Higher-order two-point efficient family of Halley type methods for simple roots

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Abstract

In this paper, we proposed a new highly efficient two-point sixth-order family of Halley type methods that do not require any second-order derivative evaluation for obtaining simple roots of nonlinear equations, numerically. In terms of computational cost, each member of the family requires two function and two first-order derivative evaluations per iteration. On the account of the results obtained, it is found that our proposed methods are efficient and show better performance than existing sixth-order methods available in the literature. Further, it is also noted that larger basins of attraction belong to our methods although the others methods are slow and has darker basins while some of the method are too sensitive upon the choice of the initial value.

Keywords: Nonlinear equations, Simple roots, Halley's method, Basins of attractions, Order of convergence

Introduction

Efficient solution techniques are required for finding simple roots of nonlinear equation of the form

$$f(x) = 0, \tag{1}$$

where $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a nonlinear sufficiently differentiable function in an interval D, which partake of scientific, engineering and various other models. One of the best known one-point optimal second-order method based on two functional evaluations is the classical Newton's method [Traub (1964); Petković et al. (2012)]. Many methods have been developed which improve the convergence rate of the Newton's method or Newton like at the expense of additional evaluations of functions or derivatives.

Halley's method [Traub (1964); Petković et al. (2012)] is the third-order modification of Newton's method, which is defined as follows:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{f(x_n)f''(x_n) - 2\{f'(x_n)\}^2}. \tag{2}$$

Despite the cubic convergence, this method is considered less practical from a computational point of view because of the costly second-order derivative evaluation. Therefore, researchers introduced multi-point methods and the primarily aim of these methods is to achieve as high as possible order of convergence using a fixed number of function evaluations. However, multi-point methods do not use higher order derivatives and has great practical importance since they overcome the theoretical limitations of one-point methods regarding their convergence order and computational efficiency.

Therefore, a number of sixth-order methods are also appearing as the extensions of Newton's method or Newton like method to solve nonlinear equation (1). In [Neta (1979)] given a three-point sixth-order general iteration scheme for obtaining simple roots of nonlinear equations, which is defined as follows:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \dfrac{1+\beta u}{1+(\beta-2)u}\dfrac{f(y_n)}{f'(x_n)} = y_n - G_f(u)\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{1-u}{1-3u}\dfrac{f(z_n)}{f'(x_n)}, \end{cases} \tag{3}$$

where $u = \frac{f(y_n)}{f(x_n)}$, $G_f(u) = \frac{u(1+\beta u)}{1+(\beta-u)}$, $\beta \in \mathbb{R}$.
In [Sharma and Ghua (2011)], proposed three-point family of sixth-order methods based on fourth-order Ostrowski's method [Kanwar et al. (2011)], which is given by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(x_n)}{f(x_n)-2f(y_n)}\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{f(x_n)+af(y_n)}{f(x_n)+(a-2)f(y_n)}\dfrac{f(z_n)}{f'(x_n)}, \quad a \in \mathbb{R}. \end{cases} \tag{4}$$

On the other hand, [Wang and Liu (2009)] have constructed two three-point sixth-order families of Jarratt's method [Petković et al. (2012); Behl et al. (2013)] requiring two of functions and two of first-order derivative evaluations per iteration, one of them is defined as follows:

$$\begin{cases} y_n = x_n - \dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \dfrac{9-5w}{10-6w}\dfrac{f(x_n)}{f'(y_n)}, \quad w = \dfrac{f'(y_n)}{f'(x_n)} \\ x_{n+1} = z_n - \dfrac{a+bw}{c+dw+rw^2}\dfrac{f(z_n)}{f'(x_n)}, \end{cases} \tag{5}$$

where $a = \frac{5c+3d+r}{2}$, $b = \frac{r-3c-d}{2}$, $c+d+r \neq 0$, $a,\ b,\ c,\ d,\ r \in \mathbb{R}$.

But, the body structures of above mentioned three-point sixth-order methods are more complicated as compared with two-point methods [Kanwar et al. (2011); Behl et al. (2013)]. Further, it is very rare to find two-point methods whose order of convergence higher than four [Guem et al. (2015)]. Nowadays, obtaining new two-point methods of order six not requiring the computation of a second-order derivative, is very important and interesting task from the practical point of view,

Therefore, the principle aim of this manuscript is to provide a new highly efficient two-point sixth-order class of Halley type methods, that do not require any second-order derivative evaluation for obtaining simple roots of nonlinear equations, numerically. It is also observed that the body structures of our proposed families of methods are simpler than the existing three-point families of sixth-order methods. Further, our proposed methods are more effective in all the tested examples to the existing three-point sixth-order methods available in the literature. Further, we also compare them with two-point sixth-order methods that is very recently proposed by [Guem et al. (2015)] and it is found that our methods our better than these methods. Furthermore, the dynamic study of our methods also supports the theoretical aspects.

Development of two-point sixth-order methods

In this section, we intend to develop many new families of sixth-order Halley type methods, not requiring the computation of second-order derivative. For this purpose, we consider $w_n = x_n - \frac{f(x_n)}{f'(x_n)}$, a Newton's iterate. With the help of Taylor series, we expand the function $f(w_n)$ about a point $x = x_n$ as follows:

$$f(w_n) \approx f(x_n) + f'(x_n)(w_n - x_n) + \frac{1}{2}f''(x_n)(w_n - x_n)^2, \quad \text{which further implies}$$

$$f''(x_n) \approx \frac{2\{f'(x_n)\}^2 f(w_n)}{\{f(x_n)\}^2}. \tag{6}$$

Similarly, expanding the function $f'(w_n) = f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)$ about a point $x = x_n$ by Taylor series expansion, we have $f'(w_n) \approx f'(x_n) + f''(x_n)(w_n - x_n)$, which further yields

$$f''(x_n) \approx \frac{f'(x_n)\Big(f'(x_n) - f'(w_n)\Big)}{f(x_n)}. \tag{7}$$

Now, taking the arithmetic mean of two equations (6) and (7), we get another approximation of $f''(x_n)$ as follows:

$$f''(x_n) \approx \frac{\frac{2\{f'(x_n)\}^2 f(w_n)}{\{f(x_n)\}^2} + \frac{f'(x_n)\Big(f'(x_n) - f'(w_n)\Big)}{f(x_n)}}{2}. \tag{8}$$

Inserting this approximate value of $f''(x_n)$ in scheme (2), and using the weight function on the second step, we get

$$\begin{cases} w_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{4\{f(x_n)\}^2}{3f(x_n)f'(x_n) + f(x_n)f'(w_n) - 2f'(x_n)f(w_n)} M_f(h,\ k), \end{cases} \tag{9}$$

where the weighting function M_f is a sufficient differential function with $h = \frac{f'(x_n)\Big(f(x_n)+f(w_n)\Big)}{f(x_n)f'(w_n)}$ and $k = \frac{f(w_n)}{f(x_n)}$. Theorem 1 indicates that under what choices on the weight function (9), the order of convergence will reach at six without using any more functional evaluations.

Convergence analysis

Theorem 1 *Let a sufficiently smooth function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has a simple zero ξ in the open interval D. Assume that initial guess $x = x_0$ is sufficiently close to ξ. Then, the iterative scheme defined by (9) has sixth-order convergence when*

$$\begin{aligned}&M_{00} = 1,\ M_{01} = \frac{3}{4},\ M_{10} = -\frac{1}{4},\ M_{20} = \frac{1}{2},\ M_{11} = \frac{1}{2},\ M_{02} = -\frac{5}{2},\ M_{03} = 9(6M_{30} + 3M_{21} + 5),\\ &M_{12} = -3(3M_{30} + 2M_{21} + 3),\ M_{04} = -3\{4M_{13} + 9(2M_{22} + 4M_{31} + 3M_{40} - 8)\},\end{aligned} \tag{10}$$

where $M_{ij} = \frac{\partial^{i+j}}{\partial h^i \partial k^j} M_f(h,\ k)|_{(h=1,\ k=0)}$. It satisfies the following error equation

$$\begin{aligned}e_{n+1} = -\frac{c_2}{12}\Big[&12(M_{13} + 18M_{21} + 9M_{22} + 54M_{30} + 27M_{31} + 27M_{40} - 10)c_2^4 - \{2M_{13} + 3(24M_{21}\\ &+ 6M_{22} + 72M_{30} + 18M_{31} + 18M_{40} - 5)\}c_2^2c_3 + 6(4 + M_{21} + 3M_{30})c_3^2 - 12c_2c_4\Big]e_n^6 + O(e_n^7).\end{aligned} \tag{11}$$

Proof Let ξ be a simple zero of $f(x)$. With the help of Taylor's series, we get the following expansion of $f(x_n)$ and $f'(x_n)$ around $x = \xi$

$$f(x_n) = f'(\xi)\Big(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + e_n^5c_5 + e_n^6c_6 + O(e_n^7)\Big), \tag{12}$$

and

$$f'(x_n) = f'(\xi)\Big(1 + 2e_nc_2 + 3e_n^2c_3 + 4e_n^3c_4 + 5e_n^4c_5 + 6e_n^5c_6 + 7e_n^6c_7 + O(e_n^7)\Big), \tag{13}$$

respectively. By using equations (12)–(13), we get

$$\begin{aligned}f(w_n) =&f'(\xi)\Big(c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 - 2(6c_2^4 - 12c_2^2c_3 + 3c_3^2 + 5c_2c_4\\ &- 2c_5)e_n^5 + (28c_2^5 - 73c_2^3c_3 + 34c_2^2c_4 - 17c_3c_4 + c_2(37c_3^2 - 13c_5) + 5c_6)e_n^6 + O(e_n^7)\Big).\end{aligned} \tag{14}$$

and

$$\begin{aligned}f'(w_n) =&f'(\xi)\Big(1 + 2c_2^2e_n^2 - 4(c_2^3 - c_2c_3)e_n^3 + c_2(8c_2^3 - 11c_2c_3 + 6c_4)e_n^4 - 4c_2(4c_2^4 - 7c_2^2c_3\\ &+ 5c_2c_4 - 2c_5)e_n^5 + 2(16c_2^6 - 34c_2^4c_3 + 6c_3^3 + 30c_2^3c_4 - 13c_2^2c_5 - 8c_2c_3c_4 + 5c_2c_6)e_n^6\\ &+ O(e_n^7)\Big).\end{aligned} \tag{15}$$

By using equations (12)–(15), we obtain

$$\begin{aligned}h = \frac{f'(x_n)(f(x_n) + f(w_n))}{f(x_n)f'(w_n)} =& 1 + 3c_2e_n + (5c_3 - 3c_2^2)e_n^2 - 7(c_2c_3 - c_4)e_n^3 + (6c_2^4 - 3c_2^2c_3\\ &- 2c_3^2 - 10c_2c_4 + 9c_5)e_n^4 + \{31c_2^3c_3 - 12c_2^5 - c_2^2c_4 - 5c_3c_4\\ &- c_2(11c_3^2 + 13c_5) + 11c_6\}e_n^5 + O(e_n^6).\end{aligned} \tag{16}$$

and

$$\begin{aligned}k = \frac{f(w_n)}{f(x_n)} =& c_2e_n + (-3c_2^2 + 2c_3)e_n^2 + (8c_2^3 - 10c_2c_3 + 3c_4)e_n^3 + (-20c_2^4 + 37c_2^2c_3 - 8c_3^2 - 14c_2c_4\\ &+ 4c_5)e_n^4 + \left(48c_2^5 - 118c_2^3c_3 + 51c_2^2c_4 - 22c_3c_4 + c_2(55c_3^2 - 18c_5) + 5c_6\right)e_n^5 + O(e_n^6).\end{aligned} \tag{17}$$

Since it is clear from equations (16) – (17), $h = 1 + u$ and $k = O(e_n)$. Then, from these equations, we get the remainder $u = h - 1$ and k are infinitesimal with the same order of e_n. Therefore, we can expand weight function $M(h,\ k)$ in the neighborhood of $(1,\ 0)$ by Taylor series expansion up to fourth-order terms as follows:

$$\begin{aligned} M_f(h,\ k) = M_{00} + M_{10}u + M_{01}k + \frac{M_{20}u^2 + 2M_{11}uk + M_{02}k^2}{2!} + \frac{1}{3!}(M_{30}u^3 + 3M_{21}u^2k \\ + 3M_{12}uk^2 + M_{03}k^3) + \frac{M_{40}u^4 + 4M_{31}u^3k + 6M_{22}u^2k^2 + 4M_{13}uk^3 + M_{04}k^4}{4!} \\ + O(e_n^5). \end{aligned} \tag{18}$$

Using equations (12) – (18), in scheme (9), we obtain

$$e_{n+1} = (1 - M_{00})e_n - c_2(M_{01} + 3M_{10})e_n^2 + \sum_{l=3}^{6} H_l e_n^l, \tag{19}$$

where $H_l = H_l(c_2, c_3, \ldots, c_6)M_{ij}$, for $0 \le i,\ j \le 4$.
From the equation (19), it is clear that by substituting the following values

$$M_{00} = 1, \quad M_{01} = -3M_{10}, \tag{20}$$

we get at least third-order convergence. Further, using (20) into $H_3 = 0$, we find two independent relation as follows:

$$(1 + 4M_{10}) = 0, \quad (M_{02} + 12M_{10} + 6M_{11} + 9M_{20} - 2) = 0 \tag{21}$$

After some simplification, we get

$$M_{10} = -\frac{1}{4}, \quad M_{02} = (5 - 6M_{11} - 9M_{20}). \tag{22}$$

By substituting equations (20) and (22) into $H_4 = 0$, we have

$$(M_{11} + 3M_{20} - 2) = 0, \quad \Big(M_{03} + 9(4M_{11} + M_{12} + 12M_{20} + 3M_{21} + 3M_{30} - 4)\Big) = 0. \tag{23}$$

Solving the above equation (23) for M_{11} and M_{03}, which further yields

$$M_{20} = -\frac{1}{3}(M_{11} - 2), \quad M_{03} = -9(4 + M_{12} + 3M_{21} + 3M_{30}). \tag{24}$$

By substituting equations (20), (22) and (24) into $H_5 = 0$, we obtain

$$\begin{cases} (2M_{11} - 1) = 0, \\ (4M_{11} - M_{12} - 6M_{21} - 9M_{30} - 11) = 0, \\ \big[M_{04} + 3(168 - 48M_{11} + 24M_{12} + 4M_{13} + 144M_{21} + 18M_{22} + 216M_{30} + 36M_{31} + 27M_{40})\big] = 0. \end{cases} \tag{25}$$

Solving the above equation for $M_{20},\ M_{12}$ and M_{04}, we get

$$\begin{cases} M_{11} = \dfrac{1}{2}, \\ M_{12} = -3(3 + 2M_{21} + 3M_{30}), \\ M_{04} = -3\{4M_{13} + 9(2M_{22} + 4M_{31} + 3M_{40} - 8)\}. \end{cases} \tag{26}$$

We obtain the following error equation, by using equations (20), (22), (24) and (26) into (19)

$$e_{n+1} = -\frac{c_2}{12}\Big[12(M_{13}+18M_{21}+9M_{22}+54M_{30}+27M_{31}+27M_{40}-10)c_2^4 - \{2M_{13}+3(24M_{21} + 6M_{22}+72M_{30}+18M_{31}+18M_{40}-5)\}c_2^2c_3 + 6(4+M_{21}+3M_{30})c_3^2 - 12c_2c_4\Big]e_n^6 + O(e_n^7). \tag{27}$$

This reveals that the modified family of Halley type methods (9) reaches the order of convergence six by using only four functional evaluations(viz $f(x_n)$ $f'(x_n)$ $f(w_n)$ and $f'(w_n)$) per full iteration. This completes the proof. □

Special cases

In this section, we discuss some interesting special case of weight function defined in (9) by inserting the values of free disposable parameters and different forms of weight functions $M_f(h,\ k)$.
(1) For $M_{40}=0,\ M_{31}=0,\ M_{13}=0,\ M_{22}=4$ and $M_{30}=0$, in (18), we get the following weight-function

$$M_f(h,\ k) = 1-\frac{u}{4}+\frac{u^2}{4}+\frac{3+2u+2M_{21}u^2}{4}k - \left(\frac{5}{4}+\frac{3}{2}(3+2M_{21})u-u^2\right)k^2+\frac{3}{2}(5+3M_{21})k^3, \tag{28}$$

where M_{21} is a free variable and for the sake of simplicity $u=h-1$. This is a new two-point sixth-order family of methods. For different specific values of M_{21}, we get various cases as well as two-point methods but some of the important cases describes in the following table 1.

Table 1: Sub cases of weight function (28) **and their error equations**

Particular values of M_{21}	Sub cases and their error equation
$M_{21}=0$	$M_f(h,\ k)=1-\frac{u}{4}+\frac{u^2}{4}+\frac{3+2u}{4}k+\left(u^2-\frac{5}{4}-\frac{9u}{2}\right)k^2+\frac{15k^3}{2}$, $\left(-26c_2^5+\frac{19}{4}c_2^3c_3-2c_2c_3^2+c_2^2c_4\right)e_n^6+O(e_n^7)$.
$M_{21}=-\frac{13}{9}$	$M_f(h,\ k)=1-\frac{u}{4}+\frac{u^2}{4}+\frac{27+18u-26u^2}{36}k+\left(u^2-\frac{5}{4}-\frac{u}{6}\right)k^2+k^3$, $\left(-\frac{47}{12}c_2^3c_3-\frac{23}{18}c_2c_3^2+c_2^2c_4\right)e_n^6+O(e_n^7)$.
$M_{21}=-\frac{5}{3}$	$M_f(h,\ k)=1-\frac{u}{4}+\frac{u^2}{4}+\frac{9+6u-10u^2}{12}k+\left(u^2-\frac{5}{4}+\frac{u}{2}\right)k^2$, $\left(4c_2^5-\frac{21}{4}c_2^3c_3-\frac{7}{6}c_2c_3^2+c_2^2c_4\right)e_n^6+O(e_n^7)$.

(2) For $M_{40}=0,\ M_{31}=0,\ M_{13}=0,\ M_{22}=4$ and $M_{30}=-\frac{3M_{21}+5}{6}$ in (18), we obtain

$$M_f(h,\ k) = 1-\frac{u}{4}+\frac{u^2}{4}-\frac{3M_{21}+5}{36}u^3+\frac{3+2u+2M_{21}u^2}{4}k+\frac{4u^2-5-3(1+M_{21})u}{4}k^2, \tag{29}$$

where M_{21} is a free variable. Therefore, some of the special cases given in the following table 2.

Table 2: Sub cases of weight function (29) **and their error equations**

Particular values of M_{21}	Sub cases and their error equation
$M_{21}=0$	$M_f(h,\ k)=1-\frac{u}{4}+\frac{u^2}{4}-\frac{5u^3}{36}+\frac{3+2u}{4}k+\left(-\frac{5}{4}-\frac{3u}{4}+u^2\right)k^2$,
	$\frac{1}{4}c_2\left(76c_2^4-41c_2^2c_3-3c_3^2+4c_2c_4\right)e_n^6+O(e_n^7)$.
$M_{21}=-\frac{19}{9}$	$M_f(h,\ k)=1-\frac{u}{4}+\frac{u^2}{4}+\frac{u^3}{27}+\frac{27+18u-38u^2}{36}k+\left(-\frac{5}{4}+\frac{5u}{6}+u^2\right)k^2$,
	$\left(-\frac{47}{12}c_2^3c_3-\frac{23}{18}c_2c_3^2+c_2^2c_4\right)e_n^6+O(e_n^7)$.
$M_{21}=-\frac{5}{3}$	$M_f(h,\ k)=1-\frac{u}{4}+\frac{u^2}{4}+\frac{9+6u-10u^2}{12}k+\left(-\frac{5}{4}+\frac{u}{2}+u^2\right)k^2$,
	$\left(4c_2^5-\frac{21}{4}c_2^3c_3-\frac{7}{6}c_2c_3^2+c_2^2c_4\right)e_n^6+O(e_n^7)$.

(3) We consider following weight function, that satisfies all the conditions which are mention in theorem 1

$$M_f(h,\ k)=\frac{1}{16}\left(5+\frac{27}{1+2h}+4k-20k^2-24k^3+h(2+8k)\right). \tag{30}$$

(4) We consider another weight function, which is given by

$$M_f(h,\ k)=\frac{8+k-5k^2-6k^3+h^2(1+4k)+h\left(3+4k-10k^2-12k^3\right)}{4+8h}. \tag{31}$$

(5) We consider one more weight function, which is defined as follows:

$$M_f(h,\ k)=\frac{6+16k-11k^2-12k^3+45k^4+h^2\left(1-9k^2\right)+h\left(-3-k+24k^2\right)}{4+12k}. \tag{32}$$

Numerical experiments

In this section, we apply new methods for ($M_{21}=-\frac{13}{9}$) in scheme (28), for ($M_{21}=-\frac{19}{9}$) in scheme (29), denoted by OM_6^1 and OM_6^2 respectively, to solve some nonlinear equations given in table 3, which serve to check the validity and efficiency of theoretical results. These methods are compared with method (5) for ($c=0,\ d=1,\ r=0$), proposed by [Wang and Liu (2009)], (called WM_6) and method (3) for ($\beta=2$), proposed by [Neta (1979)], denoted by (NM_6). Finally, we will also compare our schemes with a two-point family of sixth-order methods that is very recently proposed by [Guem et al. (2015)], between them we will choose their best expression (3.4, 3.8 and 3.12) denoted by (GM_6^1, GM_6^2 and GM_6^3), respectively. For better comparisons of our proposed methods, we have given three comparison tables in each example: one is corresponding to absolute error, the second one is with respect to number of iterations and third one is regarding their computational error in table 4, 5, 6, respectively. All computations have been performed using the programming package *Mathematica* 9 with multiple precision arithmetic. We use $\epsilon=10^{-34}$ as a tolerance error. The following stopping criteria are used for computer programs:
$(i)|x_{n+1}-x_n|<\epsilon$ and $(ii)|f(x_{n+1})|<\epsilon$.

Table 3: Test problems

$f(x)$	$Root(r)$
$f_1(x) = \tan^{-1}(x^2 - x)$	1.0000000000000000000000000000000000
$f_2(x) = x^3 - 30x + 5$	5.3919091867997792317129299268950973
$f_3(x) = x^3 + \sin x + 2x$	0.0000000000000000000000000000000000
$f_4(x) = \sin x - \tan x + 1$	1.0826495247186551155684838889482183
$f_5(x) = e^{-x^2+x+2} - 1$	2.0000000000000000000000000000000000

Table 4: (Comparison of different sixth-order methods with the same total number of functional evaluations (TNFE=12))

$f(x)$	x_0	WM_6	NM_6	GM_6^1	GM_6^2	GM_6^3	OM_6^1	OM_6^2
1.	0.85	1.7e−93	2.8e−110	3.9e−95	1.1e−95	1.7e−111	6.0e−140	2.2e−161
	1.6	3.2e−72	2.8e−78	1.7e−107	2.5e−63	8.3e−70	1.9e−153	5.3e−99
2.	4.5	7.1e−20	7.6e+5	9.4e−36	8.2e−35	1.7e−57	6.1e−68	3.3e−87
	6.5	2.9e−97	2.0e−145	1.3e−95	3.2e−95	3.4e−102	1.3e−130	1.1e−147
3.	−2.0	1.3e−4	2.7e−1	8.0e−60	1.6e−43	2.3e−49	2.4e−90	7.6e−75
	−1.9	1.3e−47	1.7e−4	3.5e−67	4.3e−48	8.3e−55	3.5e−125	2.7e−75
	1.9	1.3e−47	1.7e−4	3.5e−67	4.3e−48	8.3e−55	3.5e−125	2.7e−75
	2.0	1.3e−4	2.7e−1	8.0e−60	1.6e−43	2.3e−49	2.4e−90	7.6e−75
4.	0.86	1.8e−14	C	C	C	1.3e−21	1.5e−9	1.5e−16
	1.4	3.2e−26	3.3e−3	2.1e−25	9.0e−25	1.4e−29	1.5e−69	2.4e−38
5.	1.2	1.2e−47	5.0e−52	1.1e−50	9.5e−53	4.8e−57	3.5e−76	1.5e−98
	2.25	7.5e−24	D	1.2e−30	8.9e−29	2.6e−54	1.7e−58	6.9e−74

C: stands for converge to undesired root, D: stands for divergent.

Table 5: (Comparison of different sixth-order methods with respect to number of iterations)

$f(x)$	x_0	WM_6	NM_6	GM_6^1	GM_6^2	GM_6^3	OM_6^1	OM_6^2
1.	0.85	4	4	4	4	4	4	4
	1.6	4	4	4	4	4	4	4
2.	4.5	5	9	4	4	4	4	4
	6.5	4	4	4	4	4	4	4
3.	−2.0	4	6	4	4	4	4	4
	−1.9	4	6	4	4	4	4	4
	1.9	4	6	4	4	4	4	4
	2.0	4	6	4	4	4	4	4
4.	0.86	5	C	C	C	5	5	5
	1.4	5	6	5	5	5	4	4
5.	1.2	4	4	4	4	4	4	4
	2.25	5	D	5	5	4	4	4

Table 6: (Computational order of different sixth-order methods)

$f(x)$	x_0	WM_6	NM_6	GM_6^1	GM_6^2	GM_6^3	OM_6^1	OM_6^2
1.	0.85	5.999	6.000	5.999	5.999	6.000	6.000	6.000
	1.6	6.003	6.000	6.000	5.993	5.996	6.000	5.997
2.	4.5	6.000	6.000	5.956	5.952	5.994	6.010	6.001
	6.5	6.000	6.000	6.000	5.994	6.000	6.000	6.000
3.	-2.0	6.970	6.997	8.985	7.005	7.010	7.000	7.000
	-1.9	6.982	7.000	9.000	7.003	7.006	7.000	7.000
	1.9	6.982	7.000	9.000	7.003	7.006	7.000	7.000
	2.0	6.970	6.997	8.985	7.005	7.010	7.000	7.000
4.	0.86	6.000	C	C	C	6.000	6.005	6.002
	1.4	6.000	6.000	6.000	6.000	6.000	6.001	6.011
5.	1.2	5.984	5.980	5.988	5.990	5.994	5.996	6.000
	2.25	6.000	D	6.000	6.000	5.992	6.013	6.003

Attractor basins in the complex plane

We here investigate the comparison of the attained simple root finders in the complex plane using basins of attraction. It is known that the corresponding fractal of an iterative root-finding method is a boundary set in the complex plane, which is characterized by the iterative method applied to a fixed polynomial $p(z) \in \mathbb{C}$, see e.g. [Scott et al. (2011); Neta et al. (2012); Behl and Motsa (2012)]. The aim herein is to use basin of attraction as another way for comparing the iteration algorithms.

From the dynamical point of view, we consider a rectangle $D = [-3,\ 3] \times [-3,\ 3] \in \mathbb{C}$ with a 400×400 grid, and we assign a color to each point $z_0 \in D$ according to the simple root at which the corresponding iterative method starting from z_0 converges, and we mark the point as black if the method does not converge. In this section, we consider the stopping criterion for convergence to be less than 10^{-4} wherein the maximum number of full cycles for each method is considered to be 100. In this way, we distinguish the attraction basins by their colors for different methods.

Test problem 1. Let $p_1(z) = (z^4 + 1)$, having simple zeros $\{-0.707107 - 0.707107i,\ -0.707107+0.707107i,\ 0.707107-0.707107i,\ 0.707107+0.707107i\}$. It is straight forward to see from Fig. $1-2$ that our methods, namely OM_6^1 and OM_6^2 contain lesser number of non convergent points, have a larger and brighter basin of attraction in comparison to the methods, namely WM_6, NM_6, GM_6^1, GM_6^2 and GM_6^3. Further, our methods do not show an chaotic behavior as method WM_6.

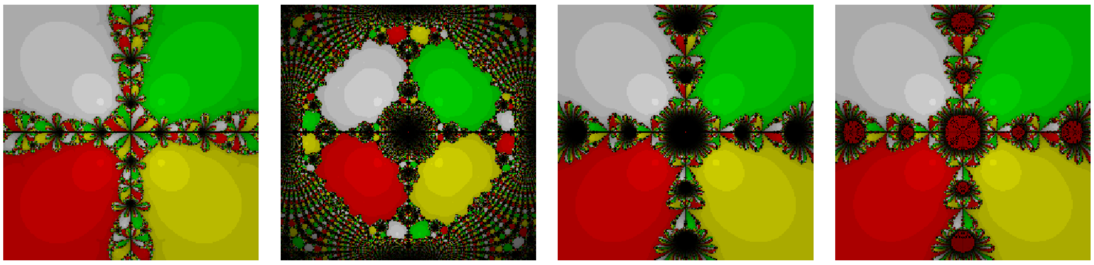

Figure 1: The methods WM_6, NM_6, GM_6^1 and GM_6^2, respectively for test problem 1.

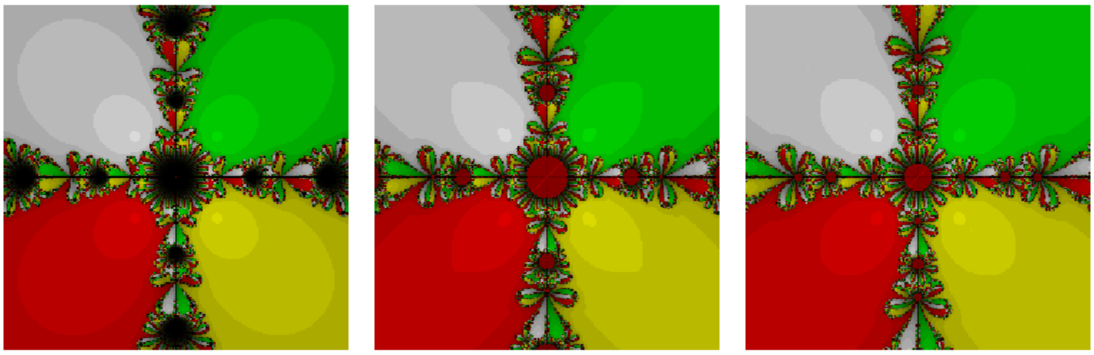

Figure 2: The methods GM_6^3, OM_6^1, and OM_6^2, respectively for test problem 1.

Test Problem 2. Let $p_2(z) = (z^6 + z)$, having simple zeros $\{-1,\ -0.309017 - 0.951057i,\ -0.309017+0.951057i,\ 0,\ 0.809017-0.587785i,\ 0.809017+0.587785i\}$. We can easily observe from Fig. 3 – 4 that our proposed methods have larger and brighter basin of attraction in comparison to the mentioned methods.

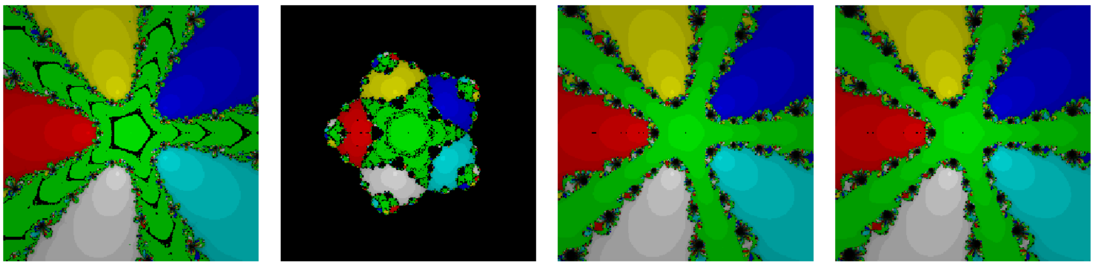

Figure 3: The methods WM_6, NM_6, GM_6^1 and GM_6^2, respectively for test problem 2.

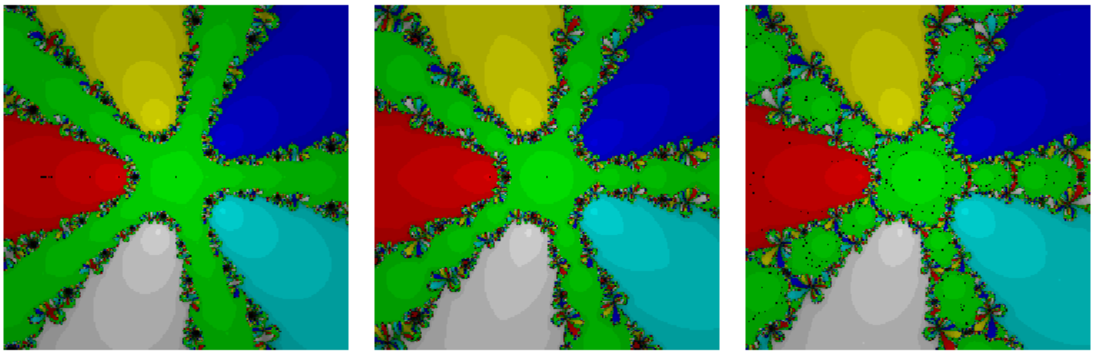

Figure 4: The methods GM_6^3, OM_6^1 and OM_6^2, respectively for test problem 2.

Conclusions

In this paper, we proposed several second-derivative free families of Halley type methods based on weight function and arithmetic means of the approximated value of the second-order derivative. We can easily get several new methods by choosing different values of the disposable parameter M_{21} in schemes (28) and (29). Further, we can also obtain several families of sixth-order Halley-type method by considering different kind of weight functions which satisfy the conditions mentioned in Theorem 1. Each member of the proposed family requires two evaluations of the function f and two of its first-order derivative f' per full step. Our proposed iterative methods are compared in their efficiency and performance to various other multi-point methods in Table 4, 5, 6 and it is observed from these tables that our proposed methods are efficient and perform better than existing methods available in the literature. Based on Figs. $1-4$, we conclude that larger basins of attraction belong to our methods namely, OM_6^1 and OM_6^2 although the others methods are slow and has darker basins while some of the method are too sensitive upon the choice of the initial value.

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