The extended Timoshenko beam element in finite element analysis for the investigation of size effects

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Abstract

In this paper an extended Timoshenko beam element is developed for the investigation of size effect via finite element analysis. The surface effect derived from initial surface stress and surface elasticity is considered as external pressure in terms of the generalized Young-Laplace equation and the virtual displacement principle. We find the size effect highly relies on the geometrical model considered in numerical simulation. For a cantilever nanowire the stocky beam suddenly becomes strengthened provided the diameter is less than a critical size, while it is weakened for slender case. These abnormal changes of stiffness can be supported by static bending tests. This method will bring useful insight into the size effect and is of importance to some engineering applications like nanofabrication and nano sensors.

Keywords: Size effects, Surface effects, Timoshenko beam, FEM

Introduction

Size effects broadly refer to the abnormal changes of mechanical properties as the structure size approaches to tens of nanometer.[1,2] Over the last couple of decades, increasing attentions have been drawn to these behaviors because nanostructures have emerged as one of the most attractive topics and size effect at nanoscale has great potential to design lightweight material and sensors.[3] Previous studies have indicated size effects are attributed to the large ratio of surface area to the material volume, in which case the interactions of superficial atoms become extremely active. However the inherent mechanism is still a challenging problem.

In general, investigations of size effects on mechanical properties can be divided into two major groups, namely experimental validation and theoretical analysis based on simple beam theories. There have been many literatures report that surge of effecitveYoung's moduli is observed experimentally as the characteristic size approaching to nanometers.[4,5] Classical continuum theory cannot formulate this size dependent characteristic since it lacks of mechanism to account for the size effects on the mechanical properties of material.[6] Many efforts have been dedicated to build analytical framework to including size effect due to the difficult in manufacturing and controlling of materials at a length scale of several tens or hundreds nanometers.[7-10] Method like classical molecular dynamics simulation[11], nonlocal theory of elasticity[12] are effective in predicting size-dependence of mechanical properties at nano scale. However, the computational cost is intensive and paradoxes arise.[13-16]

Recently by incorporating surface elasticity[17] and generalized Young-Laplace equation,[18] the analytical solutions that predict the size-dependence of effective Young's modulus of nanomaterials show a good agreement with experimental data.[8,9] A recursive algorithm that breaks the constraint on model complexity for analytical solution successfully captured size effects of continuum nanoscale solid with complex 3D topology and obtained result that matches with experimental data.[7] But computation cost and convergence issue still remain. Given the fact that a great portion of nanomaterials that ubiquitously existing in nature is consist of beam like

ligaments,[14] it is very attractive to establish an extended beam element which can formulate the size dependence of the mechanical properties and overcomes those aforementioned drawbacks.
To incarnate the size effects in the simulation model, an extended Timoshenko beam element is developed so that surface elasticity theory and generalized Young-Laplace equation are well integrated into finite element analysis. Shape-dependent pressure is introduced in the model to serve as the external loading under which the nanostructure is deformed. Theoretical prediction is verified by two numerical tests which show the softened and strengthened beams below critical dimensional size.

Surface elasticity theory and generalized Young-Laplace equation

Surface elasticity theory [19] states that the surface stress $\sigma^s_{\alpha\beta}$, a symmetric 2×2 tensor in the tangent plane, is:

$$\sigma^s_{\alpha\beta} = \frac{\partial G\left(\varepsilon^s_{\alpha\beta}\right)}{\partial \varepsilon^s_{\alpha\beta}} + \tau_0 \delta_{\alpha\beta} \quad (\alpha, \beta = 1,2,3) \tag{1}$$

here $\varepsilon^s_{\alpha\beta}$ denotes the surface strain tensor, $G(\varepsilon^s_{\alpha\beta})$ is the surface energy and $\delta_{\alpha\beta}$ is the Kronecker delta. The initial surface stress is represented by τ_0. With assumption that the surface is homogeneous, isotropic, and linearly elastic, the overall surface stress tensor can be further simplified to:

$$\sigma^s_{\alpha\beta} = \tau_0 + E_s \varepsilon^s_{\alpha\beta} \tag{2}$$

where E_s is the surface stiffness.
From generalized Young-Laplace equation [18], a stress jump normal to the interface which depends on the curvature $\kappa_{\alpha\beta}$ and surface tension $\tau_{\alpha\beta}$ occurs on the curved material surface as:

$$\Delta\sigma_{ij} n_i n_j = \tau_{\alpha\beta} \kappa_{\alpha\beta} \tag{3}$$

here n_i and n_j is the unit normal vectors of the material surface.
Equations (1), (2) and (3) formulated the surface effects as a curvature-dependent distributed load along the normal direction of beam surface, as show in Fig. 1.

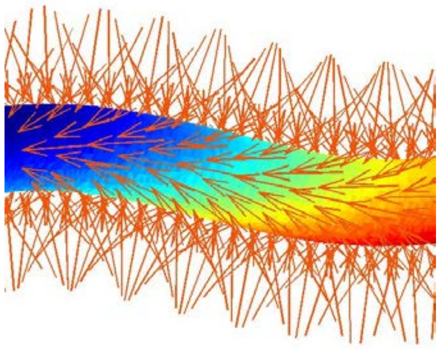

FIG. 1. A schematic of size-effect-induced pressure (red arrows) on the beam surface.

Timoshenko beam with surface effect

The formulation of the element stiffness matrix for extended Timoshenko beam element comprises contributions from axial compression, torsional and bending. Axial and torsional effects are considered in the conventional manner.
The bending contribution is formulated under Timoshenko beam theory. Element stiffness is derived from a 2D circular cross-section beam model with only bending considered for simplicity, extending to 3D is straightforward.

the axial $u(x, y)$ and transverse $v(x, y)$ displacements in the x-y plane is used to describe the motion of an arbitrary material point on the beam. Here motion in z direction is not considered. The assumption of Timoshenko beam theory can be represented as:

$$\begin{aligned} u(x, \mathrm{y}) &= -y(\frac{\partial v(x)}{\partial x} + \gamma) \\ v(x, y) &= v(x) \\ \theta_z &= \frac{\partial v}{\partial x} + \gamma = v' + \gamma \\ \gamma &= \frac{V}{GA_s} \end{aligned} \tag{4}$$

here θ_z is the rotation angle and γ is angle of shearing.

The strain can be determined by differentiating the displacement of (5) as:

$$\begin{aligned} \varepsilon_{11} &= -y\frac{\partial \theta_z}{\partial x} \\ \varepsilon_{13} &= \frac{1}{2}(\theta_z - \frac{\partial v}{\partial x}) = \frac{1}{2}\gamma \end{aligned} \tag{5}$$

The stress component given by Hooke's law is,

$$\begin{aligned} \sigma_{11} &= E\varepsilon_{11} \\ \sigma_{13} &= G\gamma \end{aligned} \tag{6}$$

here E is the elastic modulus, G is the shear modulus. The bending moment M over the cross section is the integral,

$$M = \int_A -y\sigma_{11} dA \tag{7}$$

According to the principle of virtual displacements, the virtual external work of real external forces moving through collocated virtual displacements equals the internal virtual work of real stresses in equilibrium with real forces with the virtual strains compatible with the virtual displacements integrated over the volume of the solid[20] and can be mathematically expressed as:

$$\begin{aligned} &\delta W_I = \delta W_E \\ &\int_v \sigma \cdot \delta\varepsilon dV = \int_v \delta U^T f^B dV + \int_S \delta U^{ST} f^s dV + \sum_i \delta U^{iT} F^i \end{aligned} \tag{8}$$

where δW_I is the total internal virtual work and δW_E is the total external virtual work. σ is the actual stress, $\delta\varepsilon$ is the virtual strains. f^b, f^s and F^i are the actual external body force, surface traction and concentrated force and δU^T, δU^{ST} and δU^{iF} are the corresponding virtual displacement.

The overall internal virtual work of Timoshenko beam including surface effect can be express as:

$$\delta W_I = \delta W_{IC} + \delta W_{IS} \tag{9}$$

where δW_I denote the overall internal virtual work and it is consist of the contribution from the conventional bending and shearing effects and the contribution of surface effects from initial surface tension and surface stiffness, denoted as δW_{IC} and δW_{IS} correspondingly. The Timoshenko beam theory assumes that the internal energy of beam member is due to bending and shearing which can be expressed as:

$$\delta W_{IC} = \int_0^L EI(\frac{\partial \theta_z}{\partial x})^2 dx + \int_0^L \kappa AG(\theta_z + \frac{\partial v}{\partial x})^2 dx \tag{10}$$

here I denotes the moment of inertia of the cross section. EI is the flexure rigidity, κ denotes the shear area coefficient and κ = 10/9 for solid circular sections, A is the cross section area.

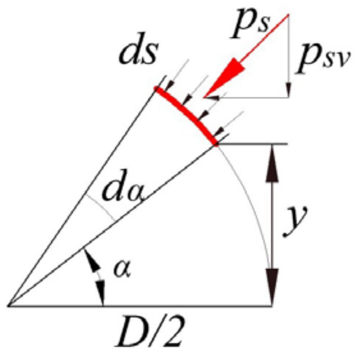

FIG. 2. Circular cross section of beam with surface effects considered.

For a representative infinitesimal surface element on the surface of the cross section display as red arc in Fig. 2, according to Eq. (6) the longitudinal strain, which is perpendicular to the cross sectional plane, is,

$$\varepsilon_s = -\frac{D}{2}\sin\alpha\frac{\partial\theta_z}{\partial x} \tag{11}$$

Base on Eq. (2), the surface stress along beam axis can be expressed as:

$$\begin{aligned}\tau_{axial} &= \tau_0 + E_s\varepsilon_s \\ &= \tau_0 + E_s\left(-\frac{D}{2}\sin\alpha\frac{\partial\theta_z}{\partial x}\right)\end{aligned} \tag{12}$$

This surface stress along beam axis introduces an extra moment on the infinitesimal surface element

$$\begin{aligned}dM_s &= -\tau_{axial}yds \\ &= -\left(\tau_0 + E_s\left(-\frac{D}{2}\sin\alpha\frac{\partial\theta_z}{\partial x}\right)\right)\frac{D}{2}\sin\alpha\frac{D}{2}d\alpha\end{aligned} \tag{13}$$

By integration along the edge of the cross section the overall moment of the surface effect at this cross section is,

$$\begin{aligned}M_s &= \int dM_s \\ &= \int_0^{2\pi} -\left(\tau_0 + E_s\left(-\frac{D}{2}\sin\alpha\frac{\partial\theta_z}{\partial x}\right)\right)\frac{D}{2}\sin\alpha\frac{D}{2}d\alpha \\ &= \frac{\pi D^3}{8}E_s\frac{\partial\theta_z}{\partial x}\end{aligned} \tag{14}$$

The contribution of the surface effects to the internal work is then determined as:

$$\delta W_{IS} = \int_0^L M_s\frac{\partial\theta_z}{\partial x}dx = \int_0^L \frac{\pi E_s D^3}{8}\left(\frac{\partial\theta_z}{\partial x}\right)^2 dx \tag{15}$$

The overall internal virtual work is obtained as:

$$\delta W_I = \delta W_{IC} + \delta W_{IS} = \int_0^L EI\left(\frac{\partial\theta_z}{\partial x}\right)^2 dx + \int_0^L \kappa AG\left(\theta_z + \frac{\partial v}{\partial x}\right)^2 dx + \int_0^L \frac{\pi E_s D^3}{8}\left(\frac{\partial\theta_z}{\partial x}\right)^2 dx \tag{16}$$

The external virtual work δW_E is also composed of the conventional and the surface effect part. The conventional part is the work done by the external load as:

$$\delta W_{EC} = \int_0^L q_c v dx \tag{17}$$

where q_c is the transverse force per unit length that acts on the beam.

From generalized Young-Laplace equation the surface tension alone beam longitudinal direction causes a force normal to the surface, as shown in Fig. 2 as red arrow, which can be expressed as:

$$p_s = \tau_{axial}\frac{\partial\theta_z}{\partial x}\frac{D}{2}\sin\alpha d\alpha \tag{18}$$

Only the force component acting in the flexure plane contributes to the external virtual work and can be obtained by decomposition as,

$$p_{s\_flexure} = p_s\sin\alpha = \tau_{axial}\frac{\partial\theta_z}{\partial x}\frac{D}{2}\sin\alpha d\alpha\sin\alpha \tag{19}$$

By integrating $p_{s\_flexure}$ around the edge of the cross section total surface effects induced transverse load at this cross section can be obtained as:

$$\begin{aligned} q_s &= \int \tau_{axial} \frac{\partial \theta_z}{\partial x} \sin \alpha ds \\ &= \int_0^{\pi} \frac{D}{2} \frac{\partial \theta_z}{\partial x} (\tau_0 - E_s \sin \alpha \frac{D}{2} \frac{\partial \theta_z}{\partial x}) \sin \alpha d\alpha - \int_{\pi}^{2\pi} \frac{D}{2} \frac{\partial \theta_z}{\partial x} (\tau_0 - E_s \sin \alpha \frac{D}{2} \frac{\partial \theta_z}{\partial x}) \sin \alpha d\alpha \\ &= 2\tau_0 D \frac{\partial \theta_z}{\partial x} \end{aligned} \tag{20}$$

the surface effects part for the external work is,

$$\begin{aligned} \delta W_{ES} &= \int_0^L q_s v dx \\ &= \int_0^L 2\tau_0 D \frac{\partial \theta_z}{\partial x} v dx \end{aligned} \tag{21}$$

The total external virtual work is then determined as:

$$\delta W_E = \delta W_{EC} + \delta W_{ES} = \int_0^L q_c v dx + \int_0^L 2\tau_0 D \frac{\partial \theta_z}{\partial x} v dx \tag{22}$$

The virtual displacement principle of Timoshenko beam with surface effect including is then obtained as:

$$\int_0^L EI (\frac{\partial \theta_z}{\partial x})^2 dx + \int_0^L \kappa AG(\theta_z + \frac{\partial v}{\partial x})^2 dx + \int_0^L \frac{\pi E_s D^3}{8} (\frac{\partial \theta_z}{\partial x})^2 dx = \int_0^L q_c v dx + \int_0^L 2\tau_0 D \frac{\partial \theta_z}{\partial x} v dx \tag{23}$$

Compared with that of ordinary beam element,[21] after some rearrangement of Eq. (24),the controlling equation that correspond to the beam element with surface effect considered becomes,

$$\int_0^L EI (\frac{\partial \theta_z}{\partial x})^2 dx + \int_0^L \kappa AG(\theta_z + \frac{\partial v}{\partial x})^2 dx + \int_0^L \frac{\pi E_s D^3}{8} (\frac{\partial \theta_z}{\partial x})^2 dx - \int_0^L 2\tau_0 D \frac{\partial \theta_z}{\partial x} v dx = \int_0^L q_c v dx \tag{24}$$

By introducing the displacement interpolation matrix and strain displacement matrix, stiffness matrix of the extended Timoshenko beam element with surface effects can be obtained. The detailed finite element implementation is out of the scope of this paper.

Case study

Here we compare the deflection of cantilever obtain from proposed extended Timoshenko element with the analytical solution with and without surface effect. For cantilever beam with unit diameter, E_s=3.63 N/m and τ_0=1.22 N/m, the deflections predicted by analytical solution which do not have surface effect and the proposed new beam element for slenderness ratio equals 5 and 16 are plotted in Fig. 3.

It can be seen from Fig. 3(a) that the deflection obtained by proposed element, as shown with red solid line, have size effect since with the size approaching to nm scale the deflection decreases which indicate a strengthen effect. As the size extending to macro scale the deflection converges to that of conventional result. The analytical solutions obtained by both Timoshenko and Euler-Bernoulli beam, on the other hand, cannot capture this size effect as the deflection is constant with the change of scale. The difference between the green and blue line here is due to the difference between beam theories for stocky beam for which the Timoshenko beam theory is more physically realistic. Fig. 3(b) is the same simulation for slender beam with L/D = 16, it can been seen that the deflection at macro scale converges to the same value which is consistent with the theory that for beam with L/D > 16 the shear effect is negligible and both Timoshenko and Euler-Bernoulli beam theory obtain the same result. Meanwhile the deflection prediction by proposed element increases with decreasing of scale which indicates a softening effect occurs.

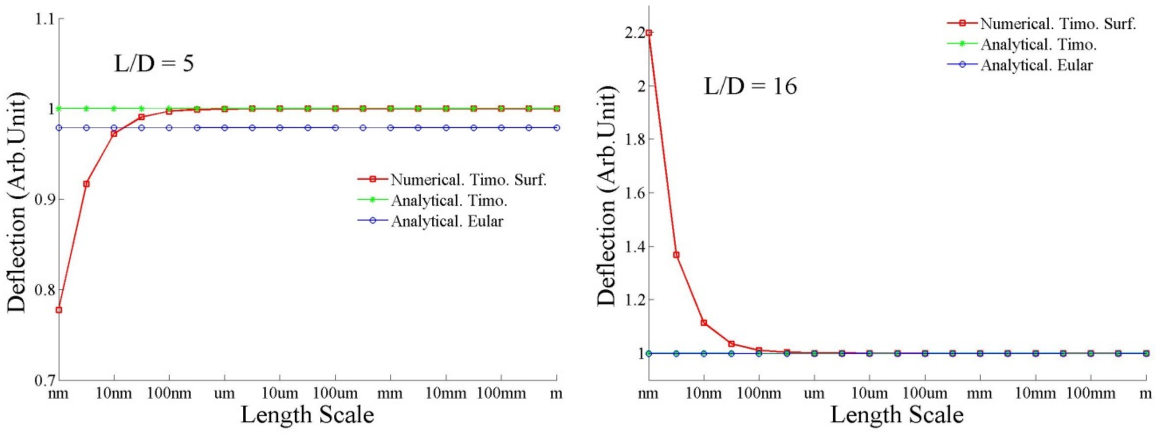

FIG. 3. Deflection prediction of proposed element and the analytical solution obtain using Timoshenko and Euler-Bernoulli beam for L/D = 5 and 16.

Conclusions

Based on the principle of virtual displacements, we derive the weak form for Timoshenko beam element with surface effects considered. Two characteristic parameters, the surface stiffness and initial surface tension, are introduced to be responsible for the size effect. Numerical simulation results successfully captured the size effect, strengthening and softening effects as the size decrease to nano meter is also observed which is consistent with theoretical prediction and experimental observation.

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