High-order algorithms for nonlinear problems and numerical instability

***José Elias Laier[1]**

[1]Department of Structural Engineering, Engineering School of São Carlos, University of São Paulo, Brazil.

*Presenting author: jelaier@sc.usp.br

Abstract

The objective of this paper is to study the numerical behavior (accuracy and numerical instability) of two high-order order single step direct integration algorithm for nonlinear dynamic. These algorithms are formulated in terms of two Hermitian finite difference operators of fifth-order local truncation error. In addition, these algorithms are unconditionally stable with no numerical damping for linear dynamic problems. The attention is devoted to the classical second-order Duffing and Van der Pol equations, as well the non-linear elastic pendulum, including the first-order Lorenz and Lotka-Volterra equations. Numerical applications compare the results including with those obtained by the second-order Newmark method

Keywords: Numerical instability, nonlinear dynamic, Hermitian finite difference algorithms

Introduction

The objective of this paper is to study the numerical behavior of two high-order order single step direct integration algorithm for nonlinear dynamic. The first one has been developed by the author [1] and the second is the classical cubic Hermitian Algorithm developed by Argyris and Mlejek [2]. These algorithms are formulated in terms of two Hermitian finite difference operators [3] of fifth-order local truncation error. In addition, these algorithms are unconditionally stable with no numerical damping for linear dynamic problems. As the analytical treatment of the numerical instability of the resultant nonlinear difference equation (i.e. the numerical version of the differential equation) is quite complex, just numerical investigation is performed.

As the high-order algorithms takes into account the repeated differentiation of the governing equation, additional nonlinear terms are required to solve nonlinear structural dynamic problems. Thus, it is interesting to consider, for example, the classic iterative procedures presented by Argyris and Mlejek [2]. Although the presence of these additional nonlinear terms increases the number of operations in the iterative operations and introduces some numerical noise in comparison to the Padè-P_{22} algorithm family [4], the reduction obtained in the matrix factorization and higher orders of the relative radii errors are interesting attributes of the proposed algorithm. Numerical applications compare the results including with those obtained by Newmark method. The results show that the accuracy of both third-order algorithms is quite similar for refined mesh, but the numerical instability (that occurs for coarse mesh) is not similar.

Hermitian Operators

The step-by-step integration algorithm to be considered in this paper takes into account the following Hermitian operators [1] [3]:

$$\begin{aligned} &Ay_i - By_{i+1} + C\Delta t\dot{y}_i + D\Delta t\dot{y}_{i+1} + E\Delta t^2\ddot{y}_i - F\Delta t^2\ddot{y}_{i+1} + G\Delta t^3\dddot{y}_i - H\Delta t^3\dddot{y}_{i+1} = 0 \\ &A_1y_i - B_1y_{i+1} + C_1\dot{y}_i - D_1\dot{y}_{i+1} + E_1\Delta t^2\ddot{y}_i + F_1\Delta t^2\ddot{y}_{i+1} + G_1\Delta t^3\dddot{y}_i - H_1\Delta t^3\dddot{y}_{i+1} = 0 \end{aligned} \quad (1)$$

where Δt is the time step, i and i+1 indicate the step, y is the function to be integrated, $\dot{y}$, $\ddot{y}$ and $\dddot{y}$ are derivatives of the function with respect to time; A, B … G_1, H_1 are combination non-dimension parameters that define the order of accuracy (local truncation error) [3]. Table 1 presents the combination parameters for the algorithms herein considered.

Duffing equation

The Duffing equation and its first time derivative can be expressed as

Table 1. Combination Parameters

	A	B	C	D	E	F	G	H
Laier [1]	12	-12	6	6	1	-1	0	0
Argyris [3]	1	-1	1	0	21/60	9/60	3/60	-2/60
Newmark	0	0	1	-1	1/2	1/2	0	0
	A_1	B_1	C_1	D_1	E_1	F_1	G_1	H_1
Laier [1]	0	0	12	-12	6	6	1	-1
Argyris [3]	0	0	1	-1	6/12	6/12	1/12	-1/12
Newmark	1	-1	1	0	1/4	1/4	0	0

$$\begin{aligned} &\ddot{y}+\delta\dot{y}+\alpha y+\beta y^3=p\cos(\omega t)\\ &\dddot{y}+\delta\ddot{y}+\alpha\dot{y}+3\beta y^2\dot{y}=-p\omega\sin(\omega t)\end{aligned} \tag{2}$$

where α, β, δ, p and ω are parameters of the equation. The second and third derivatives present in equation (2) can be explicitly written by

$$\begin{aligned} &\ddot{y}=-\delta\dot{y}-\alpha y-\beta y^3+p\cos(\omega t)\\ &\dddot{y}=-\delta\left(-d\dot{y}-\alpha y-\beta y^3+p\cos(\omega t)\right)-\alpha\dot{y}-3\beta y^2\dot{y}-p\omega\sin(\omega t)\end{aligned} \tag{3}$$

Now, taking into account equation (3) and Hermitian operators (1) the following nonlinear recurrence first-order difference equation can be written:

$$\begin{aligned} &F(y_{i+1},\dot{y}_{i+1})=Ay_i-By_{i+1}+C\Delta t\dot{y}_i+D\Delta t\dot{y}_{i+1}+E\Delta t^2\ddot{y}_i-F\Delta t^2\ddot{y}_{i+1}+G\Delta t^3\dddot{y}_i-H\Delta t^3\dddot{y}_{i+1}=0\\ &G(y_{i+1},\dot{y}_{i+1})=A_1y_i-B_1y_{i+1}+C_1\dot{y}_i-D_1\dot{y}_{i+1}+E_1\Delta t^2\ddot{y}_i+F_1\Delta t^2\ddot{y}_{i+1}+G_1\Delta t^3\dddot{y}_i-H_1\Delta t^3\dddot{y}_{i+1}=0\end{aligned} \tag{4}$$

And so, the corresponding Newton iterative formula can be expressed as:

$$\begin{Bmatrix} y_{i+1}\\ \dot{y}_{i+1}\end{Bmatrix}_{j+1}=\begin{Bmatrix} y_{i+1}\\ \dot{y}_{i+1}\end{Bmatrix}_j+\begin{vmatrix} F_{y_{i+1}}(y_{i+1},\dot{y}_{i+1}) & F_{\dot{y}_{i+1}}(y_{i+1},\dot{y}_{i+1})\\ G_{y_{i+1}}(y_{i+1},\dot{y}_{i+1}) & G_{y_{i+1}}(y_{i+1},\dot{y}_{i+1})\end{vmatrix}_j\begin{Bmatrix} F(y_{i+1},\dot{y}_{i+1})\\ G(y_{i+1},\dot{y}_{i+1})\end{Bmatrix}_j \tag{5}$$

were subscript y_{i+1} and $\dot{y}_{i+1}$ indicate the partial derivative with respect to these discrete variables and the subscript j and j+1 indicate the iteration step . Table 1 compares the first displacement peak results for three practical time-steps and the instable time step Δt limit for $\delta=0.4$, $\alpha=1.0$, $\beta=0.5$, p=0.5 and $\omega=0.5$.

Table 1. First peak displacement and instability limit

Δt	LAIER[1]	ARGYRES[3]	NEWMARK
0.2s	0.5050	0.5050	0.5041
0.1s	0.5220	0.5220	0.5217
0.05s	0.5303	0.5303	0.5302
Instability	stable	0.6622s	stable

The results show that these two third-order algorithms present the same accuracy, but the cubic algorithm presents conditional numerical stability.

Van der Pol equation

The Van der Pol equation and its first time derivative are given by

$$\begin{aligned} &\ddot{y} - \mu\left(y_0^2 - y^2\right)\dot{y} + \omega_0^2 y = 0 \\ &\dddot{y} - \mu\ddot{y}\left(y_0^2 - y^2\right) + 2\mu y\dot{y}^2 + \omega_0^2\dot{y} = 0 \end{aligned} \tag{6}$$

where μ, y_0 and ω_0 are parameters of the equation. Table 2 compares the first displacement peak results for three practical time-steps and the instable time step Δt limit for $\mu = 1.5$, $y_0 = 1$ and $\omega_0 = 1$.

Table 2. First peak displacement and instability limit

Δt	LAIER[1]	ARGYRES[3]	NEWMARK
0.2s	-0.3193	-0.3193	-0.3127
0.1s	-0.3193	-0.3193	-0.3177
0.05s	-0.3199	-0.3193	-0.3189
Instability	3.773s	11.83s	stable

The results show that also these two third-order algorithms present the same accuracy, but just Newmark method presents unconditional numerical stability.

Nonlinear pendulum

Figure 1 depicts the nonlinear pendulum that has been extensively analyzed by Argyris and Mlejek [2]. The equation of motion and its first derivative are written as:

$$\begin{aligned} &m\ddot{y} + c\dot{y} + 2ky + \left(2N_0 - 2ak\right)\left(a^2 + y^2\right)^{-0.5} y = f(t) \\ &m\dddot{y} + c\ddot{y} + 2k\dot{y} + \left(2N_0 - 2ak\right)\left(a^2 + y^2\right)^{-0.5}\dot{y} - 2\left(2N_0 - 2ak\right)\left(a^2 + y^2\right)^{-1.5} y^2\dot{y} = \dot{f}(t) \end{aligned} \tag{7}$$

where m is the mass of the pendulum, c is the damping, k is the stiffness, N_0 is the pre-tension force of the string and $f(t)$ is the excitation force. Table 3 compares the first displacement peak results for three practical time-steps and the instable time step Δt limit for $m = 500\text{Kg}$, $N_0 = 500\text{N}$, $a = 1\text{m}$, $k = 10^7\,\text{N/m}$ and $f(t) = 50\left(1 - \cos(23.73t)\right)$.

Table 3. First peak displacement and instability limit

Δt	LAIER[1]	ARGYRES[3]	NEWMARK
0.022648s	0.024645	0.024645	0.024624
0.0052958s	0.024656	0.024656	0.024656
0.0022648s	0.24656	0.024656	0.024656
Instability	stable	2.2648s	stable

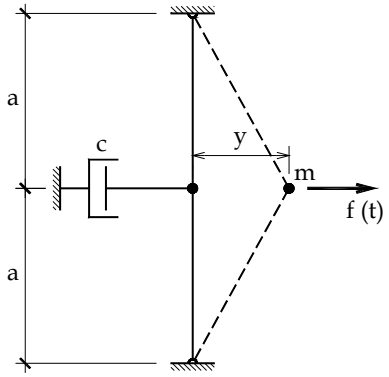

Figure 1. Nonlinear pendulum

The results shown in Table 3 indicate that the considered two third-order algorithms present the same accuracy, but the Newmark method and the algorithm developed by the author [1] present unconditional numerical stability.

Lotka-Volterra equation

The predator-prey Lotka-Volterra equation and its second and third time derivatives are given by

$$\begin{aligned}
\dot{x} &= kx - axy \\
\dot{y} &= -ly + bxy \\
\ddot{x} &= k\dot{x} - a\dot{x}y - ax\dot{y} \\
\ddot{y} &= -l\dot{y} + b\dot{x}y + bx\dot{y} \\
\dddot{x} &= k\ddot{x} - a\ddot{x}y - 2a\dot{x}\dot{y} - ax\ddot{y} \\
\dddot{y} &= -l\ddot{y} + b\ddot{x}y + 2b\dot{x}\dot{y} + bx\ddot{y}
\end{aligned} \tag{8}$$

where k, a, l and b are positive constant. As the Lotka-Volterra is of first-order just the first Hermitian operator given by equation (1) is involved. Table 4 compares the first displacement peak results for three practical time-steps and the instable time step Δt limit for k=a=l=b=1.

Table 4. First minimum peak for x function and instability limit

Δt	LAIER[1]	ARGYRES[3]	NEWMARK
0.01s	$0.560288\ 10^{-6}$	$0.560287\ 10^{-6}$	$0.549775\ 10^{-6}$
0.001s	$0.560280\ 10^{-6}$	$0.560280\ 10^{-6}$	$0.560174\ 10^{-6}$
0.0001s	$0.560280\ 10^{-6}$	$0.560280\ 10^{-6}$	$0.560279\ 10^{-6}$
Instability	0.157s	0.0952s	0.119s

The results shown in Table 4 indicate that the two third-order algorithms present again the same accuracy, but these three algorithms are not unconditional stable.

Lorenz equation

The atmospheric convection Lorenz model is governed by the equation

$$\begin{aligned} \dot{x} &= -\sigma(x - y) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned} \tag{9}$$

where σ, r and b are constant. The second and third derivatives of equation (9) are given by

$$\begin{aligned} \ddot{x} &= -\sigma(\dot{x} - \dot{y}) \\ \ddot{y} &= r\dot{x} - \dot{y} - \dot{x}z - x\dot{z} \\ \ddot{z} &= \dot{x}y + x\dot{y} - b\dot{z} \\ \dddot{x} &= -\sigma(\ddot{x} - \ddot{y}) \\ \dddot{y} &= r\ddot{x} - \ddot{y} - \ddot{x}z - 2\dot{x}\dot{z} - x\ddot{z} \\ \dddot{z} &= \ddot{x}y + 2\dot{x}\dot{y} + x\ddot{y} - b\ddot{z} \end{aligned} \tag{10}$$

As the Lorenz is of first-order just the first Hermitian operator given by equation (1) is involved Table 5 compares the first displacement peak results for three practical time-steps and the instable time step Δt limit for $\sigma = 10.0$, $r = 28.0$ and $b = 8/3$.

Table 5. First minimum peak for x function and instability limit

Δt	LAIER[1]	ARGYRES[3]	NEWMARK
0.01s	$0.203652\ 10^2$	$0.198015\ 10^2$	$0.135838\ 10^2$
0.001s	$0.200112\ 10^2$	$0.198099\ 10^2$	$0.200108\ 10^2$
0.0001s	$0.199781.10^2$	$0.198100\ 10^2$	$0.199781\ 10^2$
Instability	stable	0.00510s	0.0976s

The results presented in Table 5 show that the two third-order algorithms present quite similar accuracy, but in this case just the algorithm developed by the author is unconditionally stable.

Conclusions

The numerical applications show that the third-order algorithm developed by the author [1] and the cubic Hermitian developed by Argyris and Mlejek present as expected quite similar accuracy for refined mesh and little discrepancy for coarse mesh. The Newmark method also presents similar accuracy for refined mesh, but the discrepancy of the accuracy increase for coarse mesh. The time integration algorithm developed by the author is conditionally stable for Van der Pol and Lotka-Volterra equations. On the other hand, the Newmark method is conditionally stable for Lotka-Volterra and Lorenz equation. Finally, one has to note that the cubic Hermitian is conditionally stable for these five equations.

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