Evaluation of the Singular Stress Field of the Crack Tip of the Infinite Elastic Plate

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Abstract

In the Scaled Boundary Finite Element Method (abbr. SBFEM), the analytical advantage of the solution in the radial direction allows stress intensity factors (SIFs) to be determined directly from its definition. Therefore no special crack-tip treatment is necessary. In addition, the stiffness of infinite domain can be solved analytically. In present paper, the asymptotic fields of the crack tip of the infinite plate subjected to the different loadings are computed based on the SBFEM combining the sub-structuring technique (or super-element) , extracting the SIFs, some of higher order terms. The essential calculating formula of SBFEM is derived. The numerical results are compared with those in the literature, and the results show that SBFEM can evaluate the asymptotic fields of the crack tip with higher efficiency and accuracy. In addition, some of the higher order terms may provide evidence for the further research on the fracture characteristics of the mass concrete materials and structures.

Keywords: stress intensity factors; the scaled boundary finite element method; high order singular terms; infinite plate ；

Introduction

The analytical expression of stress fields of crack tip is presented by Williams[Williams (1957)]. It includes stress intensity factors K_I , K_{II} , T-stress and higher order terms which is named crack-tip parameters. Theoretical analysis and test can not determine parameters of the complex case, which is concerned on the material properties of structure, load form, and the initial angle of crack. Some of the numerical methods are applied to determine these parameters of practical and complex problem. The mainly numerical methods which can evaluate the singular fields of crack tip, include the finite element method(FEM), the boundary element method(BEM), the weight function method, the finite difference method and the scaled boundary finite element method(SBFEM) [(Deeks and Chidgzey(2005); Song(2005)]. FEM need discretizing crack tip area, and BEM need discretizing crack surface area. Furthermore, they can't gain analytical solution around singularity points in general case, because of their dependence on the piecewise smooth functions [Song (2002)]. The most important advantage of SBFEM is the stress singularity along the radial direction emanating from the crack tip is represented analytically. No analytical asymptotic expansion or enrichment is required. SIFs, T-stress and higher order terms are extracted directly [(Deeks and Chidgzey(2005); Song(2005)]. According to the advantages of SBFEM, it has been applied to evaluating the dynamic stress intensity factors [Song(2004;2008)], the SIF of orthotropic material[Song (2002)], the SIF under the temperature load [Song(2006)], and a unified definition of GSIFs was proposed in [Song et al (2010)]. Crack propagation was modelled in [Yang (2006);Yang and Deeks (2007); Ooi and Yang (2009; 2011a; 2011b); Bird et al (2010); Shi (2013) and Zhu(2014)]. Other applications in fracture mechanics include [Li (2014);Liu(2008)].

In the paper, the asymptotic fields of the crack tip of the infinite plate subjected to the different loadings are computed based on the SBFEM combining the sub-structuring technique (or super-element), extracting the SIFs, some of higher order terms of crack tip are provided. Numerical examples are provided to demonstrate the effectiveness and accuracy. The results are compared with those of analytical solutions and numerical solutions. The comparison shows that SBFEM can calculate the asymptotic field of crack tip with accuracy.

2 The Fundamental Equations of Scaled Boundary Finite Element Method

With emphasis placed on the two-dimensional problems the concept of the scaled boundary finite element method and the necessary equations for a bounded medium are summarized.

The governing equations of SBFEM without dynamic problems is the following

$$\left[E^0\right]\xi^2\left\{u(\xi)\right\}_{,\xi\xi}+\left(\left[E^0\right]-\left[E^1\right]+\left[E^1\right]^T\right)\xi\left\{u(\xi)\right\}_{,\xi}-\left[E^2\right]\left\{u(\xi)\right\}=0 \tag{1}$$

where $\left[E^0\right]$, $\left[E^1\right]$ and $\left[E^2\right]$ are the coefficient matrices on the boundary [Song and Wolf(1997)].

The solution of displacement field is expressed as the following

$$\left\{u(\xi)\right\}=\sum_{i=1}^{n}c_i\xi^{-\lambda_i}\left\{\phi\right\}_i \tag{2}$$

where, n is the dimension of eigen value vector matrix $\left[\Phi_{11}\right]$ whose meaning can be seen in the following formula. $\left\{\phi\right\}_i$ is i column of the matrix $\left[\Phi_{11}\right]$, c_i is the i element of the integration constant vector $\left\{c_1\right\}$. Then the radial displacement field within the sub-structure (super-element) can be obtained by interpolation through the function $\left[N^u(\eta)\right]$, therefore

$$\left\{u(\xi,\eta)\right\}=\sum_{i=1}^{n}c_i\xi^{-\lambda_i}\left[N^u(\eta)\right]\left\{\phi\right\}_i \tag{3}$$

the stress field within the sub-structure

$$\left\{\sigma(\xi,\eta)\right\}=\sum_{i=1}^{n}c_i\xi^{-\lambda_i-1}\left\{\psi(\eta)\right\}_i \tag{4}$$

where, $\left\{\psi(\eta)\right\}_i$ is the stress mode of local coordinate η, it can be calculated by the corresponding displacement mode $\left\{\phi\right\}_i$,

$$\left\{\psi(\eta)\right\}_i=\left[D\right]\left(-\lambda_i\left[B^1(\eta)\right]+\left[B^2(\eta)\right]\right)\left\{\phi\right\}_i \tag{5}$$

The solutions of displacement and stress in the sub-structure can be expressed as polar-coordinate form in order to calculate SIF. Radial coordinate can be expressed as

$$\hat{r}(\xi,\eta)=\xi r(\eta) \tag{7}$$

where, $r(\eta)=\sqrt{x^2(\eta)+y^2(\eta)}$ is the radial coordinate on the boundary of sub-structure. Angle θ is only related to η,

$$\theta(\eta)=\arctan\frac{y(\eta)}{x(\eta)} \tag{8}$$

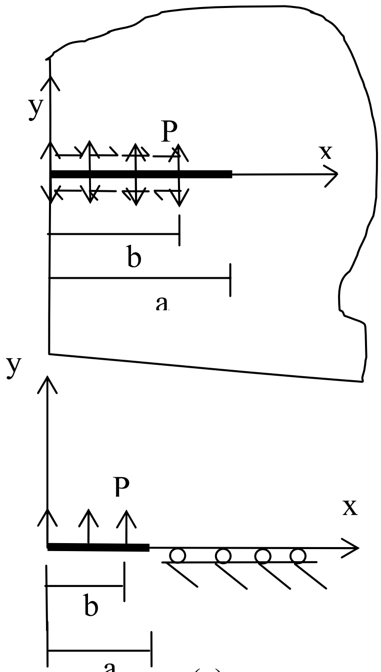
Eq.(7) is substituted into Eq.(4), we can get

$$\left\{\sigma(\hat{r},\eta)\right\}=\sum_{i=1}^{n}c_i\hat{r}^{-\lambda_i-1}\left(r(\eta)\right)^{\lambda_i+1}\left\{\psi(\eta)\right\}_i \tag{9}$$

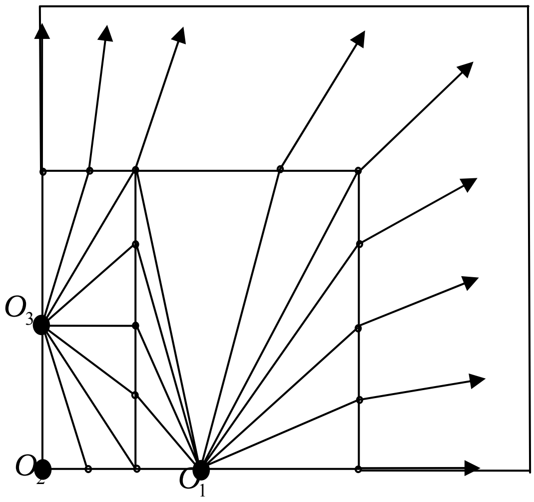
Eq.(8) and Eq.(9) constitute the stress field similar to Williams which is expressed by the series coordinate. The expression of the singular fields of crack tip of Williams is used.

3 Numerical Examples

3.1 Edge-cracked semi-infinite plate under uniform tension tractions



(a)



(b)

Fig.3a Edge-cracked semi-infinite plate under uniform tension traction

Fig.3b Boundary discretization of edge-cracked semi-infinite plate

First considering an edge-cracked semi-infinite plate subjected to uniform distributing tension p0 loaded on the crack faces as shown in Fig.3a. The semi-infinite plate is divided into three blocks (Fig.3b), and the corresponding scaling centers are $O_1(0,0)$, $O_2(-1,0)$ and $O_3(-1,2)$, respectively. The geometrical parameter of the first block is:W=1.5,H=4, the crack length is a=1. The length of loading for the crack face is b=0.35,0.5,0.75,0.85. The material properties elastic modulus E=1, Poisson's ratio ν=0.25. The Fig.3b gives the discretization model, and the total elements is 16 3-node line elements. The analytical solutions of SIF are shown in Reference[Hiroshi(2000)], and the present compared results and relative difference (RD) are shown in Table 1. The higher order terms a2,a3 are shown in Table 1. They are in good agreement and the maximum difference is less than 3%.No analytical solutions or numerical results for the a2,a3 are compared.

Table 1 Results of edge-cracked semi-infinite plate under uniform tension tractions（W=1.5，N=16）

The length of loading: b	KI	Analytical solutions	(RD)(%)	a2	a3
0.35	0.5130	0.4998	-2.6342	0.2728	-0.2698
0.5	0.7317	0.7142	-2.4510	0.3474	-0.4386
0.75	1.1385	1.1103	-2.5378	0.4376	-0.9263
0.85	1.2698	1.3100	3.0722	0.6191	-1.3251

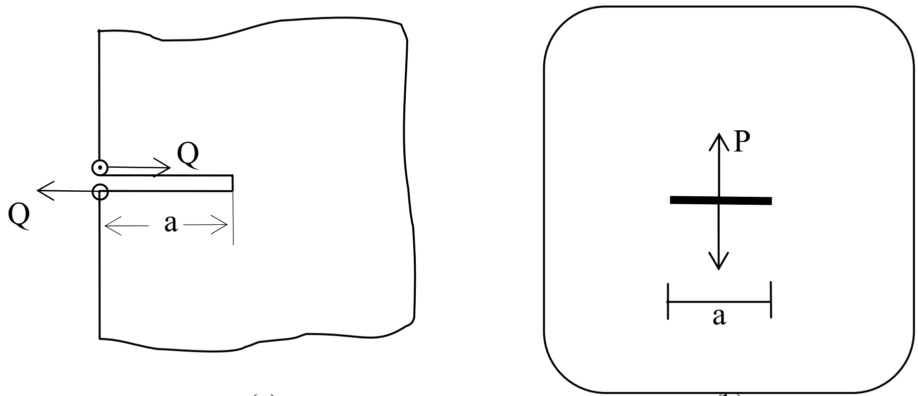
Table 2 Results of edge-cracked semi-infinite plate under uniform shear tractions

Loading length b	KII	Analytical solution	RD(%)
0.65	0.9763	0.9415	-3.7035
0.70	1.0401	1.0233	-1.6435
0.75	1.1049	1.1103	0.4871
0.80	1.1803	1.2046	2.0241
0.85	1.2879	1.3100	1.6887
0.90	1.4542	1.4336	-1.4335

3.3 Edge-cracked semi-infinite plane under uniform shear tractions

Fig.3a shows the Edge-cracked semi-infinite plate under uniform shear tractions model. The geometrical parameter is W=1.5, H=4. The length of crack a=1, the loading length of uniform shear b=0.65,0.70,0.75,0.80,0.85,0.90,0.95. The material properties elastic modulus E=1, Poisson's ratio ν=0.25. The total number of elements is 8. Reference [Hiroshi(2000)] gives the analytical solutions. The comparison of the present results with analytical solutions are shown in Table 3. Both the numerical and analytical results are in good agreement. The maximum difference is less than 4%.

3.3 Edge-cracked semi-infinite plane under concentrated shear tractions



(a) (b)

Fig.4a Edge-cracked semi-infinite plate under concentrated shear tractions

Fig.4b Edge-cracked infinite plate under concentrated tension tractions

Fig.4 shows the Edge-cracked semi-infinite plane model under concentrated shear tractions. The geometrical parameter is W=2, H=4. The length of crack is a=0.7,0.8,0.9,1.0,1.1,1.2,1.3. The material properties elastic modulus E=1, Poisson's ratio ν=0.25. The discretization is shown in Fig.3b and the total number of elements is 19. The analytical solutions of SIF are shown in Reference[Hiroshi(2000)], and the comparasion of the present results with analytical solutions are shown in Table 2. Both the numerical and analytical results are in good agreement. The maximum difference is less than 2%. The higher order terms a3 are also shown in Table 2. No analytical solutions or numerical results are compared.

Table 3 Results of edge-cracked semi-infinite plate under concentrated shear tractions

Crack length a	KII	Analytical solution	RD（%）	a3
0.7	1.7371	1.7492	0.6917	-2.1328
0.8	1.6263	1.6363	0.6108	-1.9439
0.9	1.5353	1.5427	0.4795	-1.5322
1.0	1.4598	1.4635	0.2522	-1.1236
1.1	1.3968	1.3954	-0.0990	-0.8287
1.2	1.3438	1.3360	-0.5827	-0.6297
1.3	1.2988	1.2836	-1.1865	-0.4953

Table 4 Results of SIF, a1,a2 and a3 of the edge-cracked infinite plate under concentrated tension tractions

Number of elements	KI	RD（%）（Analytical solution 0.5642）	a1	a2	a3
5	0.5605	0.6532	0.2236	0.0263	
7	0.5603	0.6864	0.2235	0.0131	-0.2904
11	0.5680	-0.6725	0.2266	0.0111	-0.2889
19	0.5707	-1.1562	0.2277	0.011	-0.2885

3.4 Edge-cracked infinite plane under concentrated tension tractions

Edge-cracked infinite plane under concentrated tension tractions model is studied, as shown in Fig.4b. The geometrical parameter is W=1.5, H=4. The length of crack a=1. The material properties elastic modulus E=1, Poisson's ratio ν=0.25. The analytical solutions of SIF are shown in Reference [Hiroshi(2000)]. The comparison of the present results with analytical solutions are shown in Table 4.The results of higher order terms a2 and a3 for the different element number are shown in Table 5. It shows that the numerical and analytical results are in good agreement. The maximum difference is less than 2%.

Conclusions

In the paper, the singular stress fields of edge-cracked infinite plate are computed based on the SBFEM, extracting the SIF and the coefficients of higher order terms. The expression of displacement field and stress field of SBFEM and the equation of asymptotic field of crack tip of fracture mechanics are derived. The results are compared with the ones of analytical solutions and some numerical results, which shows that the SBFEM can calculate the asymptotic field of crack tip with accuracy. In addition, the results of T-stress and the coefficients of higher order terms have certain significance on determining crack stability and studying the fracture characteristics of crack tip.

Acknowledgements

This work was financially supported by the National Natural Science Youth Foundation of China (51109134, 51208310, 11102118), China Postdoctoral Science Foundation (2013T60283, 2011M500557) and Program for Liaoning Excellent Talents in University (No. LJQ2011009). Thanks the Professor of the University of New South Wales, Australia, Song Chongmin, for his valuable assistance. Thanks for the International Postdoctoral Exchange Fellowship Program 2013 by the Office of China Postodoctoral Council

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