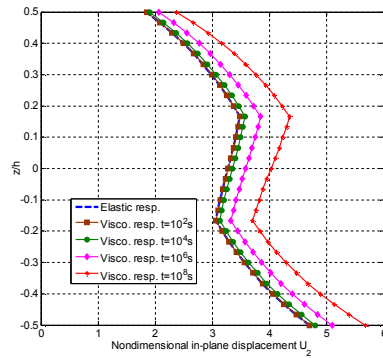


Fig. 3. The in-plane displacement U_2 distribution through the thickness of creep process

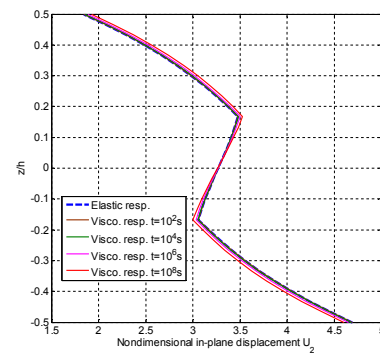


Fig. 4. The in-plane displacement U_2 distribution through the thickness of relaxation process

is smaller because of dominant elastic fiber. These responses make the structures in risks with larger deformation.

The Fig. 4 shows the distribution of in-plane displacement U_2 through the thickness for relaxation process. The viscoelastic solution at initial time is the same as elastic solution. Then, similar with 1-D viscoelastic response of relaxation, the amplitude of U_2 changes insignificantly. The amplitude lightly decreases at top and bottom of shell, and lightly increases at interface surface.

Conclusion

The viscoelastic response of laminated composite plate of EHOST has been analyzed with the help of Laplace transform without any integral transformation or any time step scheme. The accuracy and efficiency of the analysis are retained since the viscoelastic EHOST formulation was made in the linear elastic regime in the Laplace transformed domain. The numerical results for composite laminated shell, which are converted by employing inversion Laplace transform, adequately show the change of time-dependent mechanical behaviors for creep process as well as relaxation process.

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