Efficient computation of the tangency portfolio by linear programming

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ABSTRACT

In several problems of portfolio selection the reward-risk ratio criterion is optimized to search for a risky portfolio offering the maximum increase of the mean return, compared to the risk-free investment opportunities. In the classical model, following Markowitz, the risk is measured by the variance thus representing the Sharpe ratio optimization and leading to the quadratic optimization problems. Several polyhedral risk measures, being Linear Programming (LP) computable in the case of discrete random variables represented by their realizations under specified scenarios, have been introduced and aplied in portfolio optimization. The reward-risk ratio optimization with polyhedral risk meausures can be transformed into LP formulations. The LP models typically contain the number of constraints (matrix rows) proportional to the number of scenarios while the number of variables (matrix columns) proportional to the total of the number of scenarios and the number of instruments. They can effectively be solved with general purpose LP solvers provided that the number of scenarios is limited. However, real-life financial decisions are usually based on more advanced simulation models employed for scenario generation where one may get several thousands scenarios. This may lead to the LP models with huge number of variables and constraints thus decreasing their computational efficiency and making them hardly solvable by general LP tools. We show that the computational efficiency can be then dramatically improved by alternative models based on the inverse ratio minimization and taking advantages of the LP duality. In the introduced models the number of structural constraints (matrix rows) is proportional to the number of instruments thus not affecting seriously the simplex method efficiency by the number of scenarios and therefore guaranteeing easy solvability.

Keywords: Computation, Portfolio optimization, Reward-risk ratio, Tangency portfolio, Polyhedral risk measures, Fractional programming, Linear programming.

Linear programming models for reward-risk ratio optimization

Portfolio selection problems are usually tackled with the mean-risk models that characterize the uncertain returns by two scalar characteristics: the mean, which is the expected return, and the risk - a scalar measure of the variability of returns. In the original Markowitz model [8] the risk is measured by the standard deviation or variance. Several other risk measures have been later considered thus creating the entire family of mean-risk (Markowitz-type) models. While the original Markowitz model forms a quadratic programming problem, many attempts have been made to linearize the portfolio optimization procedure (c.f., [7] and references therein). The LP solvability is very important for applications to real-life financial decisions where the constructed portfolios have to meet numerous side constraints (including the minimum transaction lots, transaction costs and mutual funds characteristics) [6]. A risk measure can be LP computable in the case of discrete random variables, i.e., in the case of returns defined by their realizations under specified scenarios. Several such polyhedral risk measures have been applied to portfolio optimization [7]. Typical risk measures are deviation type. The simplest LP computable risk measures are dispersion measures similar to the variance. Konno and Yamazaki [4] introduced the portfolio selection model with the mean absolute deviation (MAD). Young [18] presented the Minimax model while earlier Yitzhaki [17] introduced the mean-risk model using Gini's mean (absolute) difference as the risk measure. The Gini's mean difference turns out to be a special aggregation technique of the multiple criteria LP model [9] based on the pointwise comparison of the absolute Lorenz curves. The latter makes the quantile shortfall risk measures directly related to the dual theory of choice under risk [15]. Recently, the second order quantile risk measures have been introduced in different ways by many authors [1][16]. The measure, usually called the Conditional Value at Risk (CVaR) or Tail VaR, represents the mean shortfall at a specified confidence level. The CVaR measures maximization is consistent with the second degree stochastic dominance [11]. The LP computable portfolio optimization models are capable to deal with non-symmetric distributions. Some of them, like the mean absolute semideviation (MAD model) can be combined with the mean itself into optimization criteria (safety or underachievement measures) that remain in harmony with the

Second order Stochastic Dominance (SSD). Some, like the conditional value at risk (CVaR) [16] having a great impact on new developments in portfolio optimization, may be interpreted as such a combined functional while allowing to distinguish the corresponding deviation type risk measure.

Having given the risk-free rate of return r_0, a risky portfolio $\mathbf{x}$ may be sought that maximizes ratio between the increase of the mean return $\mu(\mathbf{x})$ relative to r_0 and the corresponding increase of the risk measure $\varrho(\mathbf{x})$, compared to the risk-free investment opportunities. Namely, a performance measure of the reward-risk ratio is defined $(\mu(\mathbf{x}) - r_0)/\varrho(\mathbf{x})$ to be maximized. The optimal solution of the corresponding problem is usually called the tangency portfolio as it corresponds to the tangency point of the so-called capital market line drawn from the intercept r_0 and passing tangent to the risk/return frontier. For the LP computable risk measures the reward-risk ratio optimization problem can be converted into an LP form [5]. The reward-risk ratio is well defined for the deviation type risk measures. Therefore while dealing with the CVaR or Minimax risk model we must replace this performance measure (coherent risk measure) $C(\mathbf{x})$ with its complementary deviation representation $\mu(\mathbf{x}) - C(\mathbf{x})$ [3][5]. The reward-risk ratio optimization with polyhedral risk meausures can be transformed into LP formulations. Such an LP model, for instance for the CVaR risk measure, contains then T auxiliary variables as well as T corresponding linear inequalities. Actually, the number of structural constraints in the LP model (matrix rows) is proportional to the number of scenarios T, while the number of variables (matrix columns) is proportional to the total of the number of scenarios and the number of instruments $T+n$. Hence, its dimensionality is proportional to the number of scenarios T. It does not cause any computational difficulties for a few hundreds scenarios as in computational analysis based on historical data. However, real-life financial analysis must be usually based on more advanced simulation models employed for scenario generation [2]. One may get then several thousands scenarios [14] thus leading to the LP model with huge number of auxiliary variables and constraints and thereby hardly solvable by general LP tools. Similar difficulty for the standard minimum risk portfolio selection have been effectively resolved by taking advantages of the LP duality to reduce the number of structural constraints to the number of instruments [12][13]. For the linearized reward-risk ratio models such an approach does not work. Although, for the CVaR risk measure we have shown [10] that while taking advantages of possible inverse formulation of the reward-risk ratio optimization as ratio $\varrho(\mathbf{x})/(\mu(\mathbf{x}) - r_0)$ to be minimized and the LP dual of th elinearized problem one can get the number of constraints limited to the number of instruments.

In this paper we analyze efficient optimization of reward-risk ratio for various LP computable risk measures. Taking advantages of possible inverse formulation of the reward-risk ratio optimization as ratio $\varrho(\mathbf{x})/(\mu(\mathbf{x}) - r_0)$ to be minimized, we show that (under natural assumptions) this ratio optimization is consistent with the SSD rules, despite that the ratio does not represent a coherent risk measure [1]. Further, while transforming this ratio optimization to an LP model, we take advantages of the LP duality to get a model formulation providing higher computational efficiency. For the MAD and Minimax measures the number of structural constraints in the introduced model is proportional to the number of instruments n while only the number of variables is proportional to the number of scenarios T thus not affecting so seriously the simplex method efficiency. Therefore, the model can effectively be solved with general LP solvers even for very large numbers of scenarios. Indeed, the computation time for the case of fifty thousand scenarios and one hundred instruments is then below a minute. The reformulation can also be applied to more complex quantile risk measures. The Tail Gini's measures or the Weighted CVaR measures defined as combinations of CVaR measures for m tolerance levels lead originally to LP models with the number of structural constraints (matrix rows) proportional to the respectively multiplied number of scenarios mT. In the alternative model taking advantages of the LP duality the number of structural constraints is proportional to the total of the number of instruments and number of tolerance levels $n+m$. This guarantees a high computational efficiency of the dual model even for a very large number of scenarios. The standard LP models for the Gini's mean difference require T^2 auxiliary constraints which makes them hard already for medium numbers of scenarios, like a few hundred scenarios given by historical data. The models taking advantages of the LP duality allow one to limit the number of structural constraints making it proportional to the number of scenarios T thus increasing dramatically computational performances for medium numbers of scenarios although still remaining hard for very large numbers of scenarios.

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References

[1] Artzner, P., Delbaen, F., Eber, J.-M. and Heath, D. (1999) Coherent measures of risk. *Mathematical Finance* **9**, 203–228.

[2] Carino, D.R., Myers, D.H. and Ziemba, W.T. (1998) Concepts, technical issues and uses of the Russel-Yasuda Kasai financial planning model. *Operations Research* **46**, 450–463.

[3] Chekhlov, A., Uryasev, S. and Zabarankin, M. (2002) Drawdown measure in portfolio optimization. *International Journal of Theoretical and Applied Finance* **26**, 1443–1471.

[4] Konno, H. and Yamazaki, H. (1991) Mean-absolute deviation portfolio optimization model and its application to Tokyo stock market. *Management Science* **37**, 519–531.

[5] Mansini, R., Ogryczak, W. and Speranza, M.G. (2003) On LP solvable models for portfolio optimization. *Informatica* **14**, 37–62.

[6] Mansini, R., Ogryczak, W. and Speranza, M.G. (2015) *Linear and Mixed Integer Programming for Portfolio Optimization*. Springer, Cham, Switzerland.

[7] Mansini, R., Ogryczak, W. and Speranza, M.G. (2014) Twenty years of linear programming based portfolio optimization. *European Journal of Operational Research* **234**, 518–535.

[8] Markowitz, H.M. (1952) Portfolio selection. *Journal of Finance*, **7**, 77–91.

[9] Ogryczak, W. (2000) Multiple criteria linear programming model for portfolio selection. *Annals of Operations Research*, **97**, 143–162.

[10] Ogryczak, W., Przyłuski, M. and Śliwiński, T. (2015) Portfolio Optimization with Reward-Risk Ratio Measure based on the Conditional Value-at-Risk. *Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering and Computer Science 2015, WCECS 2015, 21-23 October, 2015, San Francisco, USA*, 913-918.

[11] Ogryczak, W. and Ruszczyński, A. (2002) Dual stochastic dominance and related mean-risk models. *SIAM Journal on Optimization* **13**, 60–78.

[12] Ogryczak, W. and Śliwiński, T. (2011) On solving the dual for portfolio selection by optimizing Conditional Value at Risk. *Computational Optimization and Applications* **50**, 591–595.

[13] Ogryczak, W. and Śliwiński, T. (2011) On dual approaches to efficient optimization of LP computable risk measures for portfolio selection. *Asia-Pacific Journal of Operational Research* **28**, 41–63.

[14] Pflug, G.Ch. (2001) Scenario tree generation for multiperiod financial optimization by optimal discretization. *Mathematical Programming* **89**, 251–271.

[15] Quiggin, J. (1982) A theory of anticipated utility. *Journal of Economic Behavior and Organization* **3**, 323–343.

[16] Rockafellar, R.T. and Uryasev, S. (2002) Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance* **26**, 1443–1471.

[17] Yitzhaki, S. (1982) Stochastic dominance, mean variance, and Gini's mean difference. *The American Economic Revue* **72**, 178–185.

[18] Young, M.R. (1998) A minimax portfolio selection rule with linear programming solution. *Management Science* **44**, 673–683.