Non Linear Strain Integral Damping (S.I.D.)

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Abstract

Forces are generally defined in physics as functions of position (Newton: gravity) or velocity (Laplace: magnetic force on a moving electric charge). Damping forces are little known even today and represent one of the most intriguing subjects of physics. Maxwell elements and fractional derivatives are used to modelize time domain natural hysteretic damping. The resulting models are comparatively complicated and have a limited domain of validity especially when strong non-linearity is involved. The mathematical model we use is based on the introduction of a new state variable and is particularly suitable in the non-linear vibration case. S.I.D. (Strain Integral Damping: see ref. [2]) is a very suitable mean to modelize natural hysteretic damping in the time domain and for nonlinear rubber elements in particular. In the present paper the stress is on modelling of nonlinear elements. The effectiveness of SID is shown by an example concerning a strongly non-linear spring. A ''Scilab'' script is provided to better explain.

Keywords: Natural hysteretic damping, Computational model, Vibration, Engine, Driveline, Startup

Introduction

Natural damping is only seldom viscous. Natural hysteretic damping is much more common and can be described as follows in the frequency domain.
If:
$M = Mass\ Matrix \quad K = Stiffness\ Matrix \quad f = force\ vector \quad \xi = displacement\ vector$
$\omega = angular\ frequency$

$$\left[-\omega^2 M + K \cdot \left(1 + j \cdot \mathrm{tg}(\varphi)\right)\right]\xi = f \tag{1}$$

Where an imaginary part of the stiffness matrix is introduced (tg(φ)). We shall call this I.S.D. (Imaginary Stiffness Damping) in the following.
Such a formulation is much used in the frequency domain because it is simple and practical to use and not because there is a real physical theory behind it. S.I.D. (Strain Integral Damping) wants to be as simple and practical to use for hysteretic damping modeling in the time domain. The formulation of SID will be now briefly recalled. See references [2] and [3].

1 SID Formulation

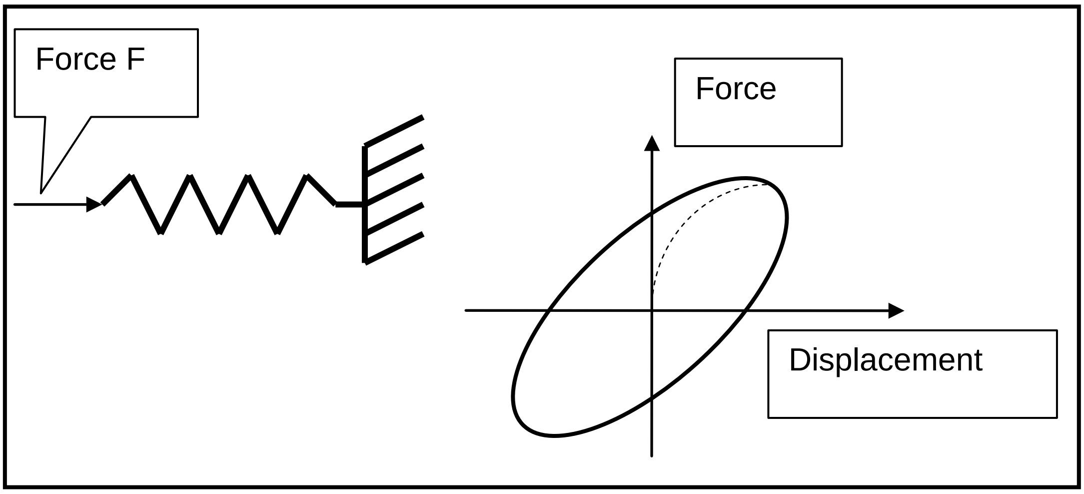

Figure 1. Hysteresis of a spring-damper

Let us consider the spring-damper of FIG. 1. If $x(t)$ is the displacement at time ‘’t’’ and we apply a sinusoidal force we shall obtain:

$$x(t) = X\ \cos(\omega t) \tag{2}$$

Velocity ‘’v’’ and acceleration ‘’a’’:

$$v(t) = \frac{dx}{dt} = -\omega\ X\ \sin(\omega t) \tag{3}$$

$$a(t) = \frac{d^2x}{dt^2} = -\omega^2\ X\ \cos(\omega t) \tag{4}$$

The applied force will be:

$$f(t) = F\ \cos(\omega t + \varphi) \tag{5}$$

‘’F’’ being the force amplitude. We can rewrite:

$$f(t) = F\cos(\omega t)\cos(\varphi) - F\ \sin(\omega t)\sin(\varphi) \tag{6}$$

Following equation (1) the springer-damper stiffness ‘’k’’ is defined by:

$$F\ \cos(\varphi) = kX \tag{7}$$

We can then write equation (6) in the form:

$$f(t) = kX\ \cos(\omega t) - k\ \mathrm{tg}(\varphi)\ X\ \sin(\omega t) \tag{8}$$

By substituting equations (2) and (3) in equation (8) we obtain:

$$f(t) = k\left(x(t) + \mathrm{tg}(\varphi)\frac{v(t)}{\omega}\right) \tag{9}$$

Where we have introduced the same $\mathrm{tg}(\varphi)$ factor of eq. (1).
Now we must express the ''ω'' of eq. (9) as a function of state variables only. We may think of expressing the ''1/ω'' factor of eq. (9) as the ratio:

$$\frac{1}{\omega} = \left|\frac{x(t)}{a(t)}\right|^{1/2} \tag{10}$$

But as it is shown in reference [1], forces cannot in general be expressed as functions of the accelerations and this leads us to define a new state variable which is the solution of the differential eq.:

$$\frac{dy}{dt} = -\omega_1 \cdot y + x(t) \tag{11}$$

The solution is:

$$y(t) = \int_0^t e^{-\omega_1(t-\tau)} x(\tau)d\tau \tag{12}$$

The constant ''ω_1'' is introduced to define as ''remote past'' all events for which:

$$(t-\tau) \gg 1/\omega_1 \tag{13}$$

Such events will have negligible effect on ''y'' (strain integral) and, as a consequence, on the damping force. We must remark that if ''ω_1'' is zero, ''y'' goes to infinity for all x(t) whose average is not zero (spring preloading). This of course wouldn't be physical. So ''ω_1'' can be seen as a high pass filter parameter: it has the same physical dimensions as a frequency and it must be set well lower than the frequencies of interest but it must not be negligible in comparison with the frequencies of interest to avoid ''y'' to go to infinity. We can better understand this by writing eq. (11) in the frequency domain:

$$\frac{Y}{X} = \frac{1}{\omega_1 + j\omega} \tag{14}$$

Where X and Y are the complex amplitudes of ''x'' and ''y'' respectively. We can see from this formula that if ω is an angular frequency of interest, it must be $\omega \gg \omega_1$ for ''y'' to be close to the integral of ''x''. $\omega_1 = 1\%(\omega)$ is a possible value.
We can then assume:

$$\frac{1}{\omega} \cong \left|\frac{y(t)}{v(t)}\right|^{1/2} \tag{15}$$

By substituting eq. (15) into eq. (9) we easily obtain:

$$f(t) = k\left(x + tg(\varphi)\ \mathrm{sign}(v(t))\ |y(t)\ v(t)|^{1/2}\right) \tag{16}$$

We remark that the term $|y(t)\ v(t)|^{1/2}$ has the physical dimensions of a displacement but is ''phased'' like a velocity.

We assume as initial condition for ''y'':

$$(t = 0) \Rightarrow (y = 0) \tag{17}$$

We can easily see that, with this initial condition, the cycle starts at the origin like the dotted curve shown in Fig. 1.
Work experience has shown that the introduction of factor ''ω_1'' in eq. (11) is not enough to avoid that ''y'' goes to infinity. This problem of course can only exist in case of spring (engine mount) preloading. The problem is easily solved by the introduction of a moving average in equation (11):

$$\frac{dy}{dt} = -\omega_1 \cdot y + \left(x(t) - \overline{z(t)}\right) \tag{18}$$

The moving average $\overline{z(t)}$ is defined as the solution of the following differential equation:

$$\frac{dz}{dt} = -\omega_2 \cdot z + x \tag{19}$$

The solution is:

$$z(t) = \int_0^t e^{-\omega_2 (t-\tau)}\ x(\tau)\ d\tau \tag{20}$$

And the corresponding weighted moving average:

$$\overline{z(t)} = \int_0^t e^{-\omega_2 (t-\tau)}\ x(\tau)\ d\tau \cdot \omega_2 / (1 - e^{-\omega_2 \times t}) \tag{21}$$

Where:

$$\omega_2 / (1 - e^{-\omega_2 \times t}) \tag{22}$$

Is the normalization factor. We can see from eq. (21) that such an average has the important property that all events of the ''past'' that happened at time τ such that:

$$(t - \tau) \gg 1/\omega_2 \tag{23}$$

Are ''squeezed'' by the weighting factor:

$$e^{-\omega_2(t-\tau)} \ll 1 \tag{24}$$

So that only the most ‘’recent’’ events are really included in the average.
The fact that factor (22) goes to infinity when t=0 is generally avoided by the substitution:

$$\omega_2 \ / \ \max(0.0001, 1 - e^{-\omega_2 \times t}) \tag{25}$$

In practice we often assume:

$$\omega_2 = \omega_1 \tag{26}$$

But this is not a general rule: $\boldsymbol{\omega_2}$ depends on the speed by which the “quasi static” preloading varies and must be set accordingly. Sometimes static preloads are not really constant. For example the engine torque varies depending on how much the driver presses on the accelerator and ‘’static’’ loads on the mounts will vary accordingly. In the driveline model of section 4.2 for example we had:

$$\omega_1 = 0.01; \qquad \omega_2 = 8$$

Because of the quickly increasing engine torque due to quickly mounting RPM. The RPM rose quickly because of simulation of a steep sloping down startup of the vehicle. Consider for example the famous “Gross Glochner” very steep descent in Austria.

2 Nonlinearity

SID is most useful in nonlinear problems. To introduce non linearity we only need modifying eq. (16) as follows:

$$f(t) = \mathrm{spl}(x) + k_s \cdot \mathrm{tg}(\varphi)\ \mathrm{sign}\big(v(t)\big)\ |y(t)\ v(t)|^{1/2} \tag{27}$$

Where ‘’spl’’ is a spline representing the non-linear spring and k_S is the secant stiffness (very seldom the tangent stiffness as explained in reference [3]). In references [2] and [3] the user is provided with useful advice and cautions concerning the practical use of SID. For example the “boxcar effect” [4] needs sometimes being considered in analyzing results obtained by time step integration. It must be remarked that assuming the secant stiffness (load divided by displacement) to drive the damping phenomenon corresponds to assuming damping forces to be proportional to the loads acting on the nonlinear element. In the author’s experience such an assumption is often closer to reality than assuming damping forces to be proportional to the differential stiffness.

3 Frequency independence of SID nonlinear cycles (Numerical example)

We are now going to present with a numerical example concerning the property of a SID spring hysteresis cycle to remain the same whatever the frequency of the imposed displacement. Such a property is a feature of natural damping as it is observed in physical reality. SID has the remarkable power of insuring that such a property is verified also in the case of calculation of a strongly nonlinear spring. The Scilab script in the appendix was used to perform the calculations. By setting the imposed displacement frequency at 20, 40 and 60 Hz we are now going to see that the cycle doesn’t change. We can see that the cycle in figure (5) is practically identical to that in figure (3) although the frequency is twice and that the cycle in figure (7) is again practically identical to that in figure (3) although the frequency is 3 times higher.

3.1 Calculation at 20 Hertz.

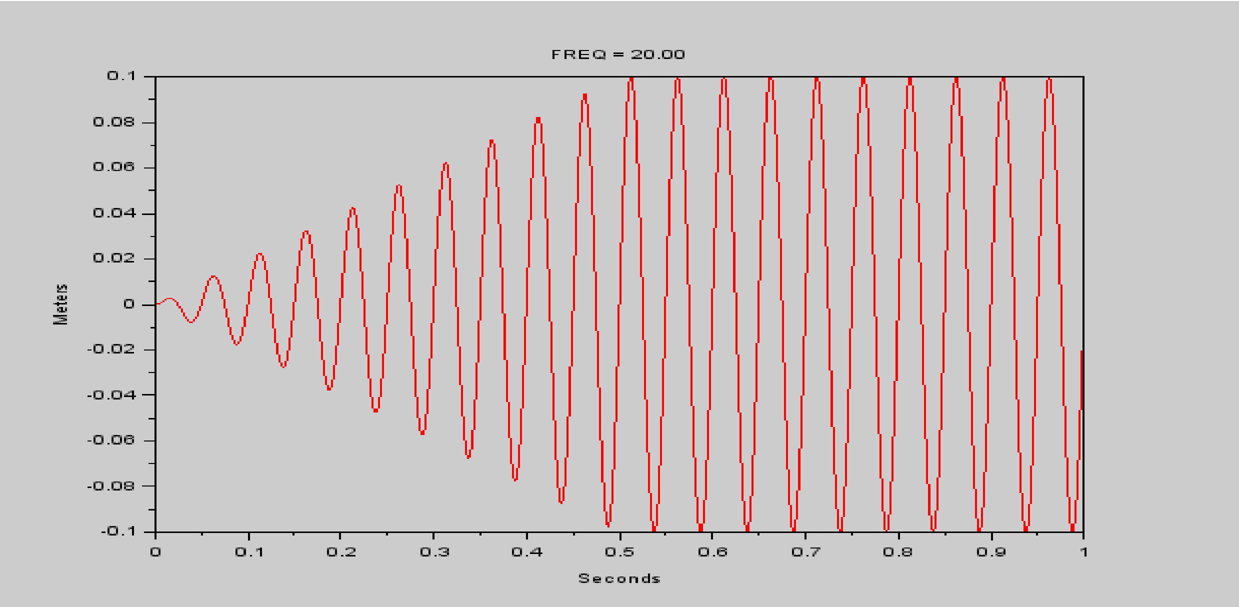

Figure 2. Displacement

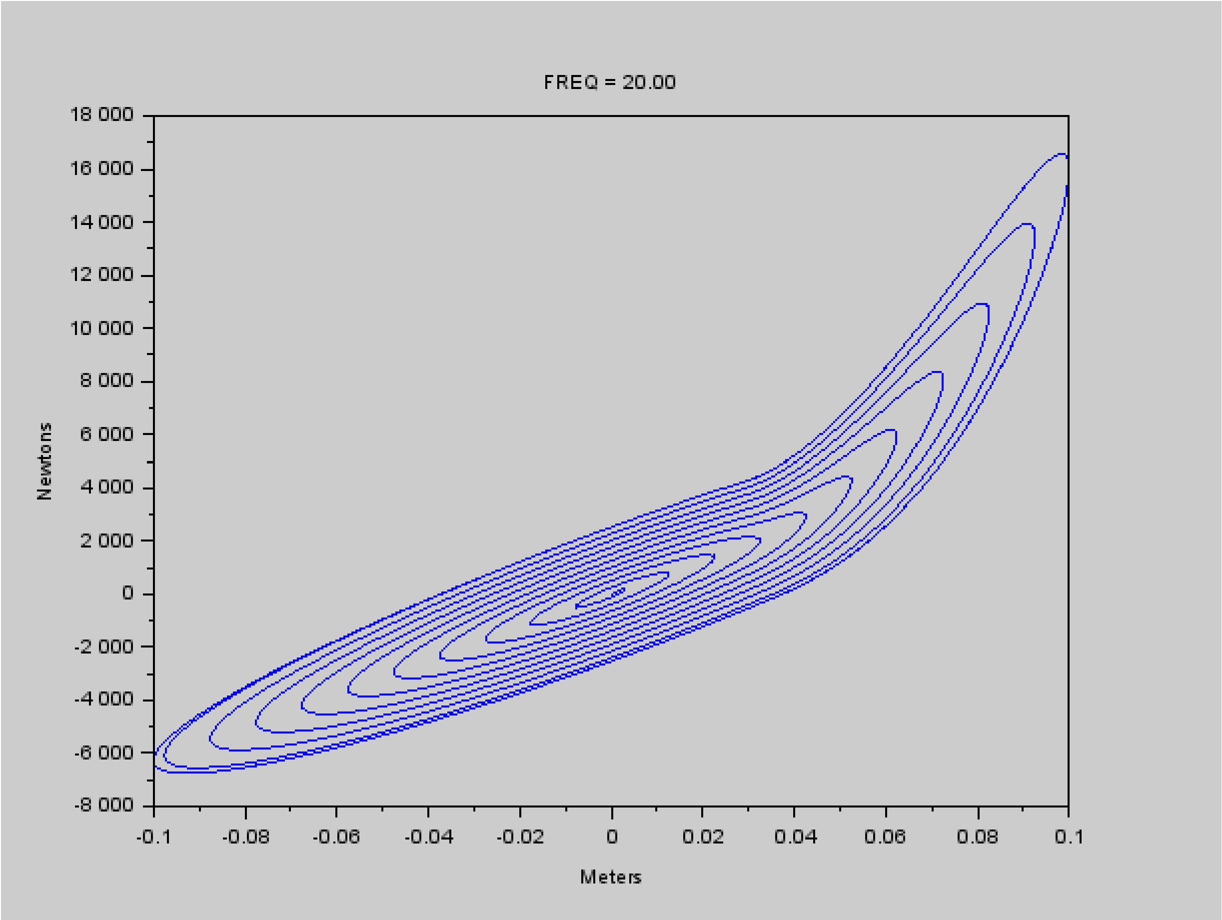

Figure 3. Cycle

3.2 Calculation at 40 Hertz.

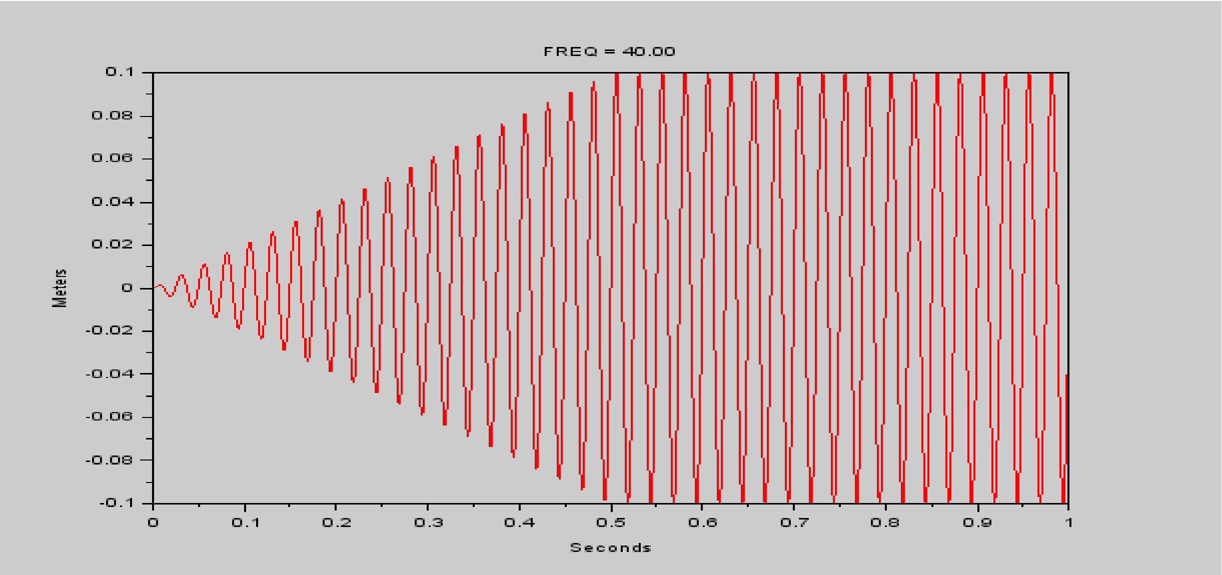

Figure 4. Displacement

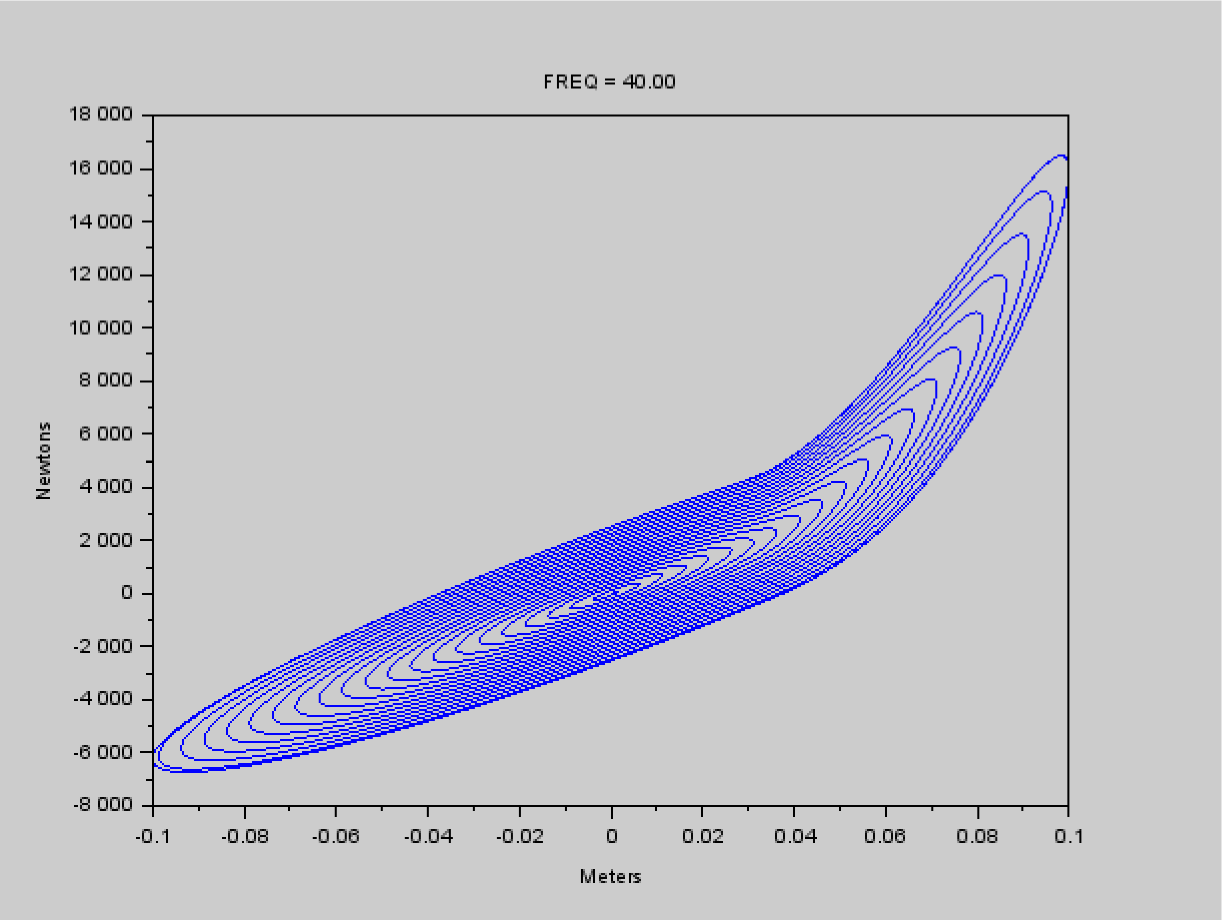

Figure 5. Cycle

3.3 Calculation at 60 Hertz.

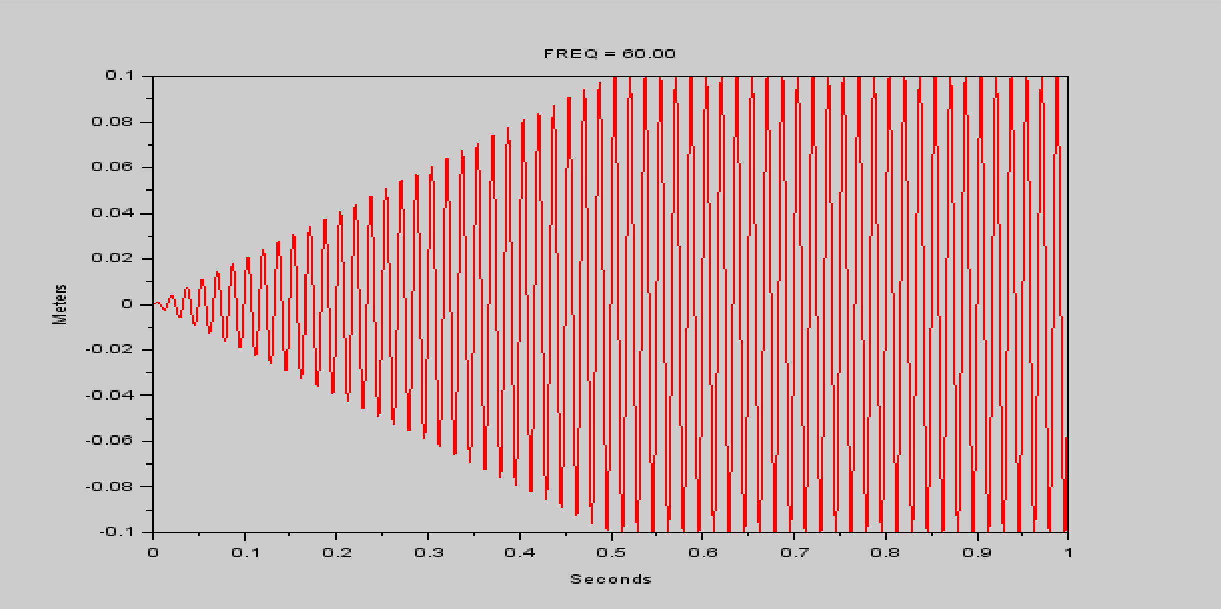

Figure 6. Displacement

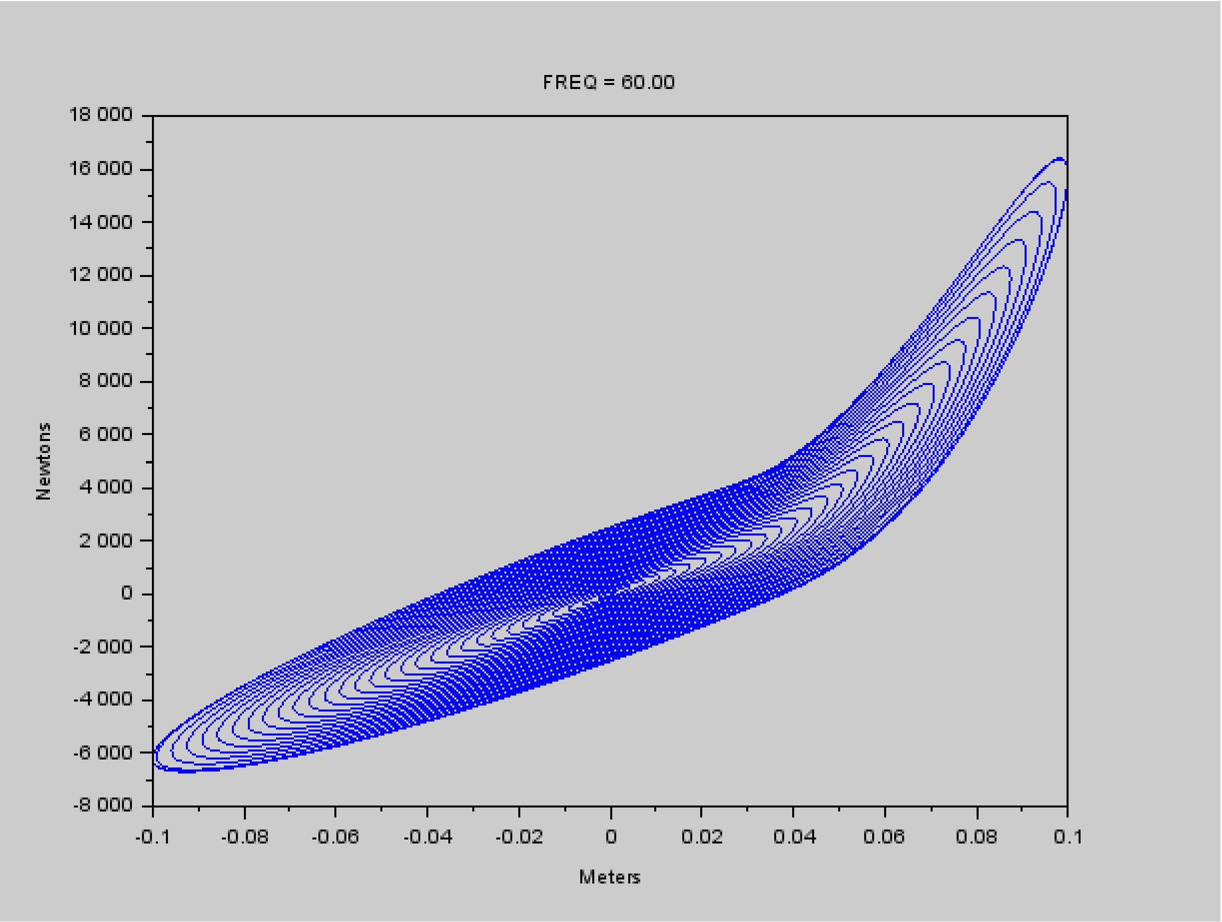

Figure 7. Cycle

4 The kind of models SID is used in

4.1 “Global car model”

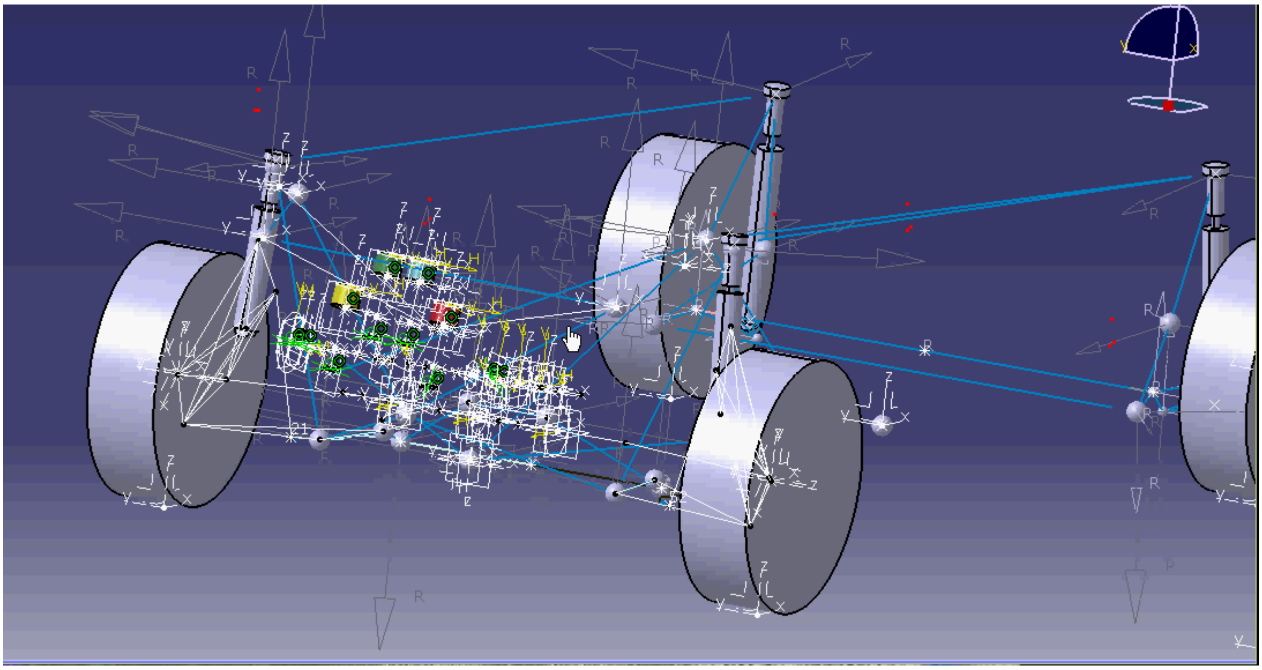

Figure 10. “Global car model”

Figure 10 shows a “VeLab” model inclusive of practically everything which is needed to predict a vehicle startup behavior. Models of the following subsystems are included:

- Starter
- Engine, pistons, crankshaft, links, engine mounts
- Clutch
- Gearbox, gears, differential, transmissions
- Suspensions, dampers, steering apparatus
- Wheels
- Tires
- Rigid or flexible car body and suspension frameworks
- Torsional dampers

4.1.1 Applicability

It is generally possible to devise and validate such subsystems separately and then assemble them in the global model. Such “global” models are seldom used except for special problems involving the whole of the vehicle like for example the study of vibration energy transmission from the engine through the suspensions and to the car body. Animation of this model helps understanding “global” problems sometimes.

4.2 “Driveline model”

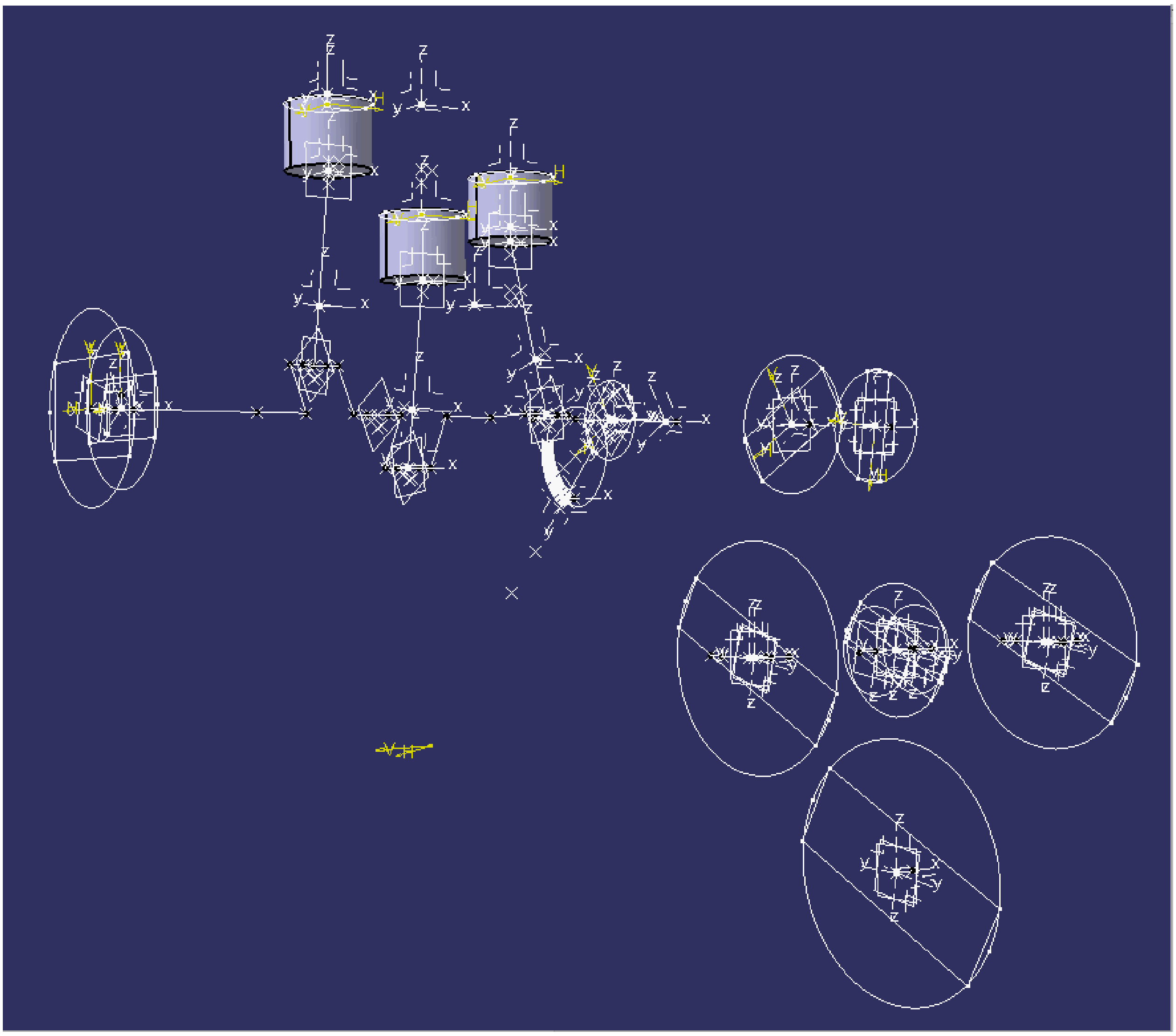

Figure 11. “Driveline model”

Figure 1 shows a driveline model that was used to study a pendulum damper dynamic behavior. The starter, gear, differential and wheels are modelized together with the vehicle which is represented by a big flywheel in such a model.

4.2.1 Applicability

Such models are more often used than the “global” one. The effects of the SHR (wheel longitudinal vibration) mode can be studied by such a model and SID is used to modelize practically everything flexible in the model. Only the tire model also includes viscous damping, tire longitudinal stiffness being concerned.

5 General remarks

SID is of great help in preparing such models because it provides the desired natural damping behavior. Using viscous damping for example would require adapting the damping to the new situation every time some eigenfrequencies change because of structural modification. Viscous damping cycles are in facts strongly frequency dependent. Once the 3 parameters governing SID are set, instead one can almost forget damping modelling and go on trying new solutions in a most expedite way. SID also has the prize of simplicity: it would be very difficult obtaining the same result displayed in figures 3-5-7 by other methods and by 18 lines of code only (see the script in the appendix). One can quickly prepare macros that formulate SID for all elastic elements in a model. It is very important to remark that the phenomena dealt with by such models all start by low levels of vibration and then soar to higher vibration levels as the transient goes on: this is precisely the kind of phenomena SID was born to deal with. This is also the reason for the "rising amplitude imposed displacement" (figures 2-4-6) chosen for the examples of figures 3-5-7 and the corresponding SciLab script in the appendix.

Conclusions

The validity of a theory can only be proved by its agreement with reliable experimental results like the well-known result of eq. (5). In this sense SID has been shown to give the kind of results we expect (see figures 3-5-7). We don't know whether SID is a ''beable'' which is what the physicists call something that has a real link with physical reality: only the future can say. We can say however that it is a very practical and easy method that corresponds, in the time domain, to the imaginary stiffness damping of eq. (1) in the frequency domain: no physical base to it but everybody uses it because it is simple and practical (see the general remarks of paragraph 5). SID only needs three parameters to be defined. SID is a suitable mathematical description of hysteretic damping and gives fairly physical results when applied to non-linear problems (see figures 3-5-7).

References

[1] L.A. Pars, *A Treatise on Analytical Dynamics*, Ox Bow Press.
[2] U. Tornar, *Modelizing Structural Hysteretic Damping by Means of MDI ADAMS*, Proceedings of the ADAMS 1997 Users Conference, Marburg, German
[3] U. Tornar, *Application of Strain Integral Damping to nonlinear Engine Mounts*, ISMA 2014, Leuven, Belgium
[4] R.K. Otnes and L. Enochson, *Applied Time Series Analysis*, Wiley-Interscience

Appendix

In the following script the variables correspond to:

freqq = frequency

tt = time

dd = displacement

cs1 = natural hysteretic damping

k1 = linear stiffness

dt = time differential

zh1 = SID ω_2 parameter of eq. (19)

h1 = SID ω_1 parameter of eq. (11)

va = velocity

z1 = solution of eq. (19)

ss1 = solution of eq. (11)

z1av = moving average of eq. (21)

force1 = force of nonlinear spring: spline spl(x) of eq. (27)

secstiff = secant stiffness: (force/displacement) that is k_s of eq. (27)

force1 = after definition of secstiff, it is the total force including damping force

SCILAB SCRIPT

```
clear;
freqq = 20; fig1 = 1; fig2 = 2;
tt=(1:4096)/4096; ll = 2* %pi; dd = sin(ll*tt*freqq)/10;
for kk = 1 : 2048; dd(kk) = dd(kk) * tt(kk)/tt(2048); end;
figure(fig1); title('FREQ = ' + msprintf('%.2f',freqq)); plot(tt,dd,'r'); xlabel('Seconds'); ylabel('Meters');
csi1 = 0.4; m1 = 400; f1 = 2; k1 = (ll*f1)*(ll*f1)*m1; dt = 1/4096;
ss1 = 0; h1 = 0.2; z1 = 0; zh1 = 0.0001; force1 = 0.;
for kk = 1 : 4096 - 1;
va = (dd(kk+1) - dd(kk))/dt; z1 = z1 + (-z1 * zh1 + dd(kk)) * dt;
z1av = z1 * zh1 / max(0.0001,1-exp(-zh1*tt(kk)));
ss1 = ss1 + (-ss1 * h1 + dd(kk)-z1av) * dt;
force1 = (dd(kk))*k1 + 2.*((dd(kk)) > 0.03)*(dd(kk)-0.03)**2*1000000;
secstif = abs(force1/ (dd(kk)));
force1 = force1 + (-0.*0.15+sign(va))*(abs(ss1 * va))**0.5 * secstif * csi1;
force(kk) = force1;
end
dd = dd(1:length(dd)-1); figure(fig2); title('FREQ = ' + msprintf('%.2f',freqq)); plot((dd),force,'b');
xlabel('Meters'); ylabel('Newtons');
```