A unified computational method of differential analysis for solving the Navier-Stokes equations.

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Abstract

Certain traditional methods of Calculus for solving DEs and systems of DEs in engineering analysis depend in one form or another on the use of some general initially assumed analytical representation of the intended solution. Unfortunately this often leads to defining one or several integrals that cannot always be resolved exactly. In order to avoid this complication we propose that the complete "*differential*" of a general initially assumed analytical representation of the intended solution with unknown coefficients to solve for be used instead as a means of solving for DEs and systems of DEs. Such a novel method of differential analysis has led to the development of what appears to be some form of a unified theory of integration. This would represent the greatest opportunity by which the complete Navier-Stokes equations for incompressible flow in the presence of any external forces may be investigated for the existence of any "*generalized*" analytical solutions under the three most commonly used coordinate systems.

Keywords: Universal Polynomial Transform, ODEs, PDEs, Multinomial Expansion Theorem**,** Quantum Physics, Quantum computers, Navier-Stokes equations, Theory of everything.

Introduction

Such a non-traditional method of using this unique form of differential analysis in Calculus would have the real potential of defining integrals that can be completely resolved because a certain number of these initially assumed "*differentials*" are expected to become "*exact*" from the application of a well defined computational process. This would represent a very significant departure from current traditional methods of engineering analysis favoring a purely "*numerical*" method of integration in cases by which no real analytical solution to many fundamental DEs and systems of DEs in engineering science is possible. The greatest advantage of performing such a type of analysis strictly at the differential level has led to the development of some type of a unified theory of integration that can be applied for finding approximate or in some cases exact analytical solutions to "*all types*" of DEs and systems of DEs encountered in engineering analysis. The entire process of analytical integration now becomes a matter of pure computational analysis just for identifying those differentials that are exact and thus completely integrable. Such a very unique method of differential analysis will be applied for the complete *analytical* solution of a number of randomly selected DEs that would include a first and second order ODE as well as a second order PDE. The outcome of having performed such a detailed differential analysis on these very simple DEs may provide us in the long term with some basic fundamental tools of analysis by which a generalized theory of the Navier-Stokes equations may be possible in the foreseeable future. Not surprisingly since such a novel method of differential analysis has led to the development of a *computational based* unified *analytical* theory of integration. Beyond the Navier-Stokes equations are other equations of significant importance to the physical sciences that would include Maxwell's equations, Einstein's field equations, the Schrödinger equation just to name a few. Each of these fundamental equations of science would define their own very unique ideology all of which may one day be consolidated into one gigantic universal theory of everything.

1. Universal differential form representation of all mathematical equations

For solving a DE or a system of DEs, an alternative representation in *complete differential form* for a generally assumed system of "k" number of implicitly defined *multivariate* mathematical equations in the form of "$f_k(z_m, x_n) = 0$" that consist of "m" number of dependent variables and "n" number of independent variables [Mikalajunas (2015)] may be *completely* defined as :

(1). *Primary Expansion:*

$$F_i\big(W_1, W_2, \ldots, W_{p+q}\big) \;=\; 0 \;=\; \sum_t^r a_{i,t}\left(\prod_j^{p+q} W_j^{E_{i,kj}}\right) \qquad (1 \le i \le k) \tag{1}$$

where "W_j" for $1 \le j \le p$ are arbitrarily defined auxiliary variables that take part in representing the complete initially assumed analytical solution of a DE or a system of DEs. For any number of basis functions that are present in a DE or a system of DEs we would have to define an additional "q" number of known *supplemental* auxiliary variables for including each of their differential expansion as part of the complete overall expansion for representing the system of "k" number of implicitly defined multivariate equations. In such cases, the total number of auxiliary variables would grow from "p" to "$p+q$" when such basis functions are present in these types of DEs. Each of the "p" number of arbitrarily defined auxiliary variables are always initially assumed as raised to some floating point number and finally, "r" refers to the total number of multivariate polynomial terms that are present in each of the "k" number of implicitly defined multivariate polynomial equations.

(2). *Secondary Expansion:*

$$dz_i \;=\; dW_i \qquad (1 \le i \le m) \tag{2}$$

$$dx_i \;=\; dW_{m+i} \qquad (1 \le i \le n) \tag{3}$$

$$\sum_{t=1}^{m} N_{i(m+n+1)-m-n-1+t}\,dz_t \;+\; \sum_{t=1}^{n} N_{i(m+n+1)-n-1+t}\,dx_t \;=$$

$$=\; N_{i(m+n+1)}\,dW_j \qquad [1 \le i \le p+q-m-n]\;\;[m+n+1 \le j \le p+q] \tag{4}$$

As in the case of the *Primary Expansion,* each of the expressions for "N_u" in equation (4) is also defined as a multivariate polynomial with unknown coefficients and floating point exponent values to solve for.

And finally we have,

$$\sum_{t=1}^{m} T_{i(m+n+1)-m-n-1+t}\,dz_t \;+\; \sum_{t=1}^{n} T_{i(m+n+1)-n-1+t}\,dx_t \;=$$

$$=\; T_{i(m+n+1)}\,dW_j \qquad [1 \le i \le q]\;\;[p \le j \le p+q] \tag{5}$$

where each of the expression for "T_u" in equation (5) is also a multivariate polynomial but this time containing only <u>*known*</u> coefficient and exponent values that are reserved exclusively for defining each of the basis functions that would be present inside a DE or a system of DEs.

At the present time there is no other known *universal* representation of <u>*all*</u> mathematical equations consisting only of algebraic and elementary basis functions other than the one suggested above.

In complete expanded form we would write this as follow:

(1). <u>*Primary Expansion:*</u>

$$F_1 = 0 = a_{1,1} W_1^{m_{11}} W_2^{m_{12}} \cdots W_{p+q}^{m_{1,p+q}} + a_{1,2} W_1^{m_{1,p+q+1}} W_2^{m_{1,p+q+2}} \cdots W_{p+q}^{m_{1,2(p+q)}} + \dots + a_{1,r} W_1^{m_{1,(p+q)(r-1)+1}} W_2^{m_{1,(p+q)(r-1)+2}} \cdots W_{p+q}^{m_{1,r(p+q)}} \qquad (6)$$

$$F_2 = 0 = a_{2,1} W_1^{m_{21}} W_2^{m_{22}} \cdots W_{p+q}^{m_{2,p+q}} + a_{2,2} W_1^{m_{2,p+q+1}} W_2^{m_{2,p+q+2}} \cdots W_{p+q}^{m_{2,2(p+q)}} + \dots + a_{2,r} W_1^{m_{2,(p+q)(r-1)+1}} W_2^{m_{2,(p+q)(r-1)+2}} \cdots W_{p+q}^{m_{2,r(p+q)}} \qquad (7)$$

. . .

. . .

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$$F_k = 0 = a_{k,1} W_1^{m_{k1}} W_2^{m_{k2}} \cdots W_{p+q}^{m_{k,p+q}} + a_{k,2} W_1^{m_{k,p+q+1}} W_2^{m_{k,p+q+2}} \cdots W_{p+q}^{m_{k,2(p+q)}} + \dots + a_{k,r} W_1^{m_{k,(p+q)(r-1)+1}} W_2^{m_{k,(p+q)(r-1)+2}} \cdots W_{p+q}^{m_{k,r(p+q)}} \qquad (8)$$

(2). <u>*Secondary Expansion:*</u>

$$dz_i = dW_i \qquad (1 \le i \le m) \qquad (9)$$

$$dx_i = dW_{m+i} \qquad (1 \le i \le n) \qquad (10)$$

$$[N_1 dz_1 + N_2 dz_2 + \dots + N_m dz_m] + [N_{m+1} dx_1 + N_{m+2} dx_2 + \dots + \dots + N_{m+n} dx_n] = N_{m+n+1} dW_{m+n+1} \qquad (11)$$

$$[N_{m+n+2} dz_1 + N_{m+n+3} dz_2 + \dots + N_{2m+n+1} dz_m] + [N_{2m+n+2} dx_1 + N_{2m+n+3} dx_2 + \dots + N_{2(m+n+1)-1} dx_n] = N_{2(m+n+1)} dW_{m+n+2} \qquad (12)$$

$$
\begin{array}{ccc}
. & . & . \\
. & . & . \\
. & . & .
\end{array}
$$

$$
\left[N_{(p+q-1)(m+n+1)+1} dz_1 + N_{(p+q-1)(m+n+1)+2} dz_2 + \ldots + N_{(p+q-1)(m+n+1)+m} dz_m \right] +
$$

$$
+ \left[N_{(p+q-1)(m+n+1)+m+1} dx_1 + N_{(p+q-1)(m+n+1)+m+2} dx_2 + \ldots + N_{(p+q)(m+n+1)-1} dx_n \right] =
$$

$$
= N_{(p+q)(m+n+1)} dW_{p+q} \qquad (13)
$$

The actual process of transforming a complete mathematical equation or a system of mathematical equations in terms of the above universal differential form representation is referred to as taking its *Multivariate Polynomial Transform.* The complete *reverse* process of going from a differential form representation back to the original complete mathematical equation or system of mathematical equations would be referred to as taking the *inverse* of a *Multivariate Polynomial Transform*. This would involve following a very unique integration process in the *Secondary Differential Expansion* for determining the complete analytical expression corresponding to each auxiliary variable. They in turn would each be substituting back into the *Primary Expansion* for arriving at the complete original expression in the form of "$f_k(z_m, x_n) = 0$".

Appendix A provides a list of the *Multivariate Polynomial Transform* corresponding to a variety of univariate and multivariate mathematical equations. For simplicity, both the Sine and Cosine function have been expressed as a rational combination of the Tangent function using the following basic trigonometric identity:

$$
Sin(x) = \frac{2Tan(x/2)}{1 + Tan^2(x/2)} \qquad (14)
$$

$$
Cos(x) = \frac{1 - Tan^2(x/2)}{1 + Tan^2(x/2)} \qquad (15)
$$

Just by increasing the total number of dependent and independent variables, the concept of a *Multivariate Polynomial Transform* is still applicable for including all *systems* of mathematical equations as well. However, space limitation prevents the inclusion of these types of mathematical equations as good illustrative examples.

2. Unique template for investigating the probable existence of complete *"general"* analytical solutions to DEs and systems of DEs by using a method of conjecture

A necessary condition for defining a complete unified analytical theory of integration is by substituting an initially assumed version with unknown coefficients to solve for of the universal differential form representation of all mathematical equations as described by equations (1) through (5) into *any type* of DEs and systems of DEs. This would always result into defining a very unique type of system of nonlinear simultaneous equations to solve for. The exact *numerical* solution sets obtained would then be used as a means of inverting the corresponding initially assumed differential expansions for arriving at an exact or approximate analytical solution that would be expressible only in terms of the algebraic and elementary basis functions.

Such an initially assumed differential expansion form would possess all the characteristics of a complete mathematical transform so we would refer to it as an *initially assumed Multivariate Polynomial Transform* or in short IAMPT.

The entire process of using an IAMPT for solving DEs and systems of DEs can be divided into two fundamental stages. The first, is the computational stage by which the corresponding nonlinear simultaneous equations of a DE or a system of DEs are numerically derived and completely solved for. The second, is the analytical stage by which every numerical solution set obtained is converted to pure analytical form. This would involve the process of identifying and solving for those exact integrals that are present in the *Secondary Expansion* which have successfully pass the complete test for exactness. From this exact integration process, the complete expression for each initially assumed set of auxiliary variables are obtained and substituted into the *Primary Expansion* for arriving at the complete analytical solution of the DE or system of DEs.

When selecting a suitable IAMPT for solving a particular DE or a system of DEs, the total number of unknown coefficients and floating point exponent values to solve for becomes purely arbitrary and should be as high as possible. This is necessary as a means of capturing those "*exact*" analytical solutions that can successfully resolve a DE or a system of DEs uniquely in terms of some combination of algebraic and elementary basis functions. The limitations on the total number of unknown coefficients and exponent values to solve for as defined from an IAMPT is generally set by the capacity of a computer system to handle extremely large numbers of very complex nonlinear simultaneous equations to solve for.

The resultant system of nonlinear simultaneous equations to solve for will always consist of an *infinite number* of exact numerical solutions sets provided that the IAMPT has been chosen large enough to contain the exact solution of the DE or system of DEs that is being solved for.

Some of the reasons that would account for the existence of such an infinite number of numerical solution sets are:

- The ability for an exact solution to a DE or a system of DEs to satisfy an infinite number of initial conditions.

- The permutation of each auxiliary variable present in both the *Primary* and *Secondary Expansion* for representing the same identical exact analytical solution of the DE or system of DEs.

- As a result of the natural computational process involved in solving for a very large number of complex nonlinear simultaneous equations, many numerical solutions sets obtained are expected to define numerous types of <u>*trivial*</u> algebraic identities from the process of inverting the corresponding IAMPT. Such type of identities will always be present in one form or another in the final representation of the analytical solution. A good example is the "$Sin^2(x) + Cos^2(x) = 1$" or any other algebraic variations of this trigonometric identify that would also include other types of basis functions as well.

- The presence of singular solutions.

- As a result of the natural computational process involved in solving for a very large number of complex nonlinear simultaneous equations, many numerical solutions sets obtained will naturally lead to the formation of one or several expressions in the *Secondary Expansion* that would be represented as a ratio of two exactly identical multivariate polynomials. These types of ratios would be considered as <u>*trivial ratios*</u> that would have to be all completely eliminated before any attempts is made for inverting a *Secondary Expansion.*

For every numerical solution set obtained as a result of solving for these nonlinear simultaneous equations there will always be a corresponding exact analytical solution satisfying a "<u>*unique*</u>" set of initial conditions. We would refer to the existence of such a type of exact analytical solution as an "*instance solution*". As there are an infinite number of possible numerical solution sets of the nonlinear simultaneous equations this will give rise to an infinite number of such *instance solutions*.

By consolidating a sufficient number of such instance solutions we can by using a method of conjecture potentially uncover more complete "*generalized*" versions of analytical solutions satisfying a *general* DE or a system of DEs. It therefore becomes quite imperative that as a result of solving for the nonlinear simultaneous equations we always continuously keep track of all instance analytical solutions obtained in the form of a table that we would like to refer as a "*numerically controlled system of analytics table*" or in short an (NCSA) table.

The following general system of PDEs of any order can be used for describing the most general case of an NCSA table:

$$G_k = G_k\left(z_1, z_2, \dots, z_m, x_1, x_2, \dots, x_n, \frac{\partial z_1}{\partial x_1}, \dots, \frac{\partial z_1}{\partial x_n}, \frac{\partial z_2}{\partial x_1}, \dots, \frac{\partial z_2}{\partial x_n}, \dots, \frac{\partial z_m}{\partial x_1}, \dots, \frac{\partial z_m}{\partial x_n}, \dots, \right.$$
$$, \dots, \frac{\partial^2 z_m}{\partial x_1 \partial x_1}, \dots, \frac{\partial^2 z_m}{\partial x_1 \partial x_n}, \dots, \frac{\partial^2 z_m}{\partial x_2 \partial x_1}, \dots, \frac{\partial^2 z_m}{\partial x_2 \partial x_n}, \dots,$$
$$\left., \dots, \frac{\partial^2 z_m}{\partial x_n^2}, \dots \dots, \frac{\partial^r z_m}{\partial x_n^r}\right) = 0 \qquad (16)$$

In this case, the NCSA table would be represented as follow:

$G_k = 0$		
Initial Conditions	*Coefficient values present in the DE or system of DEs*	*Exact analytical solution obtained using the Multivariate Polynomial Transform method*
$z_{01}, z_{02}, \dots, z_{0m}, x_{01}, \dots, x_{0n} \dots$	$a_0, b_0, c_0, \dots$	$U_1 = 0$
$z_{11}, z_{02}, \dots, z_{0m}, x_{01}, \dots, x_{0n} \dots$	$a_1, b_0, c_0, \dots$	$U_2 = 0$
$z_{01}, z_{12}, \dots, z_{0m}, x_{01}, \dots, x_{0n} \dots$	$a_0, b_1, c_0, \dots$	$U_3 = 0$
. . .	. . .	. . .
. . .	. . .	. . .
. . .	. . .	. . .

Table 2.1

where "$U_i = 0$" would then be referred to as an *instance solution* satisfying the unique set of parameters contained in this table.

Example (2.1). For the simple two dimensional case that can be represented by the following *general* first order ODE,

$$x\frac{dy}{dx} + ay + bx^n y^2 = 0 \tag{17}$$

the corresponding NCSA table may be constructed in the following manner:

$x\frac{dy}{dx} + ay + bx^n y^2 = 0$		
Initial Conditions	*Coefficient Values*	*Exact analytical solution obtained using the Multivariate Polynomial Transform method*
$x_0 = 1$ $y_0 = 1$	$a = 1.0$ $b = 1.0$ $n = -1.0$	$(-3x + x^{-1})y + 2 = 0$
$x_0 = 1$ $y_0 = 2$	$a = 1.2$ $b = -1.0$ $n = 2.0$	$(1.4x^{1.2} - x^2)y - 0.80 = 0$
$x_0 = 1$ $y_0 = -1$	$a = 1.2$ $b = 1.5$ $n = -2.0$	$(1.7x^{1.2} + 1.5^{-2})y + 3.2 = 0$
$x_0 = 1$ $y_0 = 2$	$a = 2.0$ $b = -1.0$ $n = 2.0$	$x^2 y(0.5 - \ln(x)) - 1 = 0$
$x_0 = 1$ $y_0 = -2$	$a = 1.5$ $b = 2.0$ $n = 3.0$	$(-2.75x^{1.5} + 2x^3)y - 1.5 = 0$
$x_0 = 1$ $y_0 = 1$	$a = 1.0$ $b = 1.0$ $n = 1.0$	$xy(1 + \ln(x)) - 1.0 = 0$
$x_0 = 1$ $y_0 = -1$	$a = -1.0$ $b = 1.5$ $n = -1.0$	$x^{-1}y(-1 + 1.5\ln(x)) - 1.0 = 0$

Table 2.2

The evidence gathered from each of the above *instance solutions* allows us to conclude by conjecture that:

$$f_1(x, y) = 0 = (Ax^B + Cx^D)y + E \tag{18}$$

and:

$$f_2(x, y) = 0 = x^A y(B + C\ln(x)) + D \tag{19}$$

both appear to be perfect candidates for the general exact analytical solution of the ODE where the coefficients "A", "B", "C", "D" and "E" are to be expressed in terms of the coefficients "a", "b", "n" and the initial conditions of the ODE.

By substituting any one of these generally assumed analytical solution into the ODE and equating like terms to zero, we can derive a complete relationship that can exist between the known and the unknown coefficients.

The general formula used for determining the first derivative of "y" is:

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y} \tag{20}$$

In our first assumption that "$f_1(x, y) = 0$" and upon equating like terms to zero in the ODE, this would define the following system of equations to solve for:

$$A(a - B) = 0 \tag{21}$$

$$C(a - n) - bE = 0 \tag{22}$$

$$(Ax_0^a + Cx_0^n)y_0 + E = 0 \tag{23}$$

with exact solution [Mikalajunas 2015]:

$$A \neq 0 \tag{24}$$

$$B = a \tag{25}$$

$$C = \frac{-Abx_0^a y_0}{a + bx_0^n y_0 - n} \tag{26}$$

$$E = \frac{(a - n)C}{b} \qquad (a \neq n) \tag{27}$$

Following the same type of logic for our second assumption that "$f_2(x, y) = 0$", this would define the following system of nonlinear equations to solve for:

$$B(a - n) - C - bD = 0 \tag{28}$$

$$C(a - n) = 0 \tag{29}$$

$$x_0^n y_0(B + C\ln(x_0)) + D = 0 \tag{30}$$

with exact solution [Mikalajunas 2015]:

$$D \neq 0 \tag{31}$$

$$C = -bD \tag{32}$$

$$B = \frac{-D}{x_0^n y_0} - C\ln(x_0) = \frac{-D - Cx_0^n y_0 \ln(x_0)}{x_0^n y_0} \tag{33}$$

Without having constructed the NCSA table it would have been very difficult to have correctly arrived at the complete "*general analytical solution*" of this first order ODE that would satisfy all initial conditions as well. There are currently no known traditional method of integration capable of deriving complete "*general*" closed form solutions to "*any type*" of DEs and systems of DEs that would be entirely based on the use of a well defined "*exact*" method of computational analysis such as the one being proposed in this paper.

The very unique mathematical properties of an IAMPT when substituted into a DE or a system of DEs allows for all initial conditions to be fully accounted for. This is because the exact integration process that is performed in the *Secondary Expansion* for determining an exact expression for each auxiliary variable must always include the constant of integration which in turn would automatically define each of their initial values. For every *instance solution* obtained, the overall contribution of each of these initial values for the auxiliary variables can easily succeed in completely matching the initial conditions of a DE or a system of DEs. This becomes very obvious by noticing that the *Primary Expansion* of an IAMPT is always expressed as some algebraic combination of initially assumed auxiliary variables as well as known auxiliary variables. Its the initial values of each of these auxiliary variables that can easily be adjusted numerically for satisfying the overall initial conditions of a DE or a system of DEs by solving for the type of system of nonlinear equations in which there will always be more unknowns than available equations to solve them.

Based on our previous example for the general first order ODE, we notice that every *instance solution* obtained would potentially lead towards defining a more generalized version of the exact analytical solution. It is only through the painstaking gathering of this type of information in the form of a large distribution sample of *instance solution* sets can we succeed in determining only by the method of conjecture complete general closed form solutions of a DE or a system of DEs.

The complete consolidation of a large number of these generalized exact analytical solutions which would be the result of having solved for a large number of very distinct classes of DEs and systems of DEs can potentially lead to defining some very fundamental theorems. Case in point is the *superposition theorem* being the result of having solved mostly by *trial and error* a very distinct class of linear second order ODEs.

By consolidating each of these fundamental mathematical theorems into one gigantic universal theory might represent our most *realistic* hope yet of ever arriving at some *unified theory of everything.*

3. The *theory of everything* not just about modern physics anymore

To this day, the most accepted definition of the *theory of everything* is that it must remain an integral part of modern physics on the principle of defining a unique Space-Time model that would explain all the basic laws of this universe.

However, what appears to be clearly lacking in our attempt to create such a *grandiose physical theory* for explaining everything about this universe is an equivalent *grandiose mathematical theory* that would have to succeed in explaining everything about the complete analytical integration of all types of DEs as well as all types of systems of DEs.

Because DEs are completely universal and not linked to any specific area of the physical sciences, there is really no evidence to support that modern physics is the only real subject by which a complete theory of everything may be entirely constructed from.

Rather, it would have to be through the application of some unified theory of analytical integration that a theory of everything would be achievable. This would be result of consolidating each fundamental theorem associated with a single *Unified Physical System* at a time into one gigantic theory capable of explaining everything about this physical universe.

The following block diagram suggests such a scenario by which DEs and systems of DEs would play a central role for establishing such a theory of everything where each *Unified Physical System* would have its own very unique story to tell us that in the end we would need to know about:

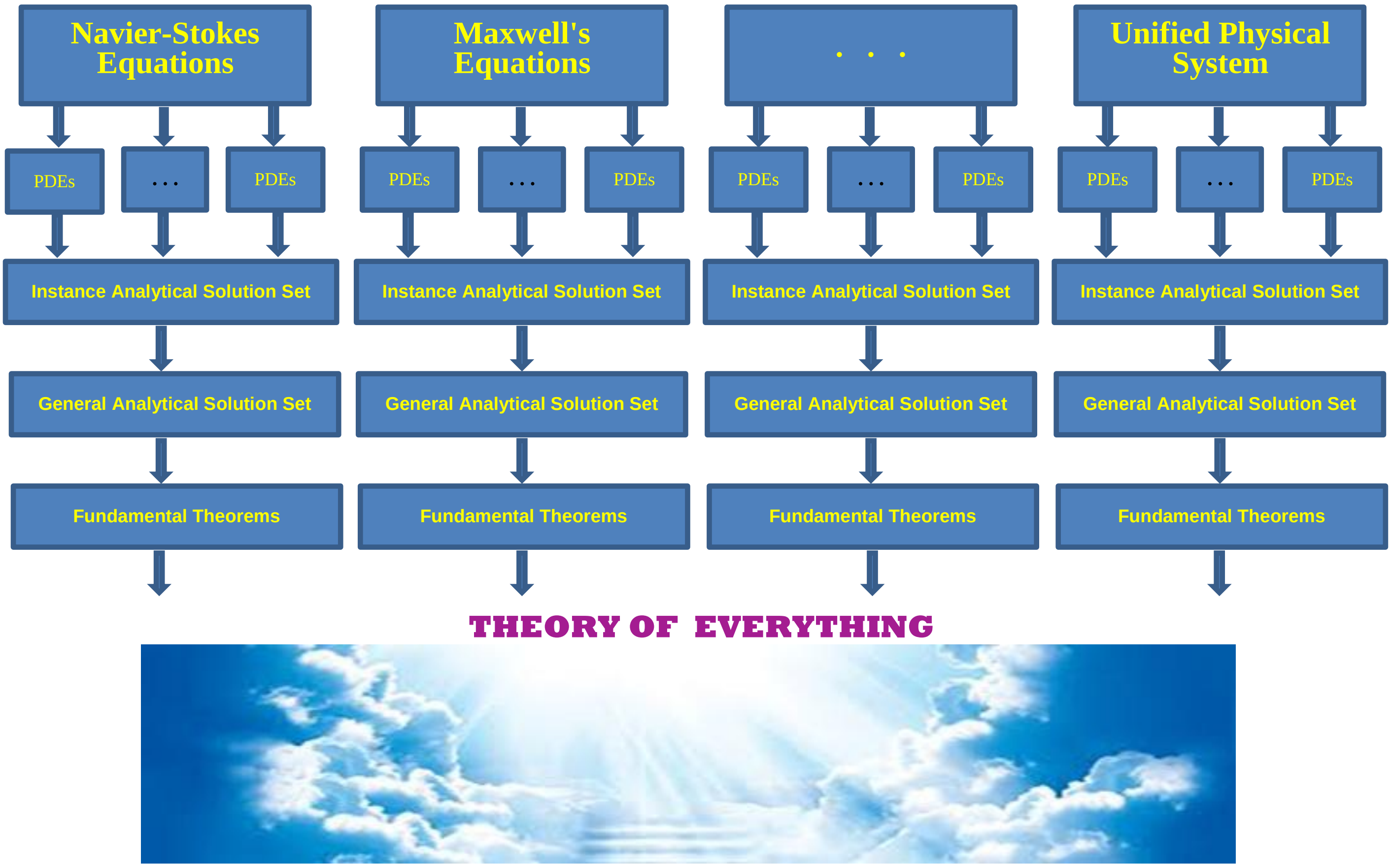

Figure 3.1

The very mathematical nature of our proposed unified theory of analytical integration is built on the principle that "***analytical solutions***" to DEs and systems of DEs *must* be constructed entirely on pure computational analysis.

In the absence of a unified theory of analytical integration, our understanding of the physical sciences cannot be complete as our method of analysis becomes reduced to a process that is mostly governed by unpredictable events. Because Calculus is so deeply embedded into all of the physical sciences, how can we expect to devise a *theory of everything* without the use of some form of a *unified analytical theory of integration that would be entirely driven by some well defined method of exact computational analysis* ?

4. Complete numerical example for a second order ODE

In our first example for the general first order ODE, we highlighted the importance of creating a special type of table called the *NCSA table* for providing much greater visibility towards the acquisition of *general* closed form solutions. Such a table would be constructed on the principle of creating a special type of database that would consist of a large number of instance solutions each satisfying a predetermined number of control parameters that would include initial conditions and all the variable coefficients that take part in defining a DE or a system of DEs.

Corresponding to a unique set of control parameters would define a unique instance solution that would be obtained as a result of substituting an IAMPT into a DE or a system of DEs and numerically solving for the resultant system of nonlinear simultaneous equations. This would be followed by the complete transformation of the resultant IAMPT into a unique instance solution.

As the number of instance solutions grows, this would allow for much greater insight in determining by method of conjecture if a more *general* analytical solution actually exists. These types of closed form solutions have a far greater capacity towards a much better understanding on the very long term behavior of a physical system. By consolidating each and every *general* analytical solutions obtained over a large class of DEs and systems of DEs into basic fundamental theorems, an even far much better understanding of the same physical system is possible. Only as we progress further in the complete formulation of a large number of such specialized fundamental theorems can we expect to move closer towards the complete development of some form of a *theory of everything*.

In the following example, we have randomly selected a second order ODE and provided a complete step by step process for arriving at its complete exact analytical solution satisfying all initial conditions.

Example (4.1). Starting with the following *second* order ODE:

$$y\frac{d^2y}{dx^2} \;-\; \left(\frac{dy}{dx}\right)^2\left\{1 \;-\; \frac{dy}{dx}Sin(y) \;-\; y\frac{dy}{dx}Cos(y)\right\} \;=\; 0 \tag{34}$$

there are two *external* inputs that are defined in terms of the Sine and Cosine function.

For the sake of simplicity in our analysis, we can use the following identities for expressing each of the two trigonometric functions as a *rational* combination of the half angle tangent function:

$$Sin(u) = \frac{2Tan(u/2)}{1+\;Tan^2(u/2)} \tag{35}$$

$$Cos(u) = \frac{1-\;Tan^2(u/2)}{1+\;Tan^2(u/2)} \tag{36}$$

Based on the use of this half angle formula for the Tangent function, we begin by selecting a much simpler alternative representation for the Sine and Cosine function by defining:

$$H \;=\; Tan(y/2) \;=\; W_{p+1} \tag{37}$$

where "p" is the total number of arbitrarily defined auxiliary variables from the IAMPT that will be selected for solving this second order ODE.

For this choice of auxiliary variable the corresponding *Multivariate Polynomial Transform* would be defined as follow:

(1). *Primary Expansion:*

$$H(W_{p+1}) = W_{p+1} \tag{38}$$

(2). *Secondary Expansion:*

$$dy = dW_2 \tag{39}$$

$$0 \cdot dx + (1 + W_{p+1}^2)dy = 2dW_{p+1} \tag{40}$$

We can arbitrarily select our IAMPT as consisting of a maximum of *five* arbitrarily defined auxiliary variables so that "$p = 5$". There will be a total number of *six* terms in the *Primary Expansion* so that "$u_P = 6$" and a total number of *four* terms in the *Secondary Expansion* so that "$u_S = 4$". Because there is only one external input in the form of the Tangent function for representing both the Sine and Cosine function, "$q = 1$" thereby bringing the total number of auxiliary variables in the entire initially assumed expansion to *six*.

For this selection of parameters, the corresponding IAMPT for solving this second order ODE can be expanded as:

(1). *Primary Expansion:*

$$F = 0 = a_1 W_1^{m_1} W_2^{m_2} \cdots W_6^{m_6} + a_2 W_1^{m_7} W_2^{m_8} \cdots W_6^{m_{12}} + \ldots + \\ + \ldots + a_6 W_1^{m_{31}} W_2^{m_{32}} \cdots W_6^{m_{36}} \tag{41}$$

(2). *Secondary Expansion:*

$$dx = dW_1 \tag{42}$$

$$dy = dW_2 \tag{43}$$

$$N_1 dx + N_2 dy = N_3 dW_3 \tag{44}$$

$$N_4 dx + N_5 dy = N_6 dW_4 \tag{45}$$

$$N_7 dx + N_8 dy = N_9 dW_5 \tag{46}$$

$$N_{10} dx + N_{11} dy = N_{12} dW_6 \tag{47}$$

where:

$$\mathrm{N}_1 = \mathrm{b}_1 \mathrm{W}_1^{\mathrm{m}_1} \mathrm{W}_2^{\mathrm{m}_2} \cdots \mathrm{W}_6^{\mathrm{m}_6} + \ldots + \mathrm{b}_4 \mathrm{W}_1^{\mathrm{m}_{19}} \mathrm{W}_2^{\mathrm{m}_{20}} \cdots \mathrm{W}_6^{\mathrm{m}_{24}} \tag{48}$$

$$\mathrm{N}_2 = \mathrm{b}_5 \mathrm{W}_1^{\mathrm{m}_{25}} \mathrm{W}_2^{\mathrm{m}_{26}} \cdots \mathrm{W}_6^{\mathrm{m}_{30}} + \ldots + \mathrm{b}_8 \mathrm{W}_1^{\mathrm{m}_{45}} \mathrm{W}_2^{\mathrm{m}_{46}} \cdots \mathrm{W}_6^{\mathrm{m}_{48}} \tag{49}$$

$$\begin{matrix} . & . & . \\ . & . & . \\ . & . & . \end{matrix}$$

$$N_9 \;=\; b_{33} W_1^{m_{193}} W_2^{m_{194}} \cdots W_6^{m_{198}} \;+\; \dots \;+\; b_{36} W_1^{m_{211}} W_2^{m_{212}} \cdots W_6^{m_{216}} \tag{50}$$

To account for the presence of both the Sine and Cosine function inside the ODE we must define the following three multivariate polynomials with *known* coefficient values:

$$N_{10} \;=\; 0 \tag{51}$$

$$N_{11} \;=\; 1 \;+\; W_{p+1}^2 \;=\; 1 \;+\; W_6^2 \tag{52}$$

and :

$$N_{12} \;=\; 2 \tag{53}$$

We can compute the total number of unknowns to solve for in our IAMPT using the following general formula with "$p = 5$", "$u_P = 6$", "$u_s = 4$" and "$q = 1$" :

$$N_{\text{Total}} \;=\; N_{\text{Primary}} \;+\; N_{\text{Secondary}} \tag{54}$$

$$= \; u_P(p+q+1) \;+\; 3\,u_S(p+q+1)(p-2) \tag{55}$$

$$= \; 6(5+1+1) \;+\; 3(4)(5+1+1)(5-2) \tag{56}$$

$$= \; 6(7) \;+\; 12(7)(3) \;=\; 42 \;+\; 252 \;=\; 294 \tag{57}$$

We can express the entire ODE in terms of the following single large multivariate polynomial by taking its complete *Multivariate Polynomial Transform* using equation (35), (36) and (37):

$$W_2 \frac{d^2Y}{dX^2} \;-\; \left(\frac{dY}{dX}\right)^2 \left\{ 1 \;-\; \frac{dY}{dX}\left(\frac{2W_{p+1}}{1+W_{p+1}^2}\right) \;-\; W_2 \frac{dY}{dX}\left(\frac{1-W_{p+1}^2}{1+W_{p+1}^2}\right) \right\} \;=\; 0 \tag{58}$$

where we have selected:

$$W_1 \;=\; X \tag{59}$$

$$W_2 \;=\; Y \tag{60}$$

and where capital letters are used to indicate that a transformation from rectangular to complete multivariate polynomial form has taken place.

A very general formula for calculating the first derivative of a general IAMPT may be defined as:

$$\frac{dY}{dX} = \frac{P_1}{Q_1} = \frac{-\dfrac{\partial F}{\partial W_1}\displaystyle\prod_{k=1}^{p+q-2} N_{3k} - \sum_{j=3}^{p+q}\left\{N_{3j-8}\frac{\partial F}{\partial W_j}\prod_{\substack{k=1\\k\neq j-2}}^{p+q-2} N_{3k}\right\}}{\dfrac{\partial F}{\partial W_2}\displaystyle\prod_{k=1}^{p+q-2} N_{3k} + \sum_{j=3}^{p+q}\left\{N_{3j-7}\frac{\partial F}{\partial W_j}\prod_{\substack{k=1\\k\neq j-2}}^{p+q-2} N_{3k}\right\}} \tag{61}$$

where both P_1 and Q_1 are each defined as a multivariate polynomial.

By expressing this equation in the following form:

$$\frac{dY}{dX}Q_1 - P_1 = 0 \tag{62}$$

we can *numerically* determine the second and higher derivatives of the dependent variable by successively differentiating both sides of this equation using the product rule and the general formula provided in equation (61).

Section 6 describes an *exact computational method* for calculating the various derivative of a product of two or more expressions using the *Multinomial Expansion Theorem* without resorting to any type of symbolic algebraic manipulation.

Our system of nonlinear simultaneous equations of interest to solve for is obtained by first taking the various derivatives of equation (58) that represents the ODE in complete multivariate polynomial form. This would include the various derivatives of each auxiliary variable that define the *Multivariate Polynomial Transform* of the single external input as provided in equations (37) through (40) which are "W_2" and "W_{p+1}".

Next, we replace the various derivatives of the dependent variable in equation (58) with the computed values obtained from the various derivatives of our IAMPT using equations (61) and (62).

The resultant nonlinear simultaneous equations can then be numerically solved for using various optimization technics where our objective function to be minimized would be represented as the sum of the squares of each of the various derivatives of equation (58):

$$G_n = \frac{d^n}{dx^n}\left[W_2\frac{d^2Y}{dX^2} - \left(\frac{dY}{dX}\right)^2\left\{1 - \frac{dY}{dX}\left(\frac{2W_{p+1}}{1+W_{p+1}^2}\right) - W_2\frac{dY}{dX}\left(\frac{1-W_{p+1}^2}{1+W_{p+1}^2}\right)\right\}\right] \tag{63}$$

Our main objective function to minimize would thus be represented as:

$$F = \sum_n G_n^2 \tag{64}$$

By succeeding in completely minimizing the above objective function to zero, the corresponding inverse *Multivariate Polynomial Transform* would define an *exact* analytical solution of the ODE that would satisfy a completely random set of initial conditions. Such a type of analytical solution obtained was earlier described as an *instance solution*. Any numerical solution set that would depart from this minima would represent only an approximation of the actual *exact* analytical solution of the ODE. The further away we are from this minima, the greater will be the error of approximation between the exact analytical solution and the one arrived at.

As we are only interested in obtaining as many exact instance solutions as possible each satisfying their own very unique initial conditions when solving for these nonlinear simultaneous equations, we must treat all initial values of the auxiliary variables as unknown coefficients to solve for in order to achieve the highest numerical solution set rate possible. It is the initial values of each auxiliary variable defined from the exact integration of a *Secondary Expansion* that when substituted into the *Primary Expansion* would completely define the initial conditions of a DE or a system of DEs. Keep in mind that our primary objective in this type of analysis is to acquire as many instance solutions as possible so that by applying a unique method of conjecture, we would be able to arrive at a more *generalized* version of the closed form solution satisfying a DE or a system of DEs.

For solving these nonlinear simultaneous equations using an optimization technic, all gradient calculations can become fairly complex quite often leading to very unpredictable results. A preferred method of optimization that generally does not require any type of gradient calculations is the *pattern search method* as described in the book by [Adby and Dempster 1974].

All calculations involving very high order partial derivatives of an IAMPT require a great deal amount of precision and thus not recommended to be performed on a regular PC. Instead, the entire computational process would become more manageable if it were conducted on a very *advanced super computer system*.

Future generations of computer hardware may begin to take full advantage of the multistate quantum bit (or Qubit) technology originating from the principles of quantum physics as they are expected to become much more powerful than the conventional types that operate only on the principle of two states being a 0 or 1. Over time the semi conductor industry that currently powers our conventional computers will eventually reach its own physical limitations in terms of its ability for designing super fast switching devices. Some estimate that because of the multi state capability of a Qubit, it would succeed in outperforming even the most powerful conventional super computer of our time in the *billion-fold* under the most demanding condition of computational requirements.

Upon the gathering of as many numerical solution sets of the nonlinear simultaneous equations as possible, the next step to follow afterwards is in the complete construction of an NCSA table that would be very specific to the particular DE or system of DEs being solved for.

For solving our second order ODE, we were able to acquire a large number of instance solutions each satisfying its own very unique set of initial conditions that would also become the initial conditions of the ODE as well. The greater the number of instance solutions that can be gathered and fully documented accordingly, the greater is the amount of information that can be made available for facilitating the entire process of deducing by *conjecture* the complete general exact analytical solution of the second order ODE.

The NCSA table for our example of a second order ODE would therefore appear as follow:

$$y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 \left\{1 - \frac{dy}{dx}Sin(y) - y\frac{dy}{dx}Cos(y)\right\} = 0$$		
Initial Conditions	*Coefficient Values*	*Exact analytical solution obtained using the Multivariate Polynomial Transform method*
$x_0 = -1.28$ $y_0 = 1.591$	N/A	$Cos(y) + x + 1.662 - 0.778\, ln(y)$
$x_0 = 0.2473$ $y_0 = 0.76$	N/A	$Cos(y) + x - 0.111 + 3.138\, ln(y)$
$x_0 = -3.2542$ $y_0 = 1.442$	N/A	$Cos(y) + x + 2.662 + 1.267\, ln(y)$
$x_0 = 1.2223$ $y_0 = 3.865$	N/A	$Cos(y) + x + 0.579 - 0.778\, ln(y)$
$x_0 = -0.837$ $y_0 = 2.691$	N/A	$Cos(y) + x - 1.051 + 2.817\, ln(y)$
$x_0 = -1.668$ $y_0 = 1.877$	N/A	$Cos(y) + x - 0.871 + 4.511\, ln(y)$

Table 4.1

Based entirely on the information provided in this table and following the same basic procedure as was done in our first example for a first order ODE, a plausible conjecture for the exact analytical solution of this second order ODE satisfying all initial conditions would be:

$$f(x,y) = 0 = Cos(y) + x + A_1 + A_2\, ln(y) \tag{65}$$

where "A_1" and "A_1" are each defined as a constant of integration.

5. Complete numerical example for a second order PDE

For PDEs and for systems of ODEs as well as for system of PDEs, the NCSA table is always constructed in pretty much the same way as we did for the first order ODE described in the first example. In all cases involved, we always allow for the initial conditions of a DEs or a system of DEs to become part of the unknown coefficients to solve for as originally defined from within an IAMPT.

In the following example, we have randomly selected a second order PDE and provided a complete step by step process for arriving at its complete exact analytical solution satisfying all initial conditions.

Example (5.1). For the following second order PDE :

$$x_2\left(\frac{\partial^2 z}{\partial x_1 \partial x_2}\right) - \frac{\partial z}{\partial x_1} - x_1 x_2^2 Sin(x_1 x_2) = 0 \tag{66}$$

there is only one external input that is defined in terms of the Sine function.

As we did in our previous example for a second order ODE, we can use the following trigonometric identity for expressing the Sine function as a *rational* combination of the tangent function:

$$f(x_1, x_2) = Sin(x_1 x_2) = \frac{2Tan(x_1 x_2/2)}{1 + Tan^2(x_1 x_2/2)} \tag{67}$$

Based on the use of this half angle formula for the Tangent function, we begin by selecting a much simpler alternative representation for the Sine function by defining:

$$H(x_1, x_2) = W_{p+1} = Tan(x_1 x_2/2) = Tan(W_2 W_3/2) \tag{68}$$

where "p" is the total number of arbitrarily defined auxiliary variables from the IAMPT that will be selected for solving this second order PDE.

For this choice of auxiliary variable the corresponding *Multivariate Polynomial Transform* would be defined as follow:

(1). <u>*Primary Expansion:*</u>

$$H(W_{p+1}) = W_{p+1} \tag{69}$$

(2). <u>*Secondary Expansion:*</u>

$$0 \cdot dz + \left(1 + W_{p+1}^2\right) W_3 dx_1 + \left(1 + W_{p+1}^2\right) W_2 dx_2 = 2dW_{p+1} \tag{70}$$

where we have selected:

$$W_1 = z \tag{71}$$

$$W_2 = x_1 \tag{72}$$

and:

$$W_3 = x_2 \tag{73}$$

We can arbitrarily select our IAMPT as consisting of a maximum of *eight* arbitrarily defined auxiliary variables so that "$p = 8$". There will be a total number of *eight* terms in the *Primary Expansion* so that "$u_P = 8$" and a total number of *four* terms in the *Secondary Expansion* so that "$u_S = 4$". Because there is only one external input in the form of the Tangent function for representing only the Sine function, "$q = 1$" thereby bringing the total number of auxiliary variables in the entire initially assumed expansion to *nine*.

For this selection of parameters, the corresponding IAMPT for solving this second order PDE can be expanded as:

(1). <u>*Primary Expansion:*</u>

$$F = 0 = a_1 W_1^{m_1} W_2^{m_2} \cdots W_9^{m_9} + a_2 W_1^{m_{10}} W_2^{m_{11}} \cdots W_9^{m_{18}} + \ldots + \\ + \ldots + a_8 W_1^{m_{64}} W_2^{m_{65}} \cdots W_9^{m_{72}} \qquad (74)$$

(2). <u>*Secondary Expansion:*</u>

$$dz = dW_1 \qquad (75)$$

$$dx_1 = dW_2 \qquad (76)$$

$$dx_2 = dW_3 \qquad (77)$$

$$N_1 dz + N_2 dx_1 + N_3 dx_2 = N_4 dW_4 \qquad (78)$$

$$N_5 dz + N_6 dx_1 + N_7 dx_2 = N_8 dW_5 \qquad (79)$$

$$N_9 dz + N_{10} dx_1 + N_{11} dx_2 = N_{12} dW_6 \qquad (80)$$

$$N_{13} dz + N_{14} dx_1 + N_{15} dx_2 = N_{16} dW_7 \qquad (81)$$

$$N_{17} dz + N_{18} dx_1 + N_{19} dx_2 = N_{20} dW_8 \qquad (82)$$

$$N_{21} dz + N_{22} dx_1 + N_{23} dx_2 = N_{24} dW_9 \qquad (83)$$

where :

$$N_1 = b_1 W_1^{m_1} W_2^{m_2} \cdots W_9^{m_9} + \ldots + b_4 W_1^{m_{28}} W_2^{m_{29}} \cdots W_9^{m_{36}} \qquad (84)$$

$$N_2 = b_5 W_1^{m_{37}} W_2^{m_{38}} \cdots W_9^{m_{45}} + \ldots + b_8 W_1^{m_{64}} W_2^{m_{65}} \cdots W_9^{m_{72}} \qquad (85)$$

$$\begin{array}{ccc} . & . & . \\ . & . & . \\ . & . & . \end{array}$$

$$N_{20} = b_{77} W_1^{m_{685}} W_2^{m_{686}} \cdots W_9^{m_{693}} + \ldots + b_{80} W_1^{m_{712}} W_2^{m_{713}} \cdots W_9^{m_{720}} \qquad (86)$$

To account for the presence of the Sine function inside the PDE we must define the following three multivariate polynomials with <u>*known*</u> coefficient values:

$$\mathrm{N}_{21} = 0 \tag{87}$$

$$\mathrm{N}_{22} = (1 + \mathrm{W}_{\mathrm{p+1}}^2)W_3 = (1 + \mathrm{W}_9^2)W_3 \tag{88}$$

$$\mathrm{N}_{23} = (1 + \mathrm{W}_{\mathrm{p+1}}^2)W_2 = (1 + \mathrm{W}_9^2)W_2 \tag{89}$$

and :

$$\mathrm{N}_{24} = 2 \tag{90}$$

We can compute the total number of unknowns to solve for in our IAMPT using the following general formula with "$n = 2$", "$p = 8$", "$u_P = 8$", "$u_s = 4$" and "$q = 1$" :

$$N_{Total} = N_{Primary} + N_{Secondary} \tag{91}$$

$$= u_P(p+q+1) + u_S(p+q+1)(n+2)(p-n-1) \tag{92}$$

$$= 8(8+1+1) + 4(8+1+1)(2+2)(8-2-1) \tag{93}$$

$$= 8(10) + 4(10)(4)(5) = 80 + 800 = 880 \tag{94}$$

As in the case for the second order ODE, the entire PDE may be expressed in terms of the following single large multivariate polynomial by taking its complete *Multivariate Polynomial Transform* using equations (68) through (73):

$$W_3\left(\frac{\partial^2 Z}{\partial W_2 \partial W_3}\right) - \frac{\partial Z}{\partial W_2} - 2W_2W_3^2\left(\frac{W_{p+1}}{1+W_{p+1}^2}\right) = 0 \tag{95}$$

where we have selected:

$$W_1 = z \tag{96}$$

$$W_2 = x_1 \tag{97}$$

$$W_3 = x_2 \tag{98}$$

and where capital letters are used to indicate that a transformation to complete multivariate polynomial form has taken place.

A very general formula for calculating the first partial derivative of our IAMPT that is based on the use of the product rule and the *Multinomial Expansion Theorem* can also be derived in a very similar manner as was done in our last example of a second order ODE which was provided in equation (61).

Our system of nonlinear simultaneous equations of interest to solve for is obtained by first taking the various partial derivatives of equation (95) that represents the PDE in complete multivariate polynomial form. This would include the various partial derivatives of each auxiliary variable that define the *Multivariate Polynomial Transform* of the single external input as provided in equations (68) through (73) which are "W_2", "W_3" and "W_{p+1}".

Next, we replace the various partial derivatives of the dependent variable in equation (95) with the computed values obtained from the various partial derivatives of our IAMPT.

The resultant nonlinear simultaneous equations can then be numerically solved for using various optimization technics where our objective function to be minimized would be represented as the sum of the squares of each of the various partial derivatives of equation (95):

$$G_i \;=\; \frac{\partial^{m_1}}{\partial W_2^{m_1}}\,\frac{\partial^{m_2}}{\partial W_3^{m_2}}\,\frac{\partial^{m_3}}{\partial W_4^{m_3}}\;\cdots\;\frac{\partial^{m_k}}{\partial W_{p+1}^{m_k}}\cdots\left\{W_3\left(\frac{\partial^2 Z}{\partial W_2 \partial W_3}\right) \;-\; \frac{\partial Z}{\partial W_2} \;-\; 2W_2W_3^2\left(\frac{W_{p+1}}{1+W_{p+1}^2}\right)\right\} \;=\; 0 \tag{99}$$

Our main objective function to minimize would therefore be represented as:

$$F \;=\; \sum_i G_i^2 \tag{100}$$

By succeeding in completely minimizing the above objective function to zero, the corresponding inverse *Multivariate Polynomial Transform* would define an *exact* analytical solution of the PDE that would satisfy a completely random set of initial conditions. Such a type of analytical solution obtained was earlier described as an *instance solution*. Any numerical solution set that would depart from this minima would represent only an approximation of the actual *exact* analytical solution of the PDE. The further away we are from this minima, the greater will be the error of approximation between the exact analytical solution and the one arrived at.

As we are only interested in obtaining as many exact instance solutions as possible each satisfying their own very unique initial conditions when solving for these nonlinear simultaneous equations, we must treat all initial values of the auxiliary variables as unknown coefficients to solve for in order to achieve the highest numerical solution set rate possible. It is the initial values of each auxiliary variable defined from the exact integration of a *Secondary Expansion* that when substituted into the *Primary Expansion* would completely define the initial conditions of a DE or a system of DEs. Keep in mind that our primary objective in this type of analysis is to acquire as many instance solutions as possible so that by applying a unique method of conjecture, we would be able to arrive at a more *generalized* version of the closed form solution satisfying a DE or a system of DEs.

For solving these nonlinear simultaneous equations using an optimization technic, all gradient calculations can become fairly complex quite often leading to very unpredictable results. A preferred method of optimization that generally does not require any type of gradient calculations is the *pattern search method* as described in the book by [Adby and Dempster 1974].

All calculations involving very high order partial derivatives of an IAMPT require a great deal amount of precision and thus not recommended to be performed on a regular PC. Instead, the entire computational process would become more manageable if it were conducted on a very *advanced super computer system*.

Upon the gathering of as many numerical solution sets of the nonlinear simultaneous equations as possible, the next step to follow afterwards is in the complete construction of an NCSA table that would be very specific to the particular DE or system of DEs being solved for.

For solving our second order PDE, we were able to acquire a large number of instance solutions each satisfying its own very unique set of initial conditions that would also become the initial conditions of the PDE as well. The greater the number of instance solutions that can be gathered and fully documented accordingly, the greater is the amount of information that can be made available for facilitating the entire process of deducing by *conjecture* the complete general exact analytical solution of the second order PDE.

The NCSA table for our example of a second order PDE would therefore appear as follow:

$x_2\left(\dfrac{\partial^2 z}{\partial x_1 \partial x_2}\right) - \dfrac{\partial z}{\partial x_1} - x_1 x_2^2 Sin(x_1 x_2) = 0$		
Initial Conditions	*Coefficient Values*	*Exact analytical solution obtained using the Multivariate Polynomial Transform method*
$x_{01} = 3.61$ $x_{02} = 1.771$	N/A	$2x_2 x_1^{1.68} + Sin(\ln[x_2^{-1.6}] + x_2^{0.78}) - Sin(x_1 x_2) - z = 0$
$x_{01} = 1.29$ $x_{02} = -1.88$	N/A	$x_2 \sqrt[6]{x_1^{0.23} + 1.78} + 1.22\ln\left(\sqrt{x_2^2+1} + 3.5\right) - Sin(x_1 x_2) - z = 0$
$x_{01} = 3.555$ $x_{02} = 2.76$	N/A	$0.56x_2 e^{x_1^{-0.46}} - 4.6Tan(x_2^{1.86} + \sqrt[4]{x_2^{1.1} - 6.1}) - Sin(x_1 x_2) - z = 0$
$x_{01} = -0.723$ $x_{02} = 1.58$	N/A	$3.06x_2 Sinh(x_1^2) - 2.45x_2^{1.46\sqrt{x_2^{3.1}-2.3}} - Sin(x_1 x_2) - z = 0$

Table 5.1

Based entirely on the information provided in the above table, there appears to be no obvious patterns by which a plausible conjecture for the exact analytical solution of this second order PDE satisfying all initial conditions can be made.

The main reason for this is that the exact analytical solution consists of a number of expressions that are completely arbitrarily defined. This would call for the development of a very sophisticated method of *comparison analysis* just for identifying those arbitrary expressions that are present in all of the instance solutions obtained. Some of these arbitrarily defined expressions may be easier to detect than others for establishing a plausible conjecture by which a complete analytical solution of the PDE satisfying all initial conditions may be arrived at.

In the final analysis, all results would be pointing towards the following expression as representing the complete exact analytical solution of the PDE satisfying all initial conditions:

$$f(z, x_1, x_2) = 0 = x_2 \varphi_1(x_1) + \varphi_2(x_2) - Sin(x_1 x_2) - z \tag{101}$$

where upon conducting such a type of special method of *comparison analysis*, each of the expression for "$\varphi_1(x_1)$" and "$\varphi_2(x_2)$" would eventually have been singled out in the end as completely arbitrarily defined.

Once again it is very important to mention that without having constructed the NCSA table it would have been virtually impossible to have correctly arrived at the complete *general* analytical solution of this second order PDE satisfying all initial conditions.

6. Exact computational method for calculating the various derivatives and partial derivatives of an initially assumed Multivariate Polynomial Transform (IAMPT)

The method of substituting an IAMPT into a DE or a system of DEs for defining a valid system of nonlinear simultaneous equations to solve for requires that the numerical values of each of the various derivatives of the DE or system of DEs become equal to that of an IAMPT. An alternative method is to substitute an IAMPT into a DE or a system of DEs and afterwards equating like multivariate polynomial terms to zero. However, this would result into defining a completely invalid system of nonlinear simultaneous equations to solve for as it would automatically impose a major restriction on each auxiliary variable for becoming totally independent from one another. The evidence is clearly provided in Appendix A where as you will notice that for the vast majority of the cases involved, it is always necessary to maintain a certain degree of dependency among auxiliary variables especially when very complex mathematical equations are involved.

The actual process of computing the _exact_ values for the various derivatives and partial derivatives of an IAMPT to any desirable order of differentiation without any loss of accuracy whatsoever can always be reduced at a *computational level.* The reason for this is that we take full advantage of a well known fact in numerical analysis that taking the various derivatives of a product of several expressions is very much similar to algebraically expanding to some exponent value the sum of several terms. The only major difference between the two is that in the case of differentiation, exponentiation becomes treated purely as an order of differentiation while all the remaining algebraic operations remain completely identical.

For the simple case of differentiating a product involving only two expressions, this would require the use of the *Binomial Expansion Theorem* which is defined by:

$$\frac{d^n}{dx^n} fg = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} \tag{102}$$

where:

$$\binom{n}{k} = B_{n,k} = \frac{n!}{k!\,(n-k)!} \tag{103}$$

are the binomial coefficients and where it is to be clearly understood that all exponent values are to be treated purely as order of differentiation.

In complete expanded form, the various derivatives of a product consisting of *two* expressions can be *symbolically* defined as :

$$[f+g]^{(n)} = f^{(0)}g^{(n)} + B_{n-1,1}f^{(1)}g^{(n-1)} + B_{n-2,2}f^{(2)}g^{(n-2)} + \ldots + f^{(n)}g^{(0)} \tag{104}$$

where the product is being substituted by the sum inside a square bracket and "n" is the order of differentiation.

When a product always involves more than two expressions, we can instead replace the *Binomial Expansion Theorem* with the following *Multinomial Expansion Theorem*:

$$(a_1 + a_2 + \cdots + a_k)^n = \sum_{\substack{n_1,n_2,\ldots,n_k \geq 0 \\ n_1+n_2+\cdots+n_k=n}} \frac{n!}{n_1!\,n_2!\cdots n_k!}\; a_1^{(n_1)}\, a_2^{(n_2)} \cdots a_k^{(n_k)} \tag{105}$$

where $n = n_1 + n_2 + \ldots + n_k$

For determining the various derivatives of a product involving any number of expressions and in accordance to our previously defined notation we can define:

$$\frac{d^n}{dx^n}(f_1 f_2 \cdots f_k) = [f_1 + f_2 + \cdots + f_k]^{(n)} \tag{106}$$

$$= \sum_{\substack{n_1, n_2, \dots, n_k \geq 0 \\ n_1 + n_2 + \cdots + n_k = n}} \frac{n!}{n_1!\, n_2! \cdots n_k!} f_1^{(n_1)} f_2^{(n_2)} \cdots f_k^{(n_k)} \tag{107}$$

where the square bracket is used to symbolize differentiation with all exponents treated as order of differentiation.

Example (6.1). To test the validity of our symbolic notation, let us consider the simple two dimensional case for calculating the various derivatives up to the 5th order at "x = 2" for the following equation:

$$y = e^{2x} = e^{-x} e^{0.5x} e^{2.5x} \tag{108}$$

Here we can start by letting:

$$f_1 = e^{-x}, \quad f_2 = e^{0.5x} \text{ and } f_3 = e^{2.5x} \tag{109}$$

so that each of their various derivatives up to 5 may be defined as:

$$f_1^{(0)} = e^{-x}, \; f_2^{(0)} = e^{0.5x} \text{ and } f_3^{(0)} = e^{2.5x} \tag{110}$$

$$f_1^{(1)} = -e^{-x}, \; f_2^{(1)} = 0.5e^{0.5x} \text{ and } f_3^{(1)} = 2.5e^{2.5x} \tag{111}$$

$$f_1^{(2)} = e^{-x}, \; f_2^{(2)} = 0.25e^{0.5x} \text{ and } f_3^{(2)} = 6.25e^{2.5x} \tag{112}$$

$$f_1^{(3)} = -e^{-x}, \; f_2^{(3)} = 0.125e^{0.5x} \text{ and } f_3^{(3)} = 15.625e^{2.5x} \tag{113}$$

$$f_1^{(4)} = e^{-x}, \; f_2^{(4)} = 0.0625e^{0.5x} \text{ and } f_3^{(4)} = 39.0625e^{2.5x} \tag{114}$$

$$f_1^{(5)} = -e^{-x}, \; f_2^{(5)} = 0.03125e^{0.5x} \text{ and } f_3^{(5)} = 97.65625e^{2.5x} \tag{115}$$

At "x = 0.5" we thus have:

$$f_1^{(0)} = e^{-0.5} = 0.607, \; f_2^{(0)} = e^{0.25} = 1.284 \text{ and } f_3^{(0)} = e^{1.25} = 3.490 \tag{116}$$

$$f_1^{(1)} = -e^{-0.5} = -0.607, \; f_2^{(1)} = 0.5e^{0.25} = 0.642 \text{ and } f_3^{(1)} = 2.5e^{1.25} = 8.726 \tag{117}$$

$$f_1^{(2)} = e^{-0.5} = 0.607, \; f_2^{(2)} = 0.25e^{0.25} = 0.321 \text{ and } f_3^{(2)} = 6.25e^{1.25} = 21.815 \tag{118}$$

$$f_1^{(3)} = -e^{-0.5} = -0.607, \; f_2^{(3)} = 0.125e^{0.25} = 0.161 \text{ and } f_3^{(3)} = 15.625e^{1.25} = 54.537 \tag{119}$$

$$f_1^{(4)} = e^{-0.5} = 0.607, \; f_2^{(4)} = 0.0625e^{0.25} = 0.080 \text{ and } f_3^{(4)} = 39.0625e^{1.25} = 136.342 \tag{120}$$

$$f_1^{(5)} = -e^{-0.5} = -0.607, \; f_2^{(5)} = 0.03125e^{0.25} = 0.040 \text{ and } f_3^{(5)} = 97.65625e^{1.25} = 340.854 \tag{121}$$

Applying the *Multinomial Expansion Theorem* on these three individual components, we arrive at:

$$\frac{d^5y}{dx^5} = [f_1 + f_2 + f_3]^{(5)} = \sum_{\substack{n_1,n_2,n_3 \geq 0 \\ n_1+n_2+n_3=5}} \frac{n!}{n_1!\,n_2!\,n_3!} f_1^{(n_1)} f_2^{(n_2)} f_3^{(n_3)} \tag{122}$$

= (**1**)(-0.607)(1.284)(3.490) + (**5**)(0.607)(0.642)(3.490) + (**10**)(-0.607)(0.321)(3.490) + (**10**)(0.607)(0.161)(3.490) +
(**5**)(-0.607)(0.080)(3.490) + (**1**)(0.607)(0.040)(3.490) + (**5**)(0.607)(1.284)(8.726) + (**20**)(-0.607)(0.642)(8.726) +
(**30**)(0.607)(0.321)(8.726) + (**20**)(-0.607)(0.161)(8.726) + (**5**)(0.607)(0.080)(8.726) + (**10**)(-0.607)(1.284)(21.815) +
(**30**)(0.607)(0.642)(21.815) + (**30**)(-0.607)(0.321)(21.815) + (**10**)(0.607)(0.161)(21.815) + (**10**)(0.607)(1.284)(54.537) +
(**20**)(-0.607)(0.642)(54.537) + (**10**)(0.607)(0.321)(54.537) + (**5**)(-0.607)(1.284)(136.342) + (**5**)(0.607)(0.642)(136.342) +
(**1**)(0.607)(1.284)(340.854) (123)

where there are a total number of 21 terms satisfying the criteria that "$n_1, n_2, n_3 \geq 0$" and "$n_1 + n_2 + n_3 = 5$".

We can define the *multinomial coefficient vector* has having a total number of 21 elements and these are:

$C_M =$ [1, 5, 10, 10, 5, 1, 5, 20, 30, 20, 5, 10, 30, 30, 10, 10, 20, 10, 5, 5, 1] (124)

We can also define the *multinomial exponent vector* as also consisting of 21 elements and they are:

$E_M = [500, 410, 320, 230, 140, 050, 401, 311, 221, 131, 041, 302, 212, 122, 032, 203, 113, 023, 104, 014, 005]$ (125)

By writing a short computer program for performing the arithmetical operation in equation (123) using equation (122) but with higher precision, the value obtained based on the *Multinomial Expansion Theorem* was determined as "$\mathbf{86.985019}$".

The 5th derivative of "e^{2x}" is "2^5e^{2x}" so that at "x = 0.5" this value becomes $32e^{2(0.5)} = 32e = \mathbf{86.9850\,1851}$ which is roughly the same value as the one computed using the *Multinomial Expansion Theorem* in equation (123).

For calculating the various <u>*partial derivatives*</u> with respect to any number of independent variables involving any number of products of multivariate expressions, the *Multinomial Expansion Theorem* is still applicable but with some minor modifications of the general formula that was derived for the two dimensional case.

The various partial derivatives of a product of several multivariate expressions may be written in a more general form as:

$$\frac{\partial^{m_1}}{\partial x_1^{m_1}} \frac{\partial^{m_2}}{\partial x_2^{m_2}} \frac{\partial^{m_3}}{\partial x_3^{m_3}} \cdots \frac{\partial^{m_k}}{\partial x_j^{m_k}} \left[f_1(x_1, x_2, \dots, x_j) \cdot f_2(x_1, x_2, \dots, x_j) \cdots f_i(x_1, x_2, \dots, x_j)\right] \tag{126}$$

which can symbolically be expanded as:

$$\left[f_1^{(0)} + f_2^{(0)} + \ldots + f_i^{(0)}\right]_{1(m_1)}^{m_1} \;\Delta\; \left[f_1^{(0)} + f_2^{(0)} + \ldots + f_i^{(0)}\right]_{2(m_2)}^{m_2} \;\Delta\; \cdots \;\Delta$$

$$\Delta\; \cdots \;\Delta\; \left[f_1^{(0)} + f_2^{(0)} + \ldots + f_i^{(0)}\right]_{j(m_k)}^{m_k} \qquad (127)$$

where "Δ" is a special operator that is used to mimic the process of algebraically expanding *term by term* the product of two or more expressions with the only exception that all exponents are to be treated as order of differentiation.

In complete notational form using the *Multinomial Expansion Theorem* this may be rewritten as:

$$\left[\sum_{\substack{n_1,n_2,\ldots,n_i\geq 0\\ n_1+n_2+\cdots+n_i=m_1}} \frac{n!}{n_1!\,n_2!\cdots n_k!}\; f_{1,1(n_1)}^{(n_1)}\, f_{2,1(n_2)}^{(n_2)} \cdots f_{i,1(n_i)}^{(n_i)}\right] \Delta$$

$$\left[\sum_{\substack{n_1,n_2,\ldots,n_i\geq 0\\ n_1+n_2+\cdots+n_i=m_2}} \frac{n!}{n_1!\,n_2!\cdots n_k!}\; f_{1,2(n_1)}^{(n_1)}\, f_{2,2(n_2)}^{(n_2)} \cdots f_{i,2(n_i)}^{(n_i)}\right] \quad \Delta\; \cdots \;\Delta$$

$$\left[\sum_{\substack{n_1,n_2,\ldots,n_i\geq 0\\ n_1+n_2+\cdots+n_i=m_k}} \frac{n!}{n_1!\,n_2!\cdots n_k!}\; f_{1,j(n_1)}^{(n_1)}\, f_{2,j(n_2)}^{(n_2)} \cdots f_{i,j(n_i)}^{(n_i)}\right]$$

(128)

When expanding the various partial derivatives of a product of several multivariate expressions using the above notational form, it is very important to insure that "*all*" the multivariate expressions present in "*each product*" are also "*all*" present in "*each term*" of the resultant expansion.

Example (6.2). Based entirely on our standard notation for representing the various partial derivatives of a product of several multivariate expressions, we will determine "$\frac{\partial f_1 f_2}{\partial x_1 \partial x_2^2}$" where "$f_1$" and "$f_2$" are each defined as arbitrary multivariate function.

$$\frac{\partial^3 f_1 f_2}{\partial x_1 \partial x_2^2} \;=\; [f_1 + f_2]_{1(1)}^{(1)} \;\Delta\; [f_1 + f_2]_{2(2)}^{(2)} \qquad (129)$$

$$=\; \left[f_{1,1(1)}^{(1)} \;+\; f_{2,1(1)}^{(1)}\right] \;\Delta\; \left[f_{1,2(2)}^{(2)} \;+\; 2f_{1,2(1)}^{(1)} f_{2,2(1)}^{(1)} +\; f_{2.2(2)}^{(2)}\right] \qquad (130)$$

Algebraically performing a term by term symbolic multiplication by treating all exponent values as order of differentiation, we obtain:

$$= \; f_{1,1(1)}^{(1)} f_{1,2(2)}^{(2)} + 2 f_{1,1(1)}^{(1)} f_{1,2(1)}^{(1)} f_{2,2(1)}^{(1)} + f_{1,1(1)}^{(1)} f_{2,2(2)}^{(2)} +$$

$$+ \; f_{2,1(1)}^{(1)} f_{1,2(2)}^{(2)} + 2 f_{2,1(1)}^{(1)} f_{1,2(1)}^{(1)} f_{2,2(1)}^{(1)} + f_{2,1(1)}^{(1)} f_{2,2(2)}^{(2)} \tag{131}$$

which in the conventional symbolic form may be translated as:

$$= \frac{\partial^3 f_1}{\partial x_1 \partial x_2^2} + 2\frac{\partial^2 f_1}{\partial x_1 \partial x_2}\frac{\partial f_2}{\partial x_2} + \frac{\partial f_1}{\partial x_1}\frac{\partial^2 f_2}{\partial x_2^2} + \frac{\partial^2 f_1}{\partial x_2^2}\frac{\partial f_2}{\partial x_1} + 2\frac{\partial f_1}{\partial x_2}\frac{\partial^2 f_2}{\partial x_1 \partial x_2} + \frac{\partial^3 f_2}{\partial x_1 \partial x_2^2} \tag{132}$$

To insure that every term in the above expansion always contains the two functions that is being differentiated, we must include " f_2" and "f_1" in the first and last term of the expansion respectively.

The final results are:

$$= \frac{\partial^3 f_1}{\partial x_1 \partial x_2^2} f_2 + 2\frac{\partial^2 f_1}{\partial x_1 \partial x_2}\frac{\partial f_2}{\partial x_2} + \frac{\partial f_1}{\partial x_1}\frac{\partial^2 f_2}{\partial x_2^2} + \frac{\partial^2 f_1}{\partial x_2^2}\frac{\partial f_2}{\partial x_1} + 2\frac{\partial f_1}{\partial x_2}\frac{\partial^2 f_2}{\partial x_1 \partial x_2} + f_1\frac{\partial^3 f_2}{\partial x_1 \partial x_2^2} \tag{133}$$

We can validate the use of our symbolic notations by performing the same operation manually and compare the results with the one obtained in the above equation:

$$\frac{\partial^2 f_1 f_2}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(\frac{\partial f_1}{\partial x_2} f_2 + f_1\frac{\partial f_2}{\partial x_2}\right) = \frac{\partial^2 f_1}{\partial x_2^2} f_2 + 2\frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_2} + f_1\frac{\partial^2 f_2}{\partial x_2^2} \tag{134}$$

$$\frac{\partial^3 f_1 f_2}{\partial x_1 \partial x_2^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial^2 f_1 f_2}{\partial x_2^2}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial^2 f_1}{\partial x_2^2} f_2 + 2\frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_2} + f_1\frac{\partial^2 f_2}{\partial x_2^2}\right) \tag{135}$$

$$= \frac{\partial^3 f_1}{\partial x_1 \partial x_2^2} f_2 + \frac{\partial^2 f_1}{\partial x_2^2}\frac{\partial f_2}{\partial x_1} + 2\frac{\partial^2 f_1}{\partial x_1 \partial x_2}\frac{\partial f_2}{\partial x_2} + 2\frac{\partial f_1}{\partial x_2}\frac{\partial^2 f_2}{\partial x_1 \partial x_2} + \frac{\partial f_1}{\partial x_1}\frac{\partial^2 f_2}{\partial x_2^2} + f_1\frac{\partial^3 f_2}{\partial x_1 \partial x_2^2} \tag{136}$$

As can be verified, the above expansion is exactly identical to the one in equation (133) thereby completely validating our standard use of special notations for taking the various partial derivatives of a product of several multivariate expressions.

The greatest advantage for using this notational convention is that it can reduce the entire process of determining the various partial derivatives of a product consisting of any number of expressions entirely on a "*computational level*".

In general, an IAMPT will always consist of multivariate polynomials as well as the differential of multivariate polynomials where each multivariate polynomial term will always be expressible as a product of several auxiliary variables. For calculating the various derivatives and partial derivatives of an IAMPT would require that each of the products of several auxiliary variables be differentiated under the product rule. So its therefore quite easy to visualize how the use of the *Multinomial Expansion Theorem* would become a very valuable tool for computing the various derivatives and partial derivatives of an IAMPT to any desirable degree of accuracy.

The complete development of all the formulas related to the calculations of the various derivatives and partial derivatives of an IAMPT for solving all types of DEs and systems of DEs is of course much beyond the scope of this paper. However, this can always be made available to anyone by special request provided you contact me at either one of the following email addresses michelmikalajunas@bellnet.ca or at jpnelson_mfc@yahoo.ca .

7. *General* closed form solutions of the Navier-Stokes equations by method of conjecture involving the use of computational differential analysis

The Navier-Stokes equations is the direct application of Newton's second law of motion for the complete analysis of both compressible and incompressible fluids.

For the case of incompressible flow and assuming constant viscosity, the equations may be described as follow:

$$\underbrace{\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right)}_{\textbf{\textit{Inertia}}} = \underbrace{-\nabla P}_{\textbf{\textit{Pressure gradient}}} + \underbrace{\mu\nabla^2\mathbf{v}}_{\textbf{\textit{Viscosity}}} + \underbrace{F}_{\textbf{\textit{Other forces}}} \tag{137}$$

along with the mass continuity equation which states that:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{138}$$

Since we will restrict our analysis to *incompressible* flow only, the density is always assumed constant so that the above equation may be rewritten as:

$$\nabla\cdot\mathbf{v} = 0 \tag{139}$$

By assuming that gravitational forces are the only external forces present, the vector equations in ***Cartesian*** coordinates expand as follow:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho g_x \tag{140}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho g_y \tag{141}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho g_z \tag{142}$$

along with the mass continuity equation defined as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{143}$$

We would construct the NCSA table by defining the ***variable*** coefficients as the fluid density "ρ", the fluid dynamic viscosity "μ" and the gravitational force components in the x, y and z direction. Since no external inputs are present in these equations other then the external forces due to gravity then we can set "q = 0" in the IAMPT that will be selected for solving these vector equations.

In the *Secondary Expansion* of our IAMPT, the first set of auxiliary variables will be used for representing the dependent and independent variables in that order. This will be followed by the remaining initially assumed auxiliary variables used for representing all basis functions in complete differential form that will be present in the exact analytical solution of the system of PDEs.

Our IAMPT will be selected on the basis of solving the above system of PDEs in terms of a system of *implicitly* defined equations that would consist only of the algebraic and elementary basis functions. The various initial conditions possible for this type of generalized flow are of course expected to be infinite. So in order to maximize our numerical solution rate of the corresponding nonlinear simultaneous equations, we can set all the coefficients defining the initial conditions in our IAMPT as part of the unknowns to solve for that would be represented by the initial values of each initially assumed auxiliary variable. Other unknowns to solve for are the variable coefficients defined in our NCSA table as well as those present in both the *Primary* and *Secondary Expansion* of our IAMPT.

Over time, the NCSA table should eventually succeed in capturing from the numerical solution set of the nonlinear simultaneous equations all those *exact instance analytical solutions* that would conform with experimental results obtained under controlled laboratory conditions.

It is only through the gathering of this type of information over a span of say many years or even many decades that a large number of *generalized* analytical solutions may potentially be uncovered. This would in the very long term enable us to acquire a far better understanding of general fluid behavior than having to depend entirely on the use of laboratory experiments as a result of the non-integrability of many integrals that would have originated from the use of conventional methods of pure mathematical analysis.

In terms of ***Cylindrical*** coordinates this would be written as:

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r}\right) = -\frac{\partial P}{\partial r} + \mu\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)\right] + \rho g_r \qquad (144)$$

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right)\right] + \rho g_\theta \qquad (145)$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right)\right] + \rho g_z \qquad (146)$$

along with the mass continuity equation defined as:

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \qquad (147)$$

Such a coordinate system may in some cases prove to be easier for the analysis of certain types of fluid motion that would mainly involve symmetry thereby allowing for the elimination of a velocity component.

A very common case is axisymmetric flow where there is no tangential velocity $(u_\theta = 0)$ and the remaining quantities are independent of θ:

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \mu\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2}\right)\right] + \rho g_r \tag{148}$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{\partial^2 u_z}{\partial z^2}\right)\right] + \rho g_z \tag{149}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = 0 \tag{150}$$

For this type of coordinate system we would proceed in constructing the NCSA table in exactly the same manner as for the *Cartesian* coordinate system where in both cases there are no external inputs so that "q = 0". This would also include managing in exactly the same manner all initial conditions and the variable coefficients defined by the fluid density "ρ", the fluid dynamic viscosity "μ" and the gravitational components in the x, y and z direction.

In terms of ***Spherical*** coordinates this would be written as:

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{rSin(\emptyset)}\frac{\partial u_r}{\partial \theta} + \frac{u_\emptyset}{r}\frac{\partial u_r}{\partial \emptyset} - \frac{u_\theta^2+u_\emptyset^2}{r}\right) = \frac{\partial P}{\partial r} + \rho g_r +$$
$$+ \mu\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2 Sin(\emptyset)^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{1}{r^2 Sin(\emptyset)}\frac{\partial}{\partial \emptyset}\left(Sin(\emptyset)\frac{\partial u_r}{\partial \emptyset}\right) - 2\left(\frac{u_r + \frac{\partial u_\emptyset}{\partial \emptyset} + u_\emptyset Cot(\emptyset)}{r^2}\right) + \frac{2}{r^2 Sin(\emptyset)}\frac{\partial u_\theta}{\partial \theta}\right\} \tag{151}$$

$$\rho\left\{\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{rSin(\emptyset)}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\emptyset}{r}\frac{\partial u_\theta}{\partial \emptyset} + \left(\frac{u_r u_\theta + u_\theta u_\emptyset Cot(\emptyset)}{r}\right)\right\} = -\frac{1}{rSin(\emptyset)}\frac{\partial P}{\partial \theta} + \rho g_\theta +$$
$$+ \mu\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2 Sin(\emptyset)^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{1}{r^2 Sin(\emptyset)}\frac{\partial}{\partial \emptyset}\left(Sin(\emptyset)\frac{\partial u_\theta}{\partial \emptyset}\right) + \left(\frac{2\frac{\partial u_r}{\partial \theta} + 2Cos(\emptyset)\frac{\partial u_\theta}{\partial \theta} - u_\theta}{r^2 Sin(\emptyset)^2}\right)\right\} \tag{152}$$

$$\rho\left\{\frac{\partial u_\emptyset}{\partial t} + u_r\frac{\partial u_\emptyset}{\partial r} + \frac{u_\theta}{rSin(\emptyset)}\frac{\partial u_\emptyset}{\partial \theta} + \frac{u_\emptyset}{r}\frac{\partial u_\emptyset}{\partial \emptyset} + \left(\frac{u_r u_\emptyset - u_\theta^2 Cot(\emptyset)}{r}\right)\right\} = -\frac{1}{r}\frac{\partial P}{\partial \emptyset} + \rho g_\emptyset +$$
$$+ \mu\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\emptyset}{\partial r}\right) + \frac{1}{r^2 Sin(\emptyset)^2}\frac{\partial^2 u_\emptyset}{\partial \theta^2} + \frac{1}{r^2 Sin(\emptyset)}\frac{\partial}{\partial \emptyset}\left(Sin(\emptyset)\frac{\partial u_\emptyset}{\partial \emptyset}\right) + \frac{2}{r^2}\frac{\partial u_r}{\partial \emptyset} - \left(\frac{u_\emptyset + 2Cos(\emptyset)\frac{\partial u_\theta}{\partial \theta}}{r^2 Sin(\emptyset)^2}\right)\right\} \tag{153}$$

along with the mass continuity equation defined as:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{rSin(\emptyset)}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{rSin(\emptyset)}\frac{\partial}{\partial \emptyset}(Sin(\emptyset)u_\emptyset) = 0 \tag{154}$$

In this coordinate system, there are two external inputs in the form of the Sine and Cosine function which according to equation (35) and (36) can each be expressed in terms of the Tangent half angle formula so that we can set "q = 1" in our IAMPT. All initial conditions and variable coefficients are handled in exactly the same manner as with the *Cartesian* and *Cylindrical* coordinate system.

Because of the universality of the new method of analytical integration we can extend this analysis to cover all possible cases for both compressible and incompressible flow where the concept of an NCSA table would still be applicable throughout.

8. The development of a universal software for the analytical solutions of all DEs and systems of DEs under a single unified theory of analytical integration

The highly computational nature of the universal differential expansion described by equations (1) through (5) for representing all mathematical equations makes it very difficult for conducting any real meaningful numerical experimentations even for solving the simplest type of DE. For solving the vast majority of DEs and systems of DEs of greatest importance to the physical sciences, super computers are by far more suitable for this type of high level and very advanced form of computational analysis.

The advent of Quantum computers in the near future could significantly improve the performance of handling even the most complex systems of PDEs. They would by far exceed the capabilities of even our most powerful super computer of our time because they would operate entirely on the fundamental principles of Quantum theory which is based on the study of energy at the atomic and subatomic level. Such advanced computer technology would allow for the capability of performing multiple tasks in parallel thereby resulting in a significant increase in the billion-fold when compared to conventional computer systems.

Among the many possible states of operation is the *binary* state of a Quantum bit or Qubit that would either be defined as spin-down or spin-up with each mode entirely controlled by a pulse of energy originating from a laser beam. Major centers of research in Quantum computing are currently in operation that would include MIT, IBM, Oxford, Harvard, Stanford and the Los Alamos National Laboratory.

The greatest advantage for having arrived at a unified theory of analytical integration is that it can be converted into a *single major universal software* by which all DEs and systems of DEs may be resolved under a single *common mathematical ideology*. Such a universal software development would be referred to as a "*Numerical Control Analytics Software*" or NCAS. It would operate on the principle of determining the existence of *general analytical solutions* to DEs and systems of DEs through the application of a very unique method of conjecture that would be driven entirely by computational analysis. This would represent a far better alternative than having to maintain a large number of highly *dispersed* mathematical theories all of which could never be consolidated in terms of a single universal software development package such as the one proposed here.

If such a Numerical Control Analytics Software would be applied only to Physics, it would certainly qualify as being **“***the complete unified theory of physics” but only in its most “raw state”. Human intervention would then only be necessary for complete translation of all computer results that would appear in the form of exact numerical computations into practical decipherable mathematical equations.*

If such a Numerical Control Analytics Software would be applied only into Engineering Science, it would become the standard method of all engineering analysis by which the concept of an IAMPT would be applied very rigorously for resolving all relevant DEs and systems of DEs in the form of general closed form solutions only. This would set the stage for the complete formulation of many fundamental key theorems similar to what the famous Superposition Theorem has succeeded in accomplishing in the general theory of linear physical systems.

9. Conclusions

The problem of integration has always presented itself as a real challenge when attempting to find closed formed solutions for the vast majorities of DEs and systems of DEs. The main reason for this is the frequent occurrences of integrals from which the vast majority of them cannot always be resolved exactly under any existing methods of mathematical analysis. This complication can be completely avoided altogether if rather than proceeding with some *initially assumed closed form solution* for attempting to solve a DE or a system of DEs, we instead work only with the complete *differential* representation of the same *initially assumed closed form solution*. The greatest advantage for proceeding in that fashion is the highest expectation that many of the assumed differentials will in the end appear *exact* and thus always completely integrable in the end. This in fact is quite achievable because every *differentiable* mathematical equation can always be converted in complete differential form by following the same basic unique mathematical structure as the one introduced by equations (1) through (5). Such a unique differential expansion form is so universal to all mathematical equations that it would certainly qualify by all mathematical standards as being a *complete unified analytical theory of integration* for resolving all types of DEs and systems of DEs in terms of closed form solutions. Many key mathematical properties of this unified analytical theory of integration have been quite extensively investigated in the past mainly by myself. But the one that stands out the most is the ability for resolving "*all types*" of DEs and systems of DEs uniquely in terms of "*general closed form solutions*" by utilizing a method of conjecture that would be driven entirely on computational analysis alone. We use the Navier-Stokes equations as a perfect model for illustrating this very unique approach of working with initially assumed differentials. In our example, we explore the various types of systems of PDEs that were developed in the past under the three most popular set of coordinate systems. In the final analysis, we were able to establish that independent of the type of flow whether compressible or incompressible, the boundary conditions and various external forces present can always be completely accounted for during the process of working with these types of initially assumed differential forms. From the very unique properties of such a proposed unified differential method of analysis, it is expected that many cases of the Navier-Stokes equations will always be completely integrable in terms of such "*general*" closed form solutions by following a very unique method of conjecture. From the Navier-Stokes equations we can apply the same type of universal differential analysis for investigating other types of *fundamental equations* that would include Maxwell's equations, Einstein's field equations, the Schrödinger equation just to name a few. Figure 3.1 provides a direct relationship between the method of universal differential analysis and the elusive "*theory of everything*". From this table, one is very tempted to conclude that for arriving at such a gigantic theory for explaining everything about our universe may no longer be just a matter for modern physics to resolve over time. Rather, it is expected that such a theory of everything may only be achievable in the end from the complete consolidation of every single theory describing its own unique physical system under one big gigantic universal theory that in the end will succeed in explaining everything about our universe.

10. Appendix A

$$(\mathbf{1.1})\quad \mathbf{f(x,y)} = \mathbf{0} = \boldsymbol{a_1x^2} + \boldsymbol{a_2y^2} + \boldsymbol{a_3xy} + \boldsymbol{a_4}$$

$W_1 = x$
$W_2 = y$

(1). *Primary Expansion:*

$$F(W_1, W_2) = 0 = a_1W_1^2 + a_2W_2^2 + a_3W_1W_2 + a_4$$

(2). *Secondary Expansion:*

$dx = dW_1$
$dy = dW_2$

$$(\mathbf{1.2})\quad \mathbf{f(x,y)} = \mathbf{0} = \boldsymbol{a_1y} + \boldsymbol{a_2e^{a_3x}Sin(a_4x)}$$

$W_1 = x$
$W_2 = y$
$W_3 = e^{a_3x}$
$W_4 = Tan(a_4x/2)$

(1). *Primary Expansion:*

$$F(W_1, W_2, W_3, W_4) = 0 = a_1W_2(1 + W_4^2) + 2a_2W_3W_4$$

(2). *Secondary Expansion:*

$$dx = dW_1$$
$$dy = dW_2$$

$$a_3 W_3 dx + 0 \cdot dy = dW_3$$
$$a_4(1 + W_4^2) dx + 0 \cdot dy = 2dW_4$$

$$\mathbf{(1.3)} \quad \mathbf{f(x,y) = 0 = x^2 + y^2\sqrt{(x-y)} + 3e^{3x}}$$

$W_1 = x$
$W_2 = y$

$W_3^2 = x - y = W_1 - W_2$

$W_4 = e^x = e^{W_1}$

(1). *Primary Expansion:*

$$F(W_1, W_2, W_3, W_4) = 0 = W_1^2 + W_2^2 W_3 + 3W_4^3$$

(2). *Secondary Expansion:*

$$dx = dW_1$$
$$dy = dW_2$$

$$dx - dy = 2W_3\, dW_3$$
$$3W_4 dx + 0 \cdot dy = dW_4$$

$$\mathbf{(1.4)} \quad \mathbf{f(x,y) = 0 = x\sqrt{x^2 + y^2} + y\sqrt{x^2 - y^2}}$$

$W_1 = x$
$W_2 = y$

$W_3^2 = W_1^2 + W_2^2$

$W_4^2 = W_1^2 - W_2^2$

(1). *Primary Expansion:*

$$F(W_1, W_2, W_3, W_4) = 0 = W_1 W_3 + W_2 W_4$$

(2). *Secondary Expansion:*

$$dx = dW_1$$
$$dy = dW_2$$

$$W_1 dx + W_2 dy = W_3 dW_3$$
$$W_1 dx - W_2 dy = W_4 dW_4$$

$$\mathbf{(1.5)} \quad \mathbf{f(x,y) = 0 = \ln\left(1 + \sqrt[3]{x+1}\right) - \sqrt[6]{y+1} - 1}$$

$W_1 = x$
$W_2 = y$

$W_3^3 = x + 1 = W_1 + 1$

$W_4 = \ln\left(1 + \sqrt[3]{x+1}\right) = \ln(1 + \mathrm{W}_3)$

$W_5^6 = y + 1 = W_2 + 1$

(1). *Primary Expansion:*

$$F(W_1, W_2, W_3, W_4, W_5) = 0 = W_4 - W_5 - 1$$

(2). *Secondary Expansion:*

$$dx = dW_1$$
$$dy = dW_2$$

$$dx + 0 \cdot dy = 3W_3^2 dW_3$$
$$dx + 0 \cdot dy = 3W_3^2(1 + W_3)dW_4$$
$$0 \cdot dx + dy = 6W_5^5 dW_5$$

$$\mathbf{(1.6)} \quad \mathbf{f(x,y) = 0 = 3Sin(x+y)} - \boldsymbol{ln\left(e^x + \sqrt{Cos(x)}\right) + ln\left(\frac{x}{y}\right) + \sqrt{ArcTan(2x)}}$$

$$W_1 = x$$
$$W_2 = y$$
$$W_3 = Tan\left(\frac{x+y}{2}\right)$$
$$W_4 = e^x$$
$$W_5 = Tan\left(\frac{x}{2}\right)$$
$$W_6^2 = Cos(x) = \frac{1 - W_5^2}{1 + W_5^2}$$
$$W_7 = ln\left(e^x + \sqrt{Cos(x)}\right) = ln(W_4 + W_6)$$
$$W_8 = ln(x)$$
$$W_9 = ln(y)$$
$$W_{10}^2 = ArcTan(2x)$$

(1). *Primary Expansion:*

$$F(W_1, W_2, \ldots, W_{10}) = 0 = \frac{6W_3}{1 + W_3^2} - W_7 + W_8 - W_9 + W_{10}$$

(2). *Secondary Expansion:*

$$dx = dW_1$$
$$dy = dW_2$$

$$(1 + W_3^2)dx + (1 + W_3^2)dy = 2dW_3$$
$$W_4 dx + 0 \cdot dy = dW_4$$
$$(1 + W_5^2)dx + 0 \cdot dy = 2dW_5$$
$$-W_5 dx + 0 \cdot dy = W_6(1 + W_5^2)dW_6$$
$$\{W_4 W_6(1 + W_5^2) - W_5\}dx + 0 \cdot dy = W_6(1 + W_5^2)(W_4 + W_6)dW_7$$
$$dx + 0 \cdot dy = W_1 dW_8$$
$$0 \cdot dx + dy = W_2 dW_9$$
$$dx + 0 \cdot dy = (1 + 4W_1^2)W_{10} dW_{10}$$

$$(2.1)\quad \mathbf{f}(\mathbf{z},\boldsymbol{x_1},\boldsymbol{x_2}) \;=\; \mathbf{0} \;=\; \mathbf{z} + \boldsymbol{z^3 x_1 x_2} - \boldsymbol{x_2} + \mathbf{1}$$

$W_1 = z$
$W_2 = x_1$
$W_3 = x_2$

(1). *Primary Expansion:*

$$F(W_1, W_2, W_3) = 0 = W_1 + W_1^3 W_2 W_3 - W_3 + 1$$

(2). *Secondary Expansion:*

$$\begin{aligned}
dz + 0 \cdot dx_1 + 0 \cdot dx_2 &= dW_1 \\
0 \cdot dz + dx_1 + 0 \cdot dx_2 &= dW_2 \\
0 \cdot dz + 0 \cdot dx_1 + dx_2 &= dW_3
\end{aligned}$$

$$(2.2)\quad \mathbf{f}(\mathbf{z},\boldsymbol{x_1},\boldsymbol{x_2},\boldsymbol{x_3},) \;=\; \mathbf{0} \;=\; \mathbf{5}\boldsymbol{x_2 x_3 Sin}(\boldsymbol{z x_1 x_2}) + (\boldsymbol{x_1} + \boldsymbol{x_2})\boldsymbol{Cos}(\boldsymbol{z} + \mathbf{3}\boldsymbol{x_2} + \mathbf{2}\boldsymbol{x_3}) + \mathbf{3}$$

$W_1 = z$

$W_2 = x_1$

$W_3 = x_2$

$W_4 = x_3$

$W_5 = Tan(z x_1 x_2 / 2)$

$$W_6 = Tan\left\{\frac{z + 3x_2 + 2x_3}{2}\right\}$$

(1). *Primary Expansion:*

$$F(W_1, W_2, W_3, W_4, W_5, W_6) = 0 = 5W_3W_4\left[\frac{2W_5}{1 + W_5^2}\right] + (W_2 + W_3)\left[\frac{1 - W_6^2}{1 + W_6^2}\right] + 3$$

(2). *Secondary Expansion:*

$$\begin{aligned}
dz + 0 \cdot dx_1 + 0 \cdot dx_2 + 0 \cdot dx_3 &= dW_1 \\
0 \cdot dz + dx_1 + 0 \cdot dx_2 + 0 \cdot dx_3 &= dW_2 \\
0 \cdot dz + 0 \cdot dx_1 + dx_2 + 0 \cdot dx_3 &= dW_3 \\
0 \cdot dz + 0 \cdot dx_1 + 0 \cdot dx_2 + dx_3 &= dW_4
\end{aligned}$$

$$(1 + W_5^2)W_2W_3dz + (1 + W_5^2)W_1W_3dx_1 + (1 + W_5^2)W_1W_2dx_2 + 0 \cdot dx_3 = 2dW_5$$

$$(1 + W_6^2)dz + 0 \cdot dx_1 + 3(1 + W_6^2)dx_2 + 2(1 + W_6^2)dx_3 = 2dW_6$$

$$(2.3) \quad \mathbf{f(x,y) \ = \ 0 \ = \ 3\ln\left(\sqrt[3]{z + x_1^2 + x_2^2} \ - \ 25e^{2zx_1x_3}\right) \ +}$$

$$\mathbf{+ \ \sqrt[5]{x_1^2 + x_2^2 + x_3^2} \ - \ 4z^3 \ + \ 1}$$

$W_1 = z$

$W_2 = x_1$

$W_3 = x_2$

$W_4 = x_3$

$W_5^3 = z + x_1^2 + x_2^2 = W_1 + W_2^2 + W_3^2$

$W_6 = e^{2zx_1x_3} = e^{2W_1W_2W_4}$

$W_7 = \ln\left(\sqrt[3]{z + x_1^2 + x_2^2} - 25e^{2zx_1x_3}\right) - \ln(W_5 - 25W_6)$

$W_8^5 = x_1^2 + x_2^2 + x_3^2 = W_2^2 + W_3^2 + W_4^2$

(1). *Primary Expansion:*

$$F(W_1, W_2, W_3, \ldots, W_8) = 0 = 3W_7 + W_8 - 4W_1^3 + 1$$

(2). *Secondary Expansion:*

$$\begin{aligned}
dz + 0 \cdot dx_1 + 0 \cdot dx_2 + 0 \cdot dx_3 &= dW_1 \\
0 \cdot dz + dx_1 + 0 \cdot dx_2 + 0 \cdot dx_3 &= dW_2 \\
0 \cdot dz + 0 \cdot dx_1 + dx_2 + 0 \cdot dx_3 &= dW_3 \\
0 \cdot dz + 0 \cdot dx_1 + 0 \cdot dx_2 + dx_3 &= dW_4
\end{aligned}$$

$$dz + 2W_2dx_1 + 2W_3dx_2 + 0 \cdot dx_3 = 3W_5dW_5^2$$

$$2W_2W_4W_6dz + 2W_1W_4W_6dx_1 + 0 \cdot dx_2 + 2W_1W_2W_6dx_3 = dW_6$$

$$(1 - 150W_2W_4W_5^2W_6)dz + (2W_2 - 150W_1W_4W_5^2W_6)dx_1 + 2W_3dx_2 -$$
$$150W_1W_2W_5^2W_6dx_3 = 3W_5^2(W_5 - 25W_6)dW_7$$

$$0 \cdot dz + W_2dx_1 + W_3dx_2 + W_4dx_3 = 2.5W_8^4dW_8$$

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