Comparisons of Limiters in Discontinuous Galerkin Method

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Abstract

The discontinuous Galerkin method (DGM) is an important numerical method in computational fluid dynamics. The characteristics of DGM include its flexibility to construct high order schemes by using high order basis functions, and its compactness regardless of basis function orders. In supersonic simulations, the DGM often perform severe oscillations in regions where strong discontinuity solutions appear, so the slope limiters become necessary. In this study, Three slope limiters are considered: TVB limiter, WENO limiter and HWENO limiter. The performance of these limiters are compared and analyzed with two dimensional supersonic cylinder flow. The results of show that all these limiters are able to stabilize the solution procedure, but the solutions show some differences between these limiters. Explanations as well as possible improvements are given.

Keywords: Discontinuous Galerkin Method, Supersonic Flow, Slope Limiter

Introduction

The discontinuous Galerkin method was first proposed by Reed and Hill [Reed and Hill (1973)] for neutron transportation problems, since then, the application of this method is widely extended. The applications include fluid simulations, MHD simulations, shallow water simulations and many others. In the area of supersonic flow simulations, the traditional methods include finite volume method and finite difference method, both of these methods have defects in modern supersonic flow simulations, the finite difference method has its weakness of dealing with complex geometric shapes, and the finite volume method has its weakness of constructing high order scheme on unstructured meshes. The DGM could overcome both of the defects of traditional methods. By introducing element-wise polynomial basis functions and inter-cell numerical fluxes, the DGM could have compact stencil on complex geometries. These characteristics make it an ideal candidate of next generation supersonic flow simulations.

When DGM is utilized in supersonic flow simulations with strong shockwaves, numerical instability is a major problem of this scheme, which will cause non-physical oscillations and divergent solutions. Many possible solutions have been proposed to overcome this defect, artificial viscosity and slope limiters are the two main approaches. In this study, slope limiters are utilized to suppress non-physical oscillations. Slope limiters were adopted into DGM by a collective effort of many researchers[Cockburn and Shu (1989, 1998); Kuzmin and Turek (2004); Xing and Shu (2006); Hong et al.(2007)]. Slope limiters will detect severe oscillations of solutions and smooth them with smoother polynomials. In this study, the performances of TVB limiter[Cockburn and Shu (1989)], WENO limiter[Xing and Shu (2006)] and HWENO limiter[Hong et al.(2007)] in shockwave regions are compared.

Governing Equations

The governing equations of two dimensional inviscid supersonic flow are Euler equations are. The conservation forms of these equations are:

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=0 \tag{1}$$

where U, F and G refer to conservative state vector, x-direction inviscid flux and y-direction inviscid flux respectively.

$$U=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho E\end{bmatrix};\ F=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ (\rho E+p)u\end{bmatrix};\ G=\begin{bmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ (\rho E+p)v\end{bmatrix} \tag{2}$$

To enclose the equation system, the equation of state is introduced.

$$p=\rho RT \tag{3}$$

Discontinuous Galerkin Method

The physical domain Ω_h is divided into non-overlapping elements K, where $\cup K=\Omega_h$. A reference element K' is introduced to simplify numerical integrations, the reference element and physical element are connected with coordinates mapping.

$$F_K : K' \to K : \xi \mapsto x = \sum_{i=1}^{m} x_i \chi_i(\xi) \tag{4}$$

where m, χ_i, and x_i refer to number of element interpolation functions, element shape functions and shape function coefficients respectively.

At any moment t, the unknowns $U_h(\xi,t)$ on reference element can be expressed in basis function space $span\{\psi_i(\xi)\}; i=1,..,m$.

$$U_h(\xi,t)=\sum_{i=1}^{m} U_i(t)\psi_i(\xi) \tag{5}$$

In each element, the weak form of governing equations is introduced.

$$\int_{K_j}\left(\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}\right)\phi_i \mathrm{d}\Omega = 0 \quad ; \quad (i=1,\ldots,m) \tag{6}$$

with some manipulations, the equations have the following form.

$$\begin{aligned}&\int_{K_j}\frac{\partial U}{\partial t}\phi_i \mathrm{d}\Omega-\int_{K_j}\left(F\frac{\partial \phi_i}{\partial x}+G\frac{\partial \phi_i}{\partial y}\right)\mathrm{d}\Omega\\ &+\int_{\partial K_j}\phi_i\left(Fn_x+Gn_y\right)\mathrm{d}S=0 \quad ; \quad (i=1,\ldots,m)\end{aligned} \tag{7}$$

where F and G refer to inter-cell fluxes, in DGM, the solution has multiple values on element boundaries, in order to determine the value of inter-cell fluxes, numerical flux functions are introduced, in this study, the Van Leer flux [Toro (2009)] is adopted to calculate fluxes F and G.

In each time step, a system of ordinary differential equation is formed:

$$\int_{K_j} \frac{\partial U}{\partial t} \phi_i \mathrm{d}\Omega = \mathrm{Res}(K_j; i); \quad (i = 1, \dots, m) \tag{8}$$

where $\mathrm{Res}(K_j; i)$ is residual term.

$$\begin{aligned} &\mathrm{Res}(K_j; i) = \\ &\int_{K_j} \left(F \frac{\partial \phi_i}{\partial x} + G \frac{\partial \phi_i}{\partial y} \right) \mathrm{d}\Omega - \int_{\partial K_j} \phi_i \left(F n_x + G n_y \right) \mathrm{d}S; \quad (i = 1, \dots, m) \end{aligned} \tag{9}$$

so the equations become:

$$M_{K_j} \frac{dU}{dt} = \mathrm{Res}(K_j; i) \quad ; \quad (i = 1, \dots, m) \tag{10}$$

or

$$\frac{dU}{dt} = M_{K_j}^{-1} \mathrm{Res}(K_j; i) \quad ; \quad (i = 1, \dots, m) \tag{11}$$

where M_{K_j} refers to the mass matrix on element K_j.

A third order explicit Runge-Kutta scheme is introduced to solve this ordinary equations system.

$$\begin{aligned} U^{(1)} &= U^n + \Delta t M^{-1} R(U^n), \\ U^{(2)} &= \frac{3}{4} U^n + \frac{1}{4} \left[U^{(1)} + \Delta t M^{-1} R(U^{(1)}) \right] \\ U^{(3)} &= \frac{1}{3} U^n + \frac{2}{3} \left[U^{(2)} + \Delta t M^{-1} R(U^{(2)}) \right] \\ U^{n+1} &= U^{(3)} \end{aligned} \tag{12}$$

where $U^n = U(t)$, $U^{n+1} = U(t + \Delta t)$.

When high order basis functions are introduced, the dissipation of DGM is not enough to suppress numerical oscillations near strong discontinuity regions. In order to eliminate non-physical oscillations in numerical solutions, slope limiters are adopted. The TVB limiter, WENO limiter and HWENO limiter are commonly used.

The TVB limiter limits the first order components of solutions.

$$\bar{m}(a_1, a_2, \cdots, a_m) = \begin{cases} a_1, & |a_1| \le M \Delta x^2 \\ \mathrm{m}(a_1, a_2, \cdots, a_m), & |a_1| > M \Delta x^2 \end{cases} \tag{13}$$

where M refers to a problem dependent constant, and m is minmod function.

$$\mathrm{m}(a_1, a_2, \cdots, a_m) = \begin{cases} s \min_i |a_i|, & \text{if} \quad s = \mathrm{sign}(a_1) = \mathrm{sign}(a_2) = \cdots = \mathrm{sign}(a_m) \\ 0, & \text{else} \end{cases} \tag{14}$$

In WENO limiter, the average solution of adjacent elements are used to reconstruct smooth solutions. if an element K_0 has three adjacent elements K_1 K_2 and K_3, the reconstruction stencils of polynomial P_1 for this element are $K_0K_1K_2$, $K_0K_1K_3$ and $K_0K_2K_3$.

$$\frac{1}{|K_0|}\int_{K_0} P_1 d\Omega = q_{K_0}$$
$$\frac{1}{|K_m|}\int_{K_m} P_1 d\Omega = q_{K_m} \tag{15}$$
$$\frac{1}{|K_n|}\int_{K_n} P_1 d\Omega = q_{K_n}, (m,n) = (1,2),(2,3),(1,3)$$

The HWENO limiter takes gradients of solutions into consideration, for an element K_0 with adjacent elements K_1, K_2 and K_3, four additional Hermite polynomials are constructed with stencils K_0K_0, K_0K_1, K_0K_2 and K_0K_3.

$$\begin{aligned} &\frac{1}{|K_0|}\int_{K_0} P_1 d\Omega = q_{K_0} \\ &\frac{1}{|K_s|}\int_{K_s} \frac{\partial P_1}{\partial x_i} d\Omega = \left.\frac{\partial P_1}{\partial x_i}\right|_{K_s}, s = 0,1,2,3 \end{aligned} \tag{16}$$

the new solution P is reconstructed based on polynomial $P^{(i)}$ and weight w^i.

$$P = \sum_{i=1}^{m} w_i P^{(i)} \tag{17}$$

The WENO and HWENO limiters get a better performance if they are activated only on strong discontinuity regions. In this study, a shock detector [Krivodonova et al. (2004)] is introduced to indicate problem elements, on which the WENO or HWENO limiter is activated.

Numerical Results

Supersonic cylinder flow is chosen as test case for the performance of limiters. There is a strong shockwave in front of the cylinder, which will test the stability of numerical schemes. The radius of cylinder is 0.01, inflow Mach number is 3, and the non-dimensional inflow parameters are: $\rho = 1$, $u = 1$, $v = 0$, $p = 1/\left(\gamma M^2\right)$, Fig.1 shows the sketch of computational mesh, in order to perform large-scale numerical simulations on parallel computers, the mesh is partitioned into 40 sub-domains using Metis software package, 40 processers are utilized to speed up the solution procedure. In order to increase the converge speed to steady state solution, local time stepping method is introduced.

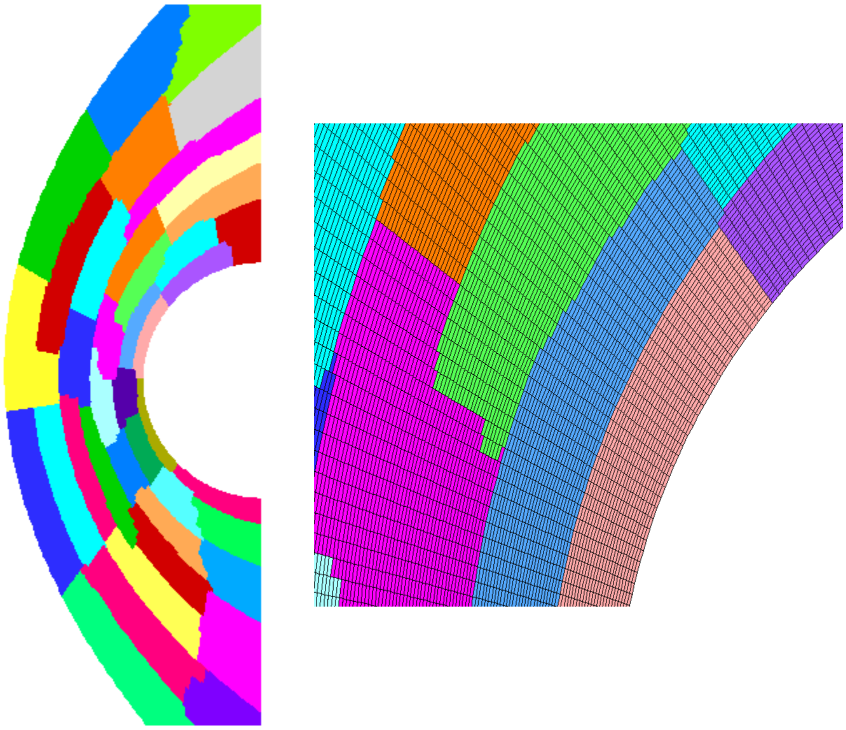

Figure 1. Computational mesh partitions(left) and its details (right)

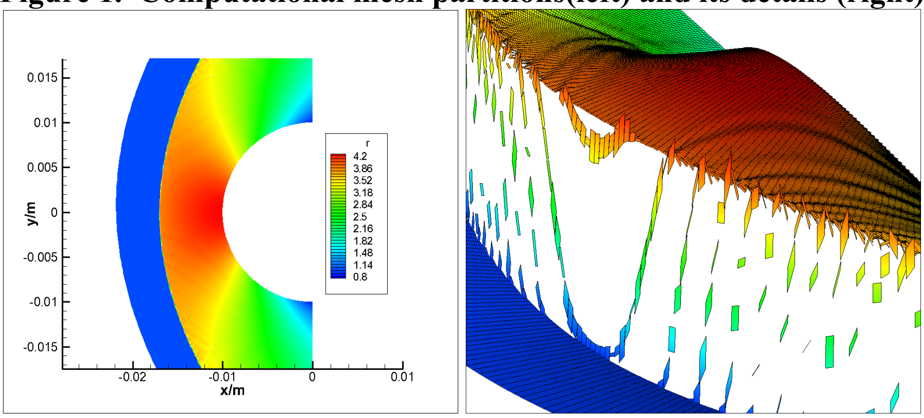

Figure 2. Density contour (left) and 3D view (right), with TVB limiter

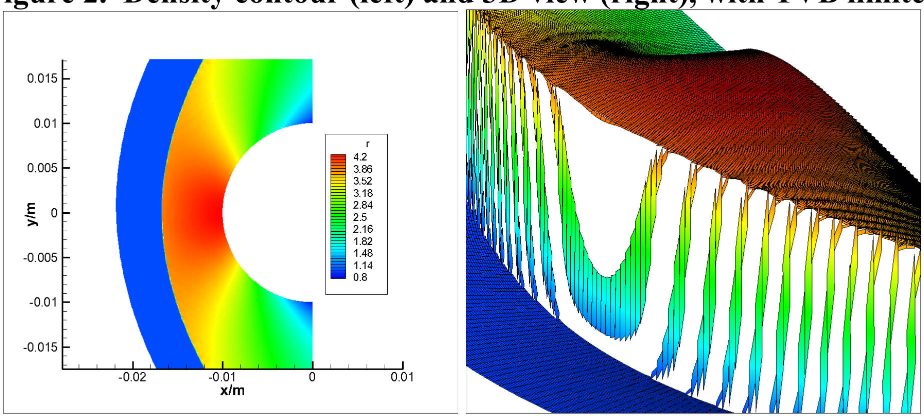

Figure 3. Density contour (left) and 3D view (right), with WENO limiter

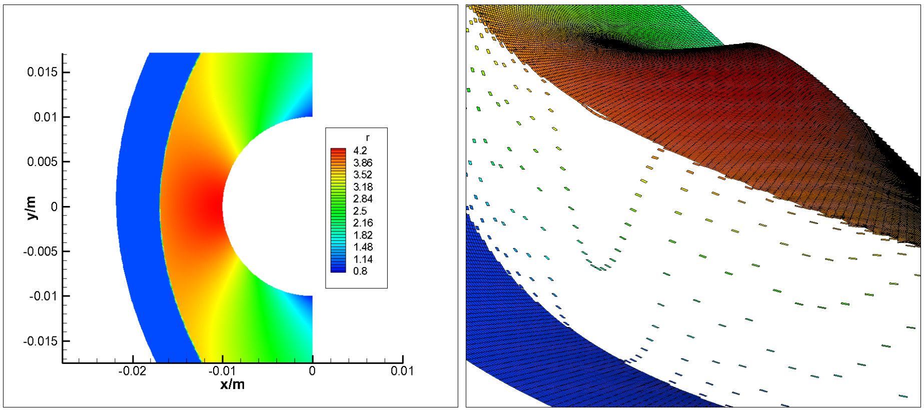

Figure 4. Density contour (left) and 3D view (right), with HWENO limiter

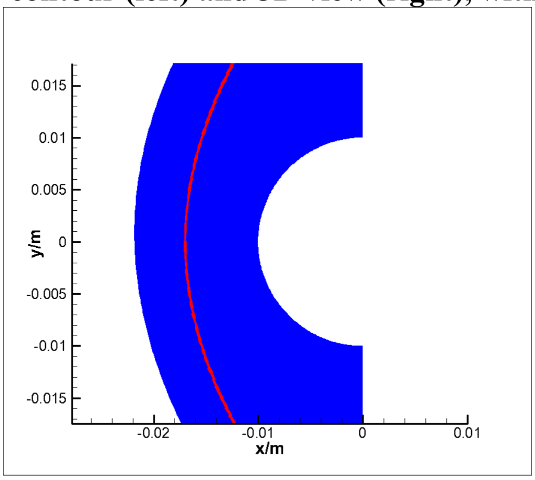

Figure 5. Elements on which limiter are activated

Density distributions and 3D density contours obtained by TVB, WENO and HWENO limiters are shown in Fig.2 to Fig.4. the results show that all these three limiters could stabilize the solution when strong shockwave appears, and capture the shockwave within few elements. In the shockwave regions, the density distribution with TVB limiter shows small overshoot, while the density distribution with WENO and HWENO limiter shows no overshoot. and the HWENO limiter gives more smooth solution than WENO and TVB limiters. Fig.5 shows the shockwave detected by shock detector, the red colored elements indicate that there are shockwaves, limiters are activated only on these elements.

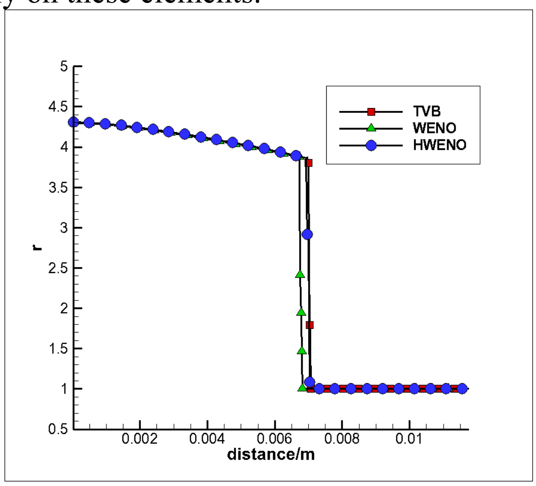

Figure 6. Density distribution along stagnation line.

The density distributions along stagnation line are shown in Fig.6, the density distributions before and after shockwave are identical for all the three limiters. But the shockwave position predicted with WENO limiter is more closer to the cylinder than the other limiters, the cause of these differences need more investigations.

Conclusions

In this study, the performance of slope limiters in discontinuous Galerkin method are compared and analyzed with two dimensional supersonic cylinder flow. The results show that all these limiters are able to stabilize the solution procedure, in shockwave regions the density fields predicted with WENO and HWENO limiters are more smoother than TVB limiter and contain no overshoot. In supersonic simulations, the WENO and HWENO limiters show better performances in suppressing non-physical oscillations and obtaining smooth solutions.

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